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ONE-BOSON-EXCHANGE RELATIVISTIC AMPLITUDE AS QUANTUM MECHANICAL POTENTIALS IN LOBACHEVSKY SPACE

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1. Introduction

Most of theoretical interpretations of data on NN and $\bar{N}N$ interactions is based on the use of the one-boson-exchange model. In this model, as the one-boson-exchange potential (OBEP), describing the nucleon-nucleon forces, the matrix elements of the relativistic scattering amplitude in the second order in the coupling constant are taken. However, this Born approximation in the form given by quantum field theory based on the four-dimensional covariant formalism little resembles the usual form of quantum-mechanical potentials and reduces to them only in the nonrelativistic limit /1/.

On the other hand, it is known /2/ that transition from the nonrelativistic theory to relativistic one is equivalent to the change of the Euclidean geometry of the three-dimensional space of 3-velocities by the Lobachevsky geometry*. Methods of the Lobachevsky geometry have been employed in papers /4,5/ to describe the processes of collisions of relativistic particles. In doing so, it has been found that application of the Lobachevsky geometry allows one to pick out an "absolute" part of the theory, i.e., such a part that does not depend on the geometry of the velocity space and has the same form both in the nonrelativistic region and in the relativistic one /5/.

A question naturally arises whether it is possible to comprehend also the matrix elements of relativistic scattering amplitude from this viewpoint. These elements can naturally be written in the Lobachevsky space as the condition for the particle momenta to be on the mass shell**

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*The three-dimensional covariant formulation of quantum field theory with the momentum space of the Lobachevsky geometry has been obtained in papers /3/.

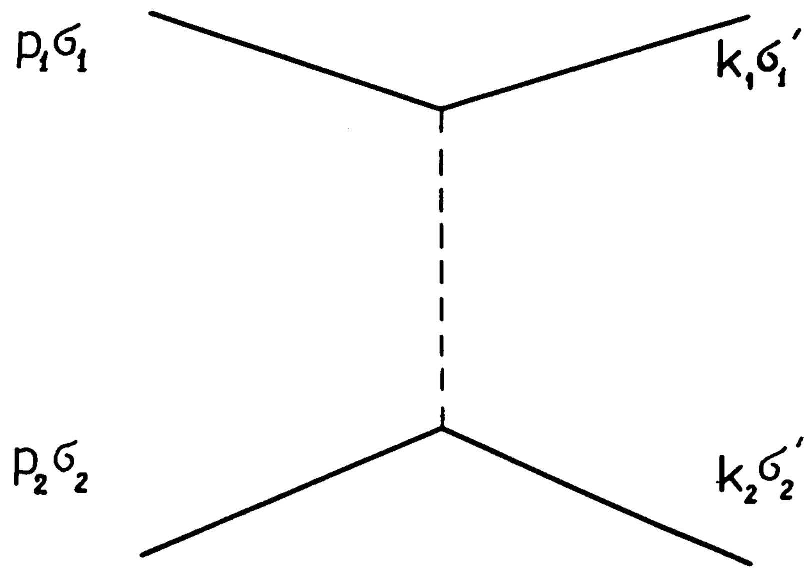
**We are using the system of units where $\hbar = c = 1$.

$$p_0^2 - \vec{p}^2 = M^2 \qquad (1)$$

defines the three-dimensional surface of the hyperboloid on the upper sheet of which the Lobachevsky geometry is just realized. In the present paper we would like to show that by using a method suggested in /6/ for describing the matrix elements of the scattering amplitude in terms of the Lobachevsky space, the relativistic OBEP can be shaped into the direct geometrical generalization of quantum-mechanical potentials. And here the important role belongs to a quantity named "the half-momentum transfer" which is an analog of the "half-velocity" of a particle, introduced by Chernikov /5/.

2. The Scalar Boson (Pseudoscalar) Exchange

Consider first the cases when the interaction proceeds through the exchange of a scalar or pseudoscalar particle with mass μ. This process is described by the Feynman diagram drawn in the figure.



The scattering amplitude in the second order in the coupling constant g is given by the expression

$$\langle \vec{p}_1, \sigma_1, \vec{p}_2\sigma_2 \,|\, T^{(2)}_{s(ps)} \,|\, \vec{k}_1\sigma_1', \vec{k}_2\sigma_2' \rangle = -g^2 \frac{j^{\sigma_1\sigma_1'}_{s(ps)}(\vec{p}_1, \vec{k}_1) \cdot j^{\sigma_2\sigma_2'}_{s(ps)}(\vec{p}_2, \vec{k}_2)}{\mu^2 - (p_1 - k_1)^2}. \qquad (2)$$

