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SOME REMARKS ON KNO SCALING

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1. The rapid increasing of information on high energy multiparticle production requires an ordering process of the data as a first step to understanding and explanation of the involved phenomena.

Recently, Koba, Nielsen and Olesen [1] using as input information the data on high energy pp-collisions, proposed the following relations

$$\frac{\langle n^q \rangle}{\langle n \rangle^q} \simeq C_q , \quad q = 1, 2, 3, \ldots \tag{1}$$

and

$$\langle n \rangle \frac{\sigma_n}{\sigma} \xrightarrow[s \to \infty]{} \Psi \left(\frac{n}{\langle n \rangle} \right) , \tag{2}$$

where C_q's are independent of s as s increases. Here σ_n is the topological cross-section for the process $a + b \to n$ charged + any neutral at the c.m. energy $s^{1/2}$, $\sigma = \sum_n \sigma_n$ is the total or inelastic cross-section, $\langle n \rangle$ is the average multiplicity and $\langle n^q \rangle$ is the qth order moment of the multiplicity distribution. Althougn the original derivation of relations (1) and (2) is founded on some questionable assumptions, namely the validity of Feynmam scaling [2] at $x \simeq 0$ ($x = 2p / \sqrt{s}$ and p is the longitudinal momentum of the corresponding particle) and a delicate uniform convergence process (see, e.g., Narayan [3]), the analyses performed by Slattery [4] and Olesen [5] showed that (1) and (2) are in good agreement with the data on $p-p$ scattering for laboratory momenta between 50 GeV/c and 300 GeV/c.

This early onset of the KNO scaling is an intriguing question and for attempts to explain this phenomena we quote the papers [6-9]. Subsequent analysis showed that KNO scaling is consistent, with the present available data on high energy $\pi - N$ collisions [10] and $\bar{p}p$ collisions [11], respectively. The function $\Psi(\frac{n}{<n>})$ seems to be universal. For a recent review of the status of KNO scaling (theory) see, e.g., Olesen [12].

The purpose of this paper is to present a model-independent qualitative discussion on KNO scaling. In this respect, we shall utilise some results in the theory of moments and complex variable theory to obtain qualitative indications on the onset of KNO scaling and the behaviour of the function $\Psi(\frac{n}{<n>})$.

2. Our starting point is the generating function

$$Q(z,y) = \sum_{n=0}^{\infty} z^n \frac{\sigma_n}{\sigma}, \tag{3}$$

where z is a complex parameter, $|z| < 1$, and $y \cong a \ln s$ is the rapidity variable. As defined $Q(z,y)$ is an entire function of the variable z, but let us remember that unitarity requires that $\sigma_n \to 0$ for $n \geq N \approx \sqrt{s}$, i.e., $Q(z,y)$ is in fact a polynomial in z. We shall come back to this question in what follows. For subsequent purpose it is useful to write Eq. (3) as follows

$$Q(z,y) = \exp \sum_{q=1}^{\infty} \chi_q \frac{(\ln z)^q}{q!}, \tag{4a}$$

$$= \exp \sum_{q=1}^{\infty} f_q \frac{(z-1)^q}{q!}, \tag{4b}$$

where χ_q's are the cummulants [14] and f_q's are the well-known integrated correlation functions. The χ_q's are defined as

$$\chi_1 = f_1 = <n>$$

$$\chi_2 = <(n - <n>)^2>$$

$$\chi_3 = <(n - <n>)^3> \tag{5}$$

$$\chi_4 = <(n - <n>)^4> - 3\chi_2^2, \text{ etc.}$$

Using the identity

$$[\exp(\ln z) - 1]^\kappa = \sum_{q=\kappa}^{\infty} \frac{(\ln z)^q}{q!} (S)_q^{(\kappa)}, \tag{6}$$

where $(S)_q^{(\kappa)}$ are the Stirling number of the second kind [15] one obtaines

$$\chi_q = \sum_{\kappa=1}^{q} f_\kappa (S)_q^{(\kappa)}. \tag{7}$$

From the relation (7) one observes that χ_q's and f_q's have the same asymptotic behaviour.

