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ON A REALIZATION OF gl(n, R) IN TERMS OF RATIONAL FUNCTIONS OF BOSE OPERATORS

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In a recent paper [1] we have constructed realization of the algebra $g\ell(n,R)$ in terms of $n-1$ pairs of Bose creation and annihilation operators. In the present note we wish to study this realization from the point of view of the representations of $g\ell(n,R)$ and to show that it leads to representations with diagonal Casimir operators such that the values of these operators appear as free parameters in the generators. We shall consider in some more details the algebra $s\ell(2,R)$.

Let b_i, a_i, $i=1 \dots n$ be a set of $2n$ pairs of creation and annihilation operators with the commutation relations:

$$[a_i, b_j] = \delta_{ij}, \quad [a_i, a_j] = [b_i, b_j] = 0. \qquad (1)$$

It is known that one can define in a rigorous way rational functions out of a_i, b_i [2]. Let D_n be the quotient division ring of a_i, b_i, i.e., the set of all rational functions of a_i, b_i, $i=1,\dots,n$. We shall show that D_n contains an isomorphic image of $g\ell(n+1,R)$. Let e_{ij}, $i,j = 1,\dots,n+1$ be the generators of the algebra $g\ell(n+1,R)$,

$$[e_{ij}, e_{k\ell}] = e_{i\ell}\delta_{jk} - e_{kj}\delta_{i\ell}. \qquad (2)$$

Introduce the matrices

$$E_n = \begin{pmatrix} e_{11} \dots e_{1n} e_{1\,n+1} \\ \dots\dots\dots\dots \\ e_{n1} \dots e_{nn} e_{n\,n+1} \\ e_{n+1\,1} \dots e_{n+1\,n} e_{n+1\,n+1} \end{pmatrix}, \quad P_n = \begin{pmatrix} e_{11} \dots e_{1n} e_{1\,n+1} \\ \dots\dots\dots\dots \\ e_{n1} \dots e_{nn} e_{n\,n+1} \end{pmatrix} \qquad (3)$$

Then the i-th order Casimir operator K_i of $g\ell(n+1,R)$ is a trace of E_n of power i,

$$K_i = \mathrm{Tr}\,\underbrace{E_n \cdot E_n \ldots E_n}_{i\ \text{times}} \equiv \mathrm{Tr}\,E^{(i)}, \quad i = 1, 2, \ldots, n+1, \qquad (4)$$

where sums and products have to be considered in the sence of the corresponding operations in the universal enveloping algebra U of $g\ell(n+1,R)$. Let D be the quotient division ring of U and let us consider the set of $n+1$ relations (4) as a system of $n+1$ equations with respect to the generators $e_{n+1,1}, \ldots, e_{n+1,n+1}$. Determining $e_{n+1,n+1}$ from $K_1 = e_{11} + \ldots + e_{n+1,n+1}$ we obtain a system of n equations which is linear in $e_{11}, \ldots, e_{nn}$. Clearly this system has a solution in D since it can be solved in terms of the operations defined in D. We obtain

$$e_{n+1,i} = f_i(K, P_n) \in D, \qquad (5)$$

where $K \equiv (K_1, \ldots, K_{n+1})$.

The operators K commute with all generators of $g\ell(n+1,R)$. Therefore replacing in (5) the operators $(K_1, \ldots, K_{n+1})$ by arbitrary numbers $a \equiv (a_1, \ldots, a_{n+1})$ we do not change the commutation relations (2) and hence obtain another realization of $g\ell(n+1,R)$ in D,

$$e_{n+1,i} = f_i(a, P_n) \in D. \qquad (6)$$

Thus we have expressed one part of the generators of $g\ell(n+1,R)$, namely $e_{n+1,i}$, $i = 1, \ldots, n+1$, through the other generators. Clearly if we write down back the Casimir operators in terms of the realization (6), we shall obtain for K_i the number a_i. Suppose that the generators of this particular realization are defined as operators in some linear space L (for instance in D). Then the necessary condition for the representation of $g\ell(n+1,R)$ in L to be irreducible will be fulfilled since the Casimir operators in L will be miltiple of the unity operator.

In order to express e_{ij} in terms of a_i, b_j we note that the mapping

$$P_n \to \begin{pmatrix} b_1 a_1 & \ldots & b_1 a_n & b_1 \\ \ldots & \ldots & \ldots & \ldots \\ b_n a_1 & \ldots & b_n a_n & b_n \end{pmatrix} \qquad (7)$$

is an isomorphism of the Lie algebra with generators P_n into D_n. Therefore, replacing in (6) the elements of P_n according to (7), we obtain a realization of the $g\ell(n+1,R)$ generators as rational functions of n pairs of creation and annihilation operators,

$$e_{ij} = b_i a_j, \quad e_{i,n+1} = b_i, \quad i,j = 1, \ldots, n \qquad (8)$$

$$e_{n+1,k} = F_k(a, a_1, \ldots, a_n, b_1, \ldots, b_n), \quad k = 1, \ldots, n+1.$$

The elements $e_{n+1,k}$ can always be written in a form

$$e_{n+1,k} = \frac{1}{P_k(a, a, b)} Q_k(a, a, b), \qquad (9)$$

where P_k and Q_k are polynomials in $a \equiv (a_1, \ldots, a_n)$ and $b \equiv (b_1, \ldots, b_n)$. If now the parameters $a_1, \ldots, a_{n+1}$ are chosen in such a way that P_k $(k = 1, \ldots, n+1)$ are functions only of a (or only of b), then the generators (8) can be turned into differential operators through the substitution

$$a_i \to \frac{\partial}{\partial x_i}, \quad b_i \to x_i \quad \left(a_i \to -x_i, \; b_i \to \frac{\partial}{\partial x_i}\right).$$