The current-matrix elements in (2)

$$j_s^{\sigma_1\sigma_1'}(\vec{p}_1, \vec{k}_1) = \langle \vec{p}_1\sigma_1 \,|\, j_s(0) \,|\, \vec{k}_1\sigma_1' \rangle = \bar{u}^{\sigma_1}(\vec{p}_1)\, u^{\sigma_1'}(\vec{k}_1)$$

$$j_{ps}^{\sigma_1\sigma_1'}(\vec{p}_1, \vec{k}_1) = \langle \vec{p}_1\sigma_1 \,|\, j_{ps}(0) \,|\, \vec{k}_1\sigma_1' \rangle = \bar{u}^{\sigma_1}(\vec{p}_1)\, \gamma_5\, u^{\sigma_1'}(\vec{k}_1)$$

can be transformed to the form local in the Lobachevsky space. For this aim we pass to bispinors defined in the rest frames*

$$\bar{u}^{\sigma}(\vec{p})\, u^{\sigma'}(\vec{k}) = \bar{u}^{\sigma}(0)\, S_p^{-1} S_k\, u^{\sigma'}(0), \qquad (3)$$

$$\bar{u}^{\sigma}(\vec{p})\, u^{\sigma'}(\vec{k}) = \bar{u}^{\sigma}(0)\, \gamma_5\, S_p^{-1} S_k\, u^{\sigma'}(0). \qquad (4)$$

The four-row bispinor transformation matrices S_p correspond to boosts Λ_p. These transformations do not

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*In the standard representation, where $\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$; $\vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$ $u^{\sigma}(0) = \sqrt{2M}\begin{pmatrix} \xi^{\sigma} \\ 0 \end{pmatrix}$. In the spinor representation $\gamma^{\mu} = \begin{pmatrix} 0 & g^{\mu\mu}\sigma^{\mu} \\ \sigma^{\mu} & 0 \end{pmatrix}$; $\gamma_5 = i\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$;

$u^{\sigma}(0) = \sqrt{M}\begin{pmatrix} \xi^{\sigma} \\ \xi^{\sigma} \end{pmatrix}$.

compose a group: their product is not the pure Lorentz transformation on the resulting vector but contains also the rotation describing the spin Tomas precession (the Wigner rotation)

$$S_p^{-1} S_k = S_{\Lambda_p^{-1} k} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot D^{1/2} \{ V^{-1}(\Lambda_p, k) \}. \tag{5}$$

Now let us introduce the following notation for 4-vector /7/ $\Lambda_p^{-1} k$.

$$\vec{k}(-)\vec{p} \equiv (\overrightarrow{\Lambda_p^{-1} k}) = \vec{k} - \frac{\vec{p}}{M}\left(k_0 - \frac{\vec{k}\vec{p}}{p_0 + M}\right) = \vec{\Delta}$$

$$(k(-)p)^0 \equiv (\Lambda_p^{-1} k)^0 = \frac{k_0 p_0 - \vec{k}\vec{p}}{M} = \sqrt{M^2 + (\vec{k}(-)\vec{p})^2} = \Delta_0. \tag{6}$$

In the nonrelativistic limit the vector $\vec{\Delta} = \vec{k}(-)\vec{p}$ reduces to the usual difference of two vectors in the Euclidean space $\vec{\Delta}_9 = \vec{k} - \vec{p}$. Thus, the vector $\vec{\Delta} = \vec{k}(-)\vec{p}$ of the Lobachevsky space can be treated as the relativistic generalization of the three-dimensional Euclidean vector of momentum transfer $\vec{\Delta}_9 = \vec{k} - \vec{p}$. In the spherical coordinates

$$p_0 = M \operatorname{ch}\chi_p \qquad k_0 = M \operatorname{ch}\chi_k \qquad \vec{n}_p = \frac{\vec{p}}{|\vec{p}|}$$

$$\vec{p} = M\vec{n}_p \operatorname{sh}\chi_p, \qquad \vec{k} = M\vec{n}_k \operatorname{sh}\chi_k$$

formula (6) turns into that of addition of the hyperbolic angles in the Lobachevsky trigonometry (see, e.g., ref. /7/)

$$\operatorname{ch}\chi_\Delta = \operatorname{ch}\chi_k \operatorname{ch}\chi_p - \operatorname{sh}\chi_k \operatorname{sh}\chi_p (\vec{n}_p \vec{n}_k). \tag{7}$$

In the space of particle velocities which is the Lobachevsky space /4,5/ an important role is played by the concept of particle half-velocity /5/. We define an analogous quantity called half-momentum transfer in the following way:

If

$$\Delta_0 = M \operatorname{ch}\chi_\Delta; \quad \vec{\Delta} = M\vec{n}_\Delta \operatorname{sh}\chi_\Delta; \quad \vec{n}_\Delta = \frac{\vec{\Delta}}{|\vec{\Delta}|},$$

then

$$\varkappa_0 = M \operatorname{ch}\chi_{\Delta/2} = M \cdot \sqrt{\frac{\Delta_0 + M}{2M}},$$

$$\vec{\varkappa} = M\vec{n}_\Delta \cdot \operatorname{sh}\chi_{\Delta/2} = \vec{\Delta} \cdot \sqrt{\frac{M}{2(\Delta_0 + M)}}. \tag{8}$$

In the nonrelativistic limit $\vec{\varkappa}$ turns into $\vec{\varkappa} \to \vec{\varkappa}_9 = \frac{\vec{k} - \vec{p}}{2}$.