Now we shall consider the following inequality [16]

$$|\chi_q| \leq q^q <|n - <n>|^q>, \quad q = 1, 2, 3, \dots \tag{8}$$

from which one obtains

$$|\chi_q| \leq q^q <|n^q - <n>^q|>$$

$$\leq q^q(<n^q> + <n>^q)$$

$$\leq q^q <n>^q (C_q + 1), \quad q = 1,2,3,\dots$$

where $C_q = \frac{<n^q>}{<n>^q}$, $q = 1,2,3,\dots$ .Therefore,

$$C_q \geq \frac{|\chi_q|}{q^q <n>^q} - 1, \quad q = 1,2,3,\dots \tag{9}$$

If $|\chi_q| \approx |b_q| <n>^q$, i.e., there are long-range correlations then

$$C_q \geq \text{const.}, \quad q = 1,2,3,\dots \tag{10}$$

More appropriately it is to consider that

$$|\chi_q| \leq |b_q|\langle n\rangle^q + |b_q^{(1)}|\langle n\rangle^{q-1} + \ldots + |b^{(q-1)}|\langle n\rangle, \tag{11}$$

i.e., there is a mixture of long-range and short-range correlations. In such a case

$$C_q \geq \text{const} + O\left(\frac{1}{\langle n\rangle}\right), \quad q = 1,2,3,\ldots \tag{12}$$

At this moment some comments are to be made:

i) One observes that KNO scaling (Eq. 1) interpreted as an extremal realization of the inequality (10) holds irrespective of the validity of Feynman scaling at $x \simeq 0$. In essence, Eq. (1) may be considered as a consequence of certain properties of moments and correlations.

ii) Strict KNO scaling is in contradiction with a polynomial generating function. This may be easily seen using

$$Q(z,y) = \sum_{q=1}^{\infty} \langle n^q\rangle \frac{(\ln z)^q}{q!} \tag{13}$$

and Eq. (1). A similar conclusion has been previously derived by Chodos et al. [17].

With these results at hand we shall proceed now to study the behaviour of the function $\psi\left(\frac{n}{\langle n\rangle}\right)$.

3. Firstly, let us consider the Wroblewski relation [18]*

$$\frac{\langle n\rangle}{D} \simeq \text{const.}, \tag{14}$$

where

$$D = \langle (n - \langle n\rangle)^2 \rangle^{1/2}.$$

One may easily show that (14) implies

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*For a recent discussion on Wroblewski relation see, e.g., Cohen [20].

$$\frac{\sigma_n}{\sigma} \leq \frac{a}{(x-1)^2}, \quad x = \frac{n}{\langle n\rangle}, \quad a > 0, \tag{15}$$

i.e., the probability distribution displays a maximum for $x \simeq 1$. Let us consider now strict KNO scaling (Eq. 1). Using

$$\langle n^q\rangle = \sum_n n^q \frac{\sigma_n}{\sigma}, \quad q = 1, 2, 3, \ldots$$

one obtains

$$\frac{\sigma_n}{\sigma} \leq \frac{C_q}{x^q}, \quad q = 1, 2, 3, \ldots \tag{16}$$

We are faced with the following (interpolation) problem. Find a function $P_n(x)$ such that $P_n(x) \leq \frac{C_q}{x^q}$, $q = 1, 2, 3, \ldots$, $x \in (0, \infty)$ and $\{C_q\}_1^\infty$ is monotonously increasing with q. The answer to this problem is the following [20].

There is a function $P(x) = \sum_{n=0}^{\ell} Q_n^{(\alpha)}(x)\, e^{-\alpha x^2}$ where $\alpha > 0$ is an appropriately constant and $Q_n^{(\alpha)}(x)$ are arbitrary polynomials in x such that

$$|P_n(x) - P(x)| \leq \epsilon.$$

Here ϵ is an arbitrary small positive constant. The function $P(x)$ may be also approximated by $\tilde{P}(x) = \Sigma\, \tilde{Q}_n^{(\beta)}(x)\, e^{-\beta x}$, $\beta > 0$, but the analyticity and convergence properties are better when one considers $P(x)$. The normalization condition restricts to a certain extent the arbitrariness of the polynomials $Q_n^{(\alpha)}(x)$ $(\tilde{Q}_n^{(\beta)}(x))$.

One observes that the answer to this problem can give a justification for the function considered by Slattery [4] and more generally for Polya-type distributions [21].

4. Now we shall consider the "inverse problem", i.e., what results shall we obtain if we start with the Eq. (2). Firstly, let us observe that Eq. (2) may be also writtem as

$$P_n = \int_0^\infty dx\, \Psi(x)\, \delta(n - x\langle n\rangle), \quad x = \frac{n}{\langle n\rangle}, \tag{17}$$

i.e., P_n may be considered as a Radon-like-transform [22-24] of $\Psi(x)$.