In the first case we obtain $\left(\frac{\partial}{\partial x} \equiv \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)\right)$:

$$e_{ij} = x_i \frac{\partial}{\partial x_j}, \quad e_{i,n+1} = x_i, \quad i,j = 1, \ldots, n$$

$$e_{n+1,k} = \frac{1}{P_k(a, x)} Q_k\left(a, \frac{\partial}{\partial x}, x\right), \quad k = 1, \ldots, n+1. \qquad (10)$$

It is important to point out that the Casimir operators obtained from the realization (10) are diagonal in any functional space L of n real variables if certainly the generators are defined as operators in L.

Let us illustrate the above results on the simplest case of $g\ell(2,R)$. We have

$$E_1 = \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix}, \quad P_1 = (e_{11}, e_{12}). \tag{11}$$

Therefore

$$K_1 = e_{11} + e_{22}$$

$$K_2 = e_{11}e_{11} + e_{12}e_{21} + e_{21}e_{12} + e_{22}e_{22}$$

and for e_{21}, e_{22} we obtain

$$e_{22} = K_1 - e_{11}$$

$$e_{21} = \frac{1}{2e_{12}}[K_2 - K_1 - K_1^2 + 2(K_1+1)e_{11} - 2e_{11}^2]. \tag{12}$$

Replacing K_1 and K_2 by arbitrary constants α_1, α_2 and using the mapping (7), namely $(e_{11}, e_{12}) \to (ba, a)$ we obtain for the generators:

$$e_{11} = ba$$

$$e_{21} = \frac{1}{2b}[\alpha_2 - \alpha_1 - \alpha_1^2 + 2(\alpha_1+1)ba - 2baba]$$

$$e_{12} = b$$

$$e_{22} = \alpha_1 - ba. \tag{13}$$

If we substitute $(a,b) \to (\frac{\partial}{\partial x}, x)$ we obtain a realization E_{ij} of the type (7). The generators E_{ij} may be defined in a space spanned on all x^n, n - integer, and for the matrix elements one gets

$$E_{11}x^n = nx^n, \quad E_{12}x^n = x^{n+1}, \quad E_{22}x^n = (\alpha_1 - n)x^n$$

$$E_{21}x^n = \frac{1}{2}[\alpha_2 - \alpha_1 - \alpha_1^2 + 2(\alpha_1+1)n - 2n^2]x^{n-1}. \tag{14}$$

It is more interesting, however, to consider the mapping $(a,b) \to (-x, \frac{\partial}{\partial x})$. In this case we obtain a realization of the type (7) if we put $\alpha_2 = \alpha_1 + \alpha_1^2$. Let us consider the real algebra $s\ell(2,R)$. For the generators $H_3 = e_{22} - e_{11}$, $H_+ = e_{12}$ and $H_- = e_{21}$ we have $(s = -\alpha - 1)$

$$H_3 = 2x\frac{\partial}{\partial x} - s + 1 \qquad H_+ = \frac{\partial}{\partial x}$$

$$H_- = -x^2\frac{\partial}{\partial x} + (s-1)x. \tag{15}$$

We consider now the representations of the group $SL(2,R)$ and the corresponding infinitesimal operators. We recall that the set of all irreducible representations is labeled by two numbers (s, ϵ), where $\epsilon = 0, 1$ and s is an arbitrary complex number [3]. The representation $\chi = (s, \epsilon)$ can be realized in the space L_χ of all infinitely differenciable functions $\phi(x)$ of one real variable such that the function $\hat{\phi}(x) = |x|^{s-1}\,\mathrm{sign}^\epsilon x\,\phi(-\frac{1}{x})$ is also infinitely differenciable. In this case the element $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2,R)$ is represented by an operator $T_\chi(g)$ according to the formula $(\phi(x) \in L_\chi)$

$$T_\chi(g)\,\phi(x) = |\beta x + \delta|^{s-1}\,\mathrm{sign}^\epsilon(\beta x + \delta)\,\phi\left(\frac{\alpha x + \gamma}{\beta x + \delta}\right). \tag{16}$$

In the lowest representation the generators H_3, $H_\pm$ can be represented by the matrices

$$h_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad h_+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h_- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{17}$$

The corresponding to them one-parameter subgroups are

$$g_3(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \quad g_+(t) = \begin{pmatrix} 1 & 0 \\ t & 0 \end{pmatrix}, \quad g_-(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}. \qquad (18)$$

Therefore the infinitesimal operator corresponding, for instance, to $g_-(t)$ is

$$\frac{\partial}{\partial t} T_\chi \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \phi(x) \Big|_{t=0} = [-x^2 \frac{\partial}{\partial x} + (s-1)\, x]\, \phi(x) = \qquad (19)$$

$$= H_- \phi(x) .$$

The other two infinitesimal operators are equal to H_3 and H_+ as given in (15). So we see that the parameters in the expressions for the generators obtained from D_1 may be chosen such that they give the infinitesimal operators for the representation $T_\chi(g)$.

One may hope that the present formalism may help to find some irreducible representations for the high-rank algebras. We should mention, however, that the solution of the system (4) of n linear equations with noncommutative coefficients is rather involved already for $n = 3$.

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