The 4-vector of momentum transfer squared is expressed through $\vec{\varkappa}$ by the formula

$$t = (p-k)^2 = 2M^2(1 - \operatorname{ch}\chi_\Delta) = -4M^2 \operatorname{sh}^2 \frac{\chi_\Delta}{2} = -4\vec{\varkappa}^2. \tag{9}$$

Note that the explicit expression for the bispinor transformation matrix contains just the particle half-velocity $\vec{\omega} = \vec{n}_p \operatorname{th}\frac{\chi_p}{2}$

$$S_p = \sqrt{\frac{p_0 + M}{2M}}\left(1 + \frac{\vec{\alpha}\vec{p}}{p_0 + M}\right) = \operatorname{ch}\chi_{p/2} + (\vec{\alpha}\vec{n}_p)\operatorname{sh}\chi_{p/2}; \; \vec{\alpha} = \gamma^0\vec{\gamma}.$$

Therefore, it is easy to show with the help of (5) and (8), that the matrix elements of currents (3) and (4) are expressed via the components of vectors of the half-momentum transfer vector $\vec{\varkappa}$

$$j_{s(ps)}^{\sigma_1 \sigma_1'}(\vec{p}, \vec{k}) = \sum_{\sigma_{1p} = -\frac{1}{2}}^{1/2} j_{s(ps)}^{\sigma_1 \sigma_{1p}}(\vec{k}(-)\vec{p}) \, D^{1/2}_{\sigma_{1p}, \sigma_1'}\{V^{-1}(\Lambda_p, k)\}, \tag{10}$$

where

$$j_s^{\sigma_1 \sigma_{1p}}(\vec{k}(-)\vec{p}) = \sqrt{2M(\Delta_{10} + M)} \; \delta_{\sigma_1 \sigma_{1p}} = 2\varkappa_0 \cdot \delta_{\sigma_1 \sigma_{1p}} \tag{11}$$

and

$$j_{ps}^{\sigma_1 \sigma_{1p}}(\vec{k}(-)\vec{p}) = \sqrt{\frac{2M}{\Delta_0 + M}} \cdot \overset{*}{\xi}{}^{\sigma_1}(\vec{\sigma}_1 \vec{\Delta}) \, \xi^{\sigma_{1p}} = 2(\vec{\sigma}_1 \vec{\varkappa})_{\sigma_1 \sigma_{1p}}. \tag{12}$$

Equations (10)-(12) allow one to write the amplitude (2) in the form:

$$\langle \vec p_1\sigma_1;\vec p_2\sigma_2|T^{(2)}_{s(ps)}|\vec k_1\sigma_1';\vec k_2\sigma_2'\rangle =$$

$$= \sum_{\sigma_{1p},\sigma_{2p}=-\frac{1}{2}}^{1/2} \langle \vec p_1\sigma_1;\vec p_2\sigma_2|T^{(2)}_{s(ps)}|\vec k_1\sigma_{1p};\vec k_2\sigma_{2p}\rangle D^{1/2}_{\sigma_{1p}\sigma_1'}\times$$

$$\times\{V^{-1}_{p_1}(\Lambda_{p_1},k_1)\}\cdot D^{1/2}_{\sigma_{2p}\sigma_2'}\{V^{-1}(\Lambda_{p_2},k_2)\}, \tag{13}$$

where

$$\langle \vec p_1\sigma_1;\vec p_2\sigma_2|T^{(2)}_{s}|\vec k_1\sigma_{1p};\vec k_2\sigma_{2p}\rangle = \tag{14}$$

$$= g^2\frac{4\varkappa_{10}\varkappa_{20}\cdot\delta_{\sigma_1\sigma_{1p}}\delta_{\sigma_2\sigma_{2p}}}{\mu^2+4\vec\varkappa^2}$$

and

$$\langle \vec p_1\sigma_1;\vec p_2\sigma_2|T^{(2)}_{ps}|\vec k_1\sigma_{1p};\vec k_2\sigma_{2p}\rangle = \tag{15}$$

$$=-g^2\frac{4(\vec\sigma_1\vec\varkappa_1)_{\sigma_1\sigma_{1p}}\cdot(\vec\sigma_2\vec\varkappa_2)_{\sigma_2\sigma_{2p}}}{\mu^2+4\vec\varkappa_1^2}.$$

The amplitudes (14) and (15), due to their dependence of the vectors $\vec\Delta=\vec k(-)\vec p$ or, as the same, on the vector of momentum half-transfer $\vec\varkappa$ are local in the Lobachevsky space. In extracting an information on two-nucleon interactions from quantum field theory one usually employes the potential [8] in the c.m.s. having the form $(\vec p_1 = -\vec p_2=\vec p;\ \vec k_1=-\vec k_2=\vec k)$

$$V^{(2)}(\vec p,\vec k) = g^2\frac{(\vec\sigma_1\vec\Delta_3)\cdot(\vec\sigma_2\vec\Delta_3)}{\mu^2+\vec\Delta_3^2} = g^2\cdot\frac{4(\vec\sigma_1\vec\varkappa_3)(\vec\sigma_2\vec\varkappa_3)}{\mu^2+4\vec\varkappa_3^2} \tag{16}$$

which can be obtained from (2) by passing to the nonrelativistic limit. The denominator in (16) is the Yukawa potential. Comparing (15) and (16) it is seen that the relativistic amplitude (15) has the form of a direct geometrical generalization of potential (16) obtained by changing the Euclidean geometry with the Lobachevsky one. Consequently, it can be said that after separating the Wigner rotation, which is due to the relativistic spin kinematics, out of the Feynman matrix element (2), the remaining part has the ''absolute'' geometrical character.