Now, in order for (17) to be well-defined we shall consider that $\Psi(x)$ belongs to $\mathcal{S}$ (the space of rapidly decreasing C^∞-functions). The unitarity supports such an assumption for $\Psi(x)$. If $\Psi(x) \in \mathcal{S}$, following Gel'fand et al. [23] we form

$$I_q(\langle n\rangle) = \int_0^\infty \Psi(x)\,(x\langle n\rangle)^q\, dx = \langle n\rangle^q C_q, \quad q=1,2,3,\ldots \tag{18}$$

$I_q(\langle n\rangle)$ is clearly a polynomial in $\langle n\rangle$ of degree $\le q$. On the other hand,

$$I_q(\langle n\rangle) = \int_0^\infty P_n n^q\, dn = \langle n^q\rangle, \quad q=1,2,3,\ldots \tag{19}$$

From the above observation and the relation (17) one may conclude that in order for P_n to be the Radon transform of a function $\Psi(x) \in \mathcal{S}$ it is necessary that $P_n \in \mathcal{S}$ and $\int_0^\infty P_n n^q\, dn$ be a polynomial in $\langle n\rangle$ of degree $\le q$, for all q (Obviously, if $\Psi(x)$ has compact support, it follows from (17) that P_n has compact support, too).

Now let us see what are the concequences of the relations (18) and (19). Eq. (19) implies

$$\langle n^q\rangle = \tilde{C}_q \langle n\rangle^q + \tilde{C}_{q-1}\langle n\rangle^{q-1} + \ldots + \tilde{C}_1\langle n\rangle + \tilde{C}_0, \tag{19'}$$

where $\tilde{C}_q$'s are constants independent of s. Therefore

$$C_q \langle n\rangle^q = \tilde{C}_q\langle n\rangle^q + \tilde{C}_{q-1}\langle n\rangle^{q-1} + \ldots + \tilde{C}_1\langle n\rangle + \tilde{C}_0,$$

i.e.,

$$C_q = \tilde{C}_q + O\left(\frac{1}{\langle n\rangle}\right), \quad q=1,2,3,\ldots \tag{20}$$

Hence Eq. (2) implies in the first approximation Eq. (1).

To conclude the paper we shall briefly discuss the problem of early onset of KNO scaling. In this respect we shall write the probability distribution in terms of the generating function

$$P_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int}\, Q(t)\, dt. \tag{21}$$

Now a well-known result in the theory of moments [16] asserts that if

$$\langle n^q\rangle < \infty, \quad q = 1, 2, 3, \ldots \tag{22}$$

then one has

$$Q(t) \cong \exp \sum_{q=1}^{r} \frac{(it)^q}{q!}\chi_q + O(|t|^{r+1}). \tag{23}$$

Obviously the probability distribution must be sufficiently smooth and vanishes sufficiently fast as $n \to \infty$, so that the inequality (22) is satisfied and the formal manipulation (21) is permissible. To meet these requirements we shall again assume that $P_n \in \mathcal{S}$. From the relations (21) and (22) one can write

$$P_n \cong \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(e^{-int} + \sum_{q=1}^{r}\frac{(it)^q}{q!}\chi_q\right) dt. \tag{24}$$

Therefore

$$\Psi(x,s) \cong \frac{1}{2\pi}\int_{-\pi\langle n\rangle}^{\pi\langle n\rangle} \exp\left(-ixt + \sum_{q=1}^{r}\frac{(it)^q}{q!}\frac{\chi_q}{\langle n\rangle^q}\right) dt. \tag{25}$$

A similar relation has been previously derived by Weingarten [8] in terms of density correlations of the hadronic gas. However, he utilised rather an ambigous assumption of fluctuations of the density of (charged) particles in order to truncate the series in (25).

Now if we consider $\Psi(x,s)$ be a smooth function (more precisely $\Psi \in \mathcal{S}_x$) and

$$\lim_{s \to \infty} \Psi(x, s) = \Psi(x) \tag{26}$$

then taking into account the above-derived results one can write

$$\Psi(x, s) \simeq \frac{1}{2\pi} \int_{-\lambda}^{\lambda} \exp\left(-ixt + \Sigma \frac{(it)^q}{q!} \frac{\chi_q}{\langle n \rangle^q}\right) dt, \tag{27}$$

where λ is an appropriately defined constant which depends on the domain considered for the variable x. The relation (27) suggests an interpretation of the early onset of KNO scaling, without considering Feynman scaling and fluctuations of particle density. The results of the paper are summarized in Fig. 1.