Expression (15) can be written in terms of the relativistic spin-vector $W^\mu(\vec p)$, introduced in [9]

$$W^0(\vec p)=\frac{\vec\sigma\vec p}{2};\quad \vec W(\vec p)=\frac{M\vec\sigma}{2}+\frac{\vec p(\frac{\vec\sigma}{2}\vec p)}{p_0+M}. \tag{17}$$

$$W^\mu(\vec p)W_\mu(\vec p) = -M^2 s(s+1) = -M^2\,1/2(1/2+1)$$

In virtue of the condition

$$p^\mu W_\mu(\vec p) = 0 \tag{18}$$

only three components of vector (17) are independent.

The vector $W^\mu(\vec p)$ can be obtained by the pure Lorentz transformation of it in the rest frame of a particle

$$W^\mu(\vec p) = (\Lambda_p)^\mu_\nu\, W^\nu(0), \tag{19}$$

where

$$W_0(0)=0;\quad \vec W(0)=\frac{M\vec\sigma}{2}. \tag{20}$$

Taking into account relations (19), (20) and definition (16) of the vector of momentum transfer $\vec\Delta$ in the Lobachevsky space, one can easily see that the equality

$$-k^\mu W_\mu(\vec p) = (\vec k(-)\vec p)\,\frac{M\vec\sigma}{2} \tag{21}$$

is valid.

From (21) and (18) the important equality

$$(p-k)^\mu W_\mu(\vec p) = (\vec k(-)\vec p)\,\frac{M\vec\sigma}{2} \tag{22}$$

follows.

If one takes into consideration that the relativistic spin vector has an extra component ruled out by condition (18), the transition from the four-dimensional scalar product to the three-dimensional one by formula (22) is equivalent to the use of the Foldy-Wouthuysen transformation. Equality (22) also determines the connection of the obtained three-dimensional parametrization of currents in the Lobachevsky space /6/ with the general parametrization of currents in terms of the 4-vector of relativistic spin $W^{\mu}(\vec{p})$, proposed by Shirokov and Cheshkov /10-11/.

The role of the Wigner rotation entering into expressions (10) and (13) consists in "transferring" the particle spin indices from the momentum $\vec{k}$ onto a momentum $\vec{p}$ in the terminology of the authors of /10-11/ (see also refs. /12-13/). Therefore, in the c.m.s. all the spin indices σ_1, σ_2 and σ_{1p}, σ_{2p} in (14) and (15) are "sitting" on one and the same momentum $\vec{p}$, i.e., under the Lorentz transformations they transform according to the small group of this vector. As is shown in /6/, in using expressions (2) as the quasipotential in the quasipotential equation the Wigner rotation in (13) transfers all these spin indices in the equation onto one momentum $\vec{p}$, and, as a result, the interaction is described by quasipotentials (14) and (15) local in the Lobachevsky space.

3. Vector-Meson Exchange

In this case the interaction Lagrangian is as follows:

$$\mathcal{L}_{Int} = g_V \bar{\Psi}_N(x)\gamma^{\mu}\Psi_N(x)\Phi_{\mu}(x) + \frac{i f_V}{2M}\bar{\Psi}_N(x)\sigma^{\mu\nu}\Psi_*(x)\times$$

$$\times \Phi_{\mu\nu}(x),$$

where

$$\Phi_{\mu\nu}(x) = \frac{\partial \Phi_{\mu}(x)}{\partial x_{\nu}} - \frac{\partial \Phi_{\nu}(x)}{\partial x_{\mu}}$$

and

$$\sigma^{\mu\nu} = \frac{\gamma^{\mu}\gamma^{\nu} - \gamma^{\nu}\gamma^{\mu}}{2}. \qquad (23)$$

In the second order in the coupling constant the matrix element of scattering amplitude corresponding to this Lagrangian has the form

$$\langle \vec{p}_1\sigma_1; \vec{p}_2\sigma_2 | T_V^{(2)} | \vec{k}_1\sigma_1'; \vec{k}_2\sigma_2'\rangle = - \frac{j^{\mu}_{\sigma_1\sigma_1'}(\vec{p}_1,\vec{k}_1)\cdot j_{\mu\,\sigma_2\sigma_2'}(\vec{p}_2,\vec{k}_2)}{\mu^2 - (p_1-k_1)^2}, \qquad (24)$$

where

$$j^{\mu}_{\sigma\sigma'}(\vec{p},\vec{k}) = \bar{u}^{\sigma}(\vec{p})\{g_V\cdot\gamma^{\mu} + \frac{\sigma^{\mu\nu}}{2M}q_{\nu}f_V\}u^{\sigma'}(\vec{k}) \qquad (25)$$

$$q_{\nu} = (p-k)_{\nu}$$

is the nucleon vector current.

Consider now the amplitude (24) assuming for the moment the constant $f_V = 0$. In formula (25), similarly to (3) and (4), we perform a transition to bispinors in the particle rest frame

$$j^{\mu}(\vec{p},\vec{k}) = \bar{u}^{\sigma}(\vec{p})\gamma^{\mu}u^{\sigma'}(\vec{k}) =$$

$$= \bar{u}^{\sigma}(0)S_p^{-1}\gamma^{\mu}S_p\cdot S_{k(-)p}\cdot I\cdot D^{1/2}\{V^{-1}(\Lambda_p,k)\}. \qquad (26)$$

Taking into account the obtained in /6/ equality

$$S_p^{-1}\gamma^{\mu}S_p = (\Lambda_p)^{\mu}_{\nu}\gamma^{\nu} = \frac{\gamma_0}{M}\{p^{\mu} + 2\gamma_5 W^{\mu}(\vec{p})\} \qquad (27)$$

that follows from the definition of $W^{\mu}(\vec{p})$ (17), (19), current (26) can be represented in the form analogous to (10):

$$j^{\mu}_{\sigma\sigma'}(\vec{p},\vec{k}) = \sum_{\sigma_p=-\frac{1}{2}}^{1/2} j^{\mu}_{\sigma\sigma_p}(\vec{k}(-)\vec{p};\vec{p})\,D^{1/2}_{\sigma_p\sigma'}\{V^{-1}(\Lambda_p,k)\}, \qquad (28)$$

where using the explicit form of the matrix $S_{k(-)p}$ in (26), we obtain

$$j^{\mu}_{\sigma\sigma_p}(\vec k(-)\vec p;\vec p) = \frac{1}{M}\overset{*}{\xi}{}^{\sigma}\{2p^{\mu}\varkappa_0 + 4W^{\mu}(\vec p)(\vec\sigma\vec\varkappa)\}\,. \qquad (29)$$

Thus, after substituting (28) into (24) we find that the matrix element of the scattering amplitude for the vector-meson exchange can also be represented in the form (13). In doing so, for the amplitude

$$\langle \vec p_1\sigma_1;\vec p_2\sigma_2|T^{(2)}_V|\vec k_1\sigma_{1p};\vec k_2\sigma_{2p}\rangle =$$

$$= \overset{*}{\xi}{}^{\sigma_1}\overset{*}{\xi}{}^{\sigma_2}\,T^{(2)}_V(\vec k(-)\vec p;\vec p)\,\xi^{\sigma_{1p}}\xi^{\sigma_{2p}} = \qquad (30)$$

$$= -g^2_V\,\frac{j^{\mu}_{\sigma_1\sigma_{1p}}(\vec k_1(-)\vec p_1;\vec p_1)\cdot j_{\mu\,\sigma_2\sigma_{2p}}(\vec k_2(-)\vec p_2;\vec p_2)}{\mu^2-(p_1-k_1)^2}$$

in the c.m.s. $\vec p_1=-\vec p_2=\vec p;\ \vec k_1=-\vec k_2=\vec k$ with the help of (28) we obtain the expression

$$T^{(2)}_V(\vec k(-)\vec p;\vec p) = -g^2_V\frac{4M^2}{\mu^2+4\vec\varkappa^2} - g^2_V\cdot\frac{4(\vec\sigma_1\vec\varkappa)(\vec\sigma_2\vec\varkappa)-(\vec\sigma_1\vec\sigma_2)\vec\varkappa^2}{\mu^2+4\vec\varkappa^2} -$$

$$- g^2_V\cdot\frac{8p_0\varkappa_0}{M^2}\cdot\frac{i\vec\sigma_1[\vec p\vec\varkappa]+i\vec\sigma_2[\vec p\vec\varkappa]}{\mu^2+4\vec\varkappa^2} -$$

$$- g^2_V\cdot\frac{8}{M^2}\cdot\frac{p_0^2\varkappa_0^2+2p_0\varkappa_0(\vec p\vec\varkappa)-M^4}{\mu^2+4\vec\varkappa^2} - \qquad (31)$$

$$- g^2_V\cdot\frac{8}{M^2}\cdot\frac{(\vec\sigma_1\vec p)(\vec\sigma_1\vec\varkappa)\cdot(\vec\sigma_2\vec p)(\vec\sigma_2\vec\varkappa)}{\mu^2+4\vec\varkappa^2}\,,$$

where

$$\vec\varkappa = \sqrt{\frac{M}{2(\Delta_0+M)}}\,(\vec k(-)\vec p)\,.$$

Now let us explain the meaning of expressions entering into (31). The first term corresponds to the interaction of spinless particles and is the relativistic generalization of the Yukawa potential. The second term describes the spin-spin interaction, the third term the spin-orbital one. The third line contains the terms describing the contribution to orbital motion. The latter term in (31) can be expanded in the following spin structures

$$\frac{1}{M^2}\cdot\frac{(\vec\sigma_1\vec p)(\vec\sigma_1\vec\varkappa)\cdot(\vec\sigma_2\vec p)(\vec\sigma_2\vec\varkappa)}{\mu^2+4\vec\varkappa^2} =$$

$$= \frac{1}{M^2}\cdot\frac{(\vec p\vec\varkappa)^2+[\vec p\vec\varkappa]^2}{\mu^2+4\vec\varkappa^2} + \frac{8\varkappa_0}{M^3}\cdot(\vec p\vec\varkappa)^2\ \times \qquad (32)$$

$$\times\ \frac{i\vec\sigma_1[\vec p\vec\varkappa]+i\vec\sigma_2[\vec p\vec\varkappa]}{\mu^2+4\vec\varkappa^2} +$$

$$+\ \frac{4}{M^2}\cdot\frac{(i\vec\sigma_1[\vec p\vec\varkappa]+i\vec\sigma_2[\vec p\vec\varkappa])^2}{\mu^2+4\vec\varkappa^2}\,.$$

In the nonrelativistic limit, when the curvature of the Lobachevsky space tends to zero and the space becomes the Euclidean one, expression (31) up to the terms of the order $\frac{1}{c^2}$ turns into:

$$T^{(2)}_{\text{nonrel}}(\vec k-\vec p;\vec p) = -g^2_V\cdot\frac{4M^2}{\mu^2+4\vec\varkappa_{\text{э}}^2} - \qquad (33)$$

$$- g^2_V\frac{4}{c^2}\cdot\frac{(\vec\sigma_1\vec\varkappa_{\text{э}})(\vec\sigma_2\vec\varkappa_{\text{э}})-(\vec\sigma_1\vec\sigma_2)\varkappa^2}{\mu^2+4\vec\varkappa_{\text{э}}^2} -$$

$$- g_V^2 \cdot \frac{8}{c^2} \cdot \frac{i\vec{\sigma}_1[\vec{p}\vec{\varkappa}_э] + i\vec{\sigma}_2[\vec{p}\vec{\varkappa}_э]}{\mu^2 + 4\vec{\varkappa}_э^2} -$$

$$- g_V^2 \cdot \frac{8}{c^2} \cdot \frac{(\vec{p} + \vec{\varkappa}_э)^2}{\mu^2 + 4\vec{\varkappa}_э^2},$$

where
$$\vec{\varkappa}_э = \frac{\vec{k} - \vec{p}}{2}.$$

Expression (32), i.e., the last line of (31), vanishes due to proportionality $1/c^4$ in the nonrelativistic limit of (31) if it is taken up to the terms of an order $1/c^2$.

Next, let us examine the part of nucleon current (25) which contains the tensor interaction

$$\frac{f_V}{2M} \cdot \bar{u}^{\sigma}(\vec{p})\, \sigma^{\mu\nu} q_\nu u^{\sigma'}(\vec{k}). \tag{34}$$

After passing in (34) to bispinors in the particle rest frames we arrive at the expression

$$\frac{f_V}{2M}\bar{u}^{\sigma}(0) S_p^{-1} \sigma^{\mu\nu} S_p \cdot q_\nu \cdot S_{k(-)p} \cdot I \cdot D^{1/2}\{V^{-1}(\Lambda_p, k)\} u^{\sigma'}(0).$$

Applying formula (27) makes it possible to get the equality

$$S_p^{-1}\sigma^{\mu\nu} S_p = -\frac{4}{M^2}\Sigma^{\mu\nu}(\vec{p}) - \frac{2}{M^2}\gamma_5(W^{\mu}(\vec{p})p^{\nu} - W^{\nu}(\vec{p})p^{\mu}), \tag{35}$$

where the operator

$$\Sigma^{\mu\nu}(\vec{p}) = \frac{W^{\mu}(\vec{p})W^{\nu}(\vec{p}) - W^{\nu}(\vec{p})W^{\mu}(\vec{p})}{2} \tag{36}$$

is constructed by analogy with $\sigma^{\mu\nu}$ (23) but instead of the γ^{μ} - matrices the relativistic spin vector $W^{\mu}(\vec{p})$ is used. Allowing for (29), (35) and (9) the nucleon current (25) takes the form

$$j^{\mu}_{\sigma\sigma'}(\vec{p}, \vec{k}) = \sum_{\sigma_p=-\frac{1}{2}}^{1/2} j^{\mu}_{\sigma\sigma_p}(\vec{k}(-)\vec{p};\vec{p})\, D^{1/2}_{\sigma_p\sigma'}\{V^{-1}(\Lambda_p, k)\}, \tag{37}$$

where

$$j^{\mu}(\vec{k}(-)\vec{p};\vec{p}) = \frac{\varkappa_0}{M}\{2p^{\mu} g_E(t) + \frac{W^{\mu}(\vec{p})}{\varkappa_0^2}(W^{\nu}(\vec{p})q_\nu)\, g_E(t) +$$

$$+ 4\Sigma^{\mu\nu}(\vec{p})q_\nu \cdot \frac{f_V}{M^2}\}. \tag{38}$$

Here we have denoted by $g_E(t)$, by analogy with the Sach's "charge" form factor $G_E(t)$, the following combination of constants

$$g_E(t) = g_V + \frac{t}{4M^2} \cdot f_V = g_V - \frac{\vec{\varkappa}^2}{M^2} f_V. \tag{39}$$

By using eq. (27) and the relation $\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu}$ it is easy to obtain the equality

$$W^{\mu}(\vec{p})W^{\nu}(\vec{p}) + W^{\nu}(\vec{p})W^{\mu}(\vec{p}) = \frac{p^{\mu}p^{\nu} - M^2 g^{\mu\nu}}{2};$$

the use of which allows us to represent the operator (36) in the form

$$\Sigma^{\mu\nu}(\vec{p}) = W^{\mu}(\vec{p})W^{\nu}(\vec{p}) + \frac{1}{4}\{M^2 g^{\mu\nu} - p^{\mu}p^{\nu}\}. \tag{40}$$

Applying of (40) to (38) results in the form of the nucleon current

$$j^{\mu}_{\sigma\sigma'}(\vec{k}(-)\vec{p};\vec{p}) =$$

$$= \frac{1}{M}\overset{*}{\xi}{}^{\sigma}\{2g_V\varkappa_0 p^{\mu} + f_V\varkappa_0 q^{\mu} + 4g_V W^{\mu}(\vec{p})(\vec{\sigma}\vec{\varkappa})\}\xi^{\sigma'}, \tag{41}$$

analogous to (29). Substituting (41) into (30) we obtain in the c.m.s. the following expression for (24)

$$\langle\vec{p}, \sigma_1\sigma_2|T_V^{(2)}|\vec{k}, \sigma_1'\sigma_2'\rangle = \sum_{\sigma_{1p}\sigma_{2p}=-\frac{1}{2}}^{1/2} \langle\vec{p}, \sigma_1\sigma_2|T_V^{(2)}|\vec{k}, \sigma_{1p}\sigma_{2p}\rangle \times$$

$$\times D^{1/2}_{\sigma_{1p}\sigma_1'}\{V^{-1}(\Lambda_p, k)\} D^{1/2}_{\sigma_{2p}\sigma_2'}\{V^{-1}(\Lambda_p, k)\} \tag{42}$$

where

$$<\vec p,\sigma_1\sigma_2\,|\,T_V^{(2)}\,|\,\vec k,\sigma_{1p}\sigma_{2p}> \equiv \overset{*}{\xi}{}^{\sigma_1}\overset{*}{\xi}{}^{\sigma_2}\,T_V^{(2)}(\vec k(-)\vec p;\vec p)\,\xi^{\sigma_{1p}}\xi^{\sigma_{2p}};$$

and

$$T_V^{(2)}(\vec k(-)\vec p;\vec p) = -g_M^2\cdot\frac{4M^2}{\mu^2+4\vec\varkappa^2} - 4g_M^2\frac{(\vec\sigma_1\vec\varkappa)(\vec\sigma_2\vec\varkappa)-(\vec\sigma_1\vec\sigma_2)\vec\varkappa^2}{\mu^2+4\vec\varkappa^2} -$$

$$-8g_M^2\cdot\frac{p_0\varkappa_0}{M^2}\;\frac{i\vec\sigma_1[\vec p\vec\varkappa]+i\vec\sigma_2[\vec p\vec\varkappa]}{\mu^2+4\vec\varkappa^2} - \tag{43}$$

$$-\frac{4}{\mu^2+4\vec\varkappa^2}\Big\{\frac{4p_0\varkappa_0}{M^2}(\vec p\vec\varkappa)\,g_Vg_M+\varkappa_0^2\,g_M^2-2M^2\,g_M^2+$$

$$+[(2p_0^2-M^2)\,g_V^2-\vec\varkappa^2f_V^2]\,\frac{\varkappa_0}{M}\Big\}-$$

$$-g_M^2\cdot\frac{8}{M^2}\cdot\frac{(\vec\sigma_1\vec p)(\vec\sigma_p\vec\varkappa)\cdot(\vec\sigma_2\vec p)(\vec\sigma_2\vec\varkappa)}{\mu+\vec\varkappa}$$

and $g_M = g_V + f_V$.

Comparing (43) with (31) we may conclude that including the term with $\sigma^{\mu\nu}$ into the current only results in an essential change of the part of the amplitude which describes the orbital motion. In other expressions only the change of the coupling constant $g_V^2 \to g_M^2$ takes place and the spin structure of the amplitude remains unchanged.

The above consideration allows one to state that the parametrization with the use of the Lobachevsky space makes it possible to represent the Born approximation for the relativistic scattering amplitude (i.e., the matrix element corresponding to a vector-particle exchange) in the form of a direct geometrical generalization of quantum mechanical potentials derived by changing the Euclidean quantities by their analogs in the Lobachevsky space. Expressions (33) and (31) are similar in form (of course, without the last term of (31)) and differ only in the geometrical nature of their quantities $\vec\varkappa$ and $\vec\varkappa_э$. Therefore, our method allows one to obtain some terms of "absolute"

geometrical nature from the "dynamical part" of (24). One might wonder what is the reason that the half-momentum transfer $\vec\varkappa$ rather than the momentum transfer $\vec\Delta = \vec k(-)\vec p$ is a more suitable quantity in the relativistic geometrical generalization. To answer this question we consider an analogous quantity, the particle half-momentum $\pi_p = (\pi_0, \vec\pi) = (M\,\mathrm{ch}\,\chi_p/2\,;\,M\vec n_p\,\mathrm{sh}\,\chi_p/2)$.

In the nonrelativistic limit, the energy of a particle moving with momentum $\vec p$ is

$$p_0 = \sqrt{M^2+\vec p^2} \approx M + \frac{\vec p^2}{2M} = M + \frac{2(\pi_э)^2}{M};\quad \vec\pi_э = \vec p/2. \tag{44}$$

On the other hand, the exact expression for the relativistic energy in terms of the half-momentum looks like:

$$p_0 = \sqrt{M^2+\vec p^2} = M + \frac{2(\vec\pi_p)^2}{M}. \tag{45}$$

The analogous relation

$$\Delta_0 = \sqrt{M^2+\vec\Delta^2} = M + \frac{2(\vec\varkappa)^2}{M} \tag{46}$$

also holds for $\vec\Delta$. Thus, it can be said that the half-momentum (or half-momentum transfer) is connected with the particle kinetic energy equal to the part of the total energy without the rest mass $W_{kin.} = p_0 - M$. It is just the kinetic energy (and not the total one) which has the non-relativistic analog, and only this one can be generalized in a geometrical way. It is seen from (44) and (45) that this generalization is achieved by introducing the half-momentum parameter which makes it possible to give the "absolute" geometrical form /5/ to the particle kinetic energy. The half-momentum transfer in (46) plays the role of the particle half-momentum $\vec\pi_p$ in (45). Consequently, just the quantities of kinetic nature can be generalized geometrically, i.e., they have the "absolute" character.

In this connection, we note that the explicit form of the functions $D^{1/2}\{V^{-1}(\Lambda_p, k)\}$ decribing the Wigner rotation is the most simple one just in terms of half-momentum of particle $\vec{\pi}_p$. Thus, the complicated in the form expression /9/

$$D^{1/2}\{V^{-1}(\Lambda_p, k)\} = \frac{(k_0+M)(p_0+M) - (\vec{\sigma}\vec{k})(\vec{\sigma}\vec{p})}{\sqrt{2(k_0+M)(p_0+M)(k_0p_0 - \vec{k}\vec{p} + M^2)}} \quad (47)$$

written in terms of $\vec{\pi}_p$ looks as follows:

$$D^{1/2}\{V^{-1}(\Lambda_p, k)\} = \frac{\pi_k^0 \pi_p^0 - (\vec{\sigma}\vec{\pi}_k)(\vec{\sigma}\vec{\pi}_p)}{M\varkappa_0} .$$

4. Conclusion

To complete the paper, we briefly summarize our consideration. As has been shown, the transition to quantities in terms of the Lobachevsky geometry allows one to represent matrix elements of the relativistic scattering amplitude in the form of direct geometrical generalization of the quantum-mechanical potentials: in the "absolute" form /5/. This makes it possible to say that the matrix elements of the relativistic scattering amplitude in the second order in coupling constant (or rather their part of the order $1/c^2$ in the nonrelativistic limit) could be obtained by direct geometrical generalization of the corresponding quantum-mechanical potentials. Therefore the developed here formalism can be used for a phenomenological description of interactions of elementary particles and bound state system in the relativistic energy range in the cases when there are difficulties connected with application of the methods of relativistic quantum field theory, but the nonrelativistic quantum-mechanical potentials, describing qualitatively the feature, are known.

The corresponding relativistic potentials can be obtained by the following recipe: at the beginning, it is necessary in the nonrelativistic pitentials taken in the momentum space, to make transition from the momentum transfer to the half-momentum transfer $\vec{\varkappa}_E = \frac{\vec{k} - \vec{p}}{2}$ and then to replace the nonrelativistic half-momentum transfer $\vec{\varkappa}_E$ by its analog in the Lobachevsky space by formula (8). To obtain the relativistically covariant expression it is also necessary to multiply the obtained expression by $D^{1/2}$ function describing the kinematical Wigner rotation.

Note that unlike the Foldy-Wouthuysen method, the transition from (24) to the three-dimensional expression (31) has been achieved in an equivalent way without expanding in powers of v^2/c^2 and losing the relativistic terms. In particular, in (31) there enters also the term (32) of the same order in $1/M^2$ as the remaining terms of (31) (however, the term (32) is not present in the nonrelativistic expression (33) due to its proportionality to $1/c^4$).

The obtained forms of the relativistic OBEP are suitable for their use as potentials in the quasipotential equations, since the momentum space of these equations is the Lobachevsky space /7/.

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