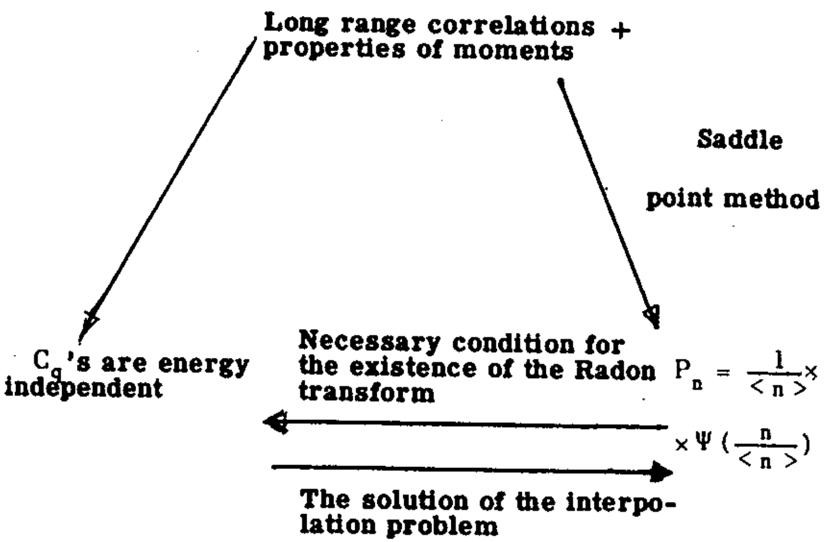

Fig. 1

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Appendix

Here a discussion will be presented on the possible connections between KNO scaling and a scaling relation proposed by Bander/25/. Utilising the fluid-analog model /26/ Bander proposed the following scaling relation

$$\frac{\partial}{\partial \langle n \rangle} \ln \sigma_n \xrightarrow[\langle n \rangle \to \infty]{} p\left(\frac{n}{\langle n \rangle}\right), \tag{A1}$$

where $p\left(\frac{n}{\langle n \rangle}\right)$ is the "presure" function defined as

$$p(z) = \lim_{y \to \infty} \frac{\ln Q(z, y)}{y}. \tag{A2}$$

Although Bander concluded that KNO scaling and his scaling relation are inconsistent as they are derived from different assumptions, we shall show that KNO scaling always implies the Bander relation. We start with the relation

$$\frac{\sigma_n}{\sigma} \xrightarrow[\langle n \rangle \to \infty]{} \frac{1}{\langle n \rangle} \Psi\left(\frac{n}{\langle n \rangle}\right)$$

and we shall assume that $\langle n \rangle$ is a continuous variable. Therefore*

$$\frac{\partial}{\partial \langle n \rangle} \ln \sigma_n \simeq -\frac{x}{\langle n \rangle} \frac{\partial \ln \Psi(x)}{\partial x} - \frac{1}{\langle n \rangle}, \quad x = \frac{n}{\langle n \rangle},$$

i.e.,

$$p(x) \simeq \frac{1}{\langle n \rangle}\left(-\frac{\partial \ln \Psi(x)}{\partial \ln x} - 1\right) = \frac{1}{\langle n \rangle} \Psi_1(x). \tag{A3}$$

We obtained the following result. If the multiplicity distribution behaves like $\frac{1}{\langle n \rangle}\Psi(x)$ when $\langle n \rangle \to \infty$, then the pressure function behaves like $\frac{1}{\langle n \rangle} \Psi_1(x)$ in the same limit. One can show that the "density" function

$\rho(z) = \frac{\partial p}{\partial \ln z}$ displays a similar behaviour, i.e.,

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* Here we have assumed that we can neglect, in the first approximation, the $\langle n \rangle$ dependence of σ.

$$\rho(z,s) \xrightarrow[\langle n\rangle\to\infty]{} \frac{1}{\langle n\rangle}\Psi_2(x). \qquad \text{(A4)}$$

where $\Psi_2(x)$ is defined in terms of $\Psi_1(x)$.

Now let us consider a short-range correlation model, e.g., the Poisson distribution. In such a case one may easily observe that the Bander relation is not an asymptotic one, i.e.,

$$\Psi(x,s) = \frac{\langle n\rangle^{n+1} e^{-\langle n\rangle}}{n!}. \qquad \text{(A5)}$$

Using the relation (A3) one derives*

$$p(x) = (x-1). \qquad \text{(A6)}$$

The same result is obtained if one utilises the Bander relation (Eq. (A1)). The same conclusion may be derived for any other short-range correlation model. One may conclude that the Bander relation is equivalent to KNO scaling for the pressure function. (in the case of short-range correlations).

Let us go further and consider a long-range correlation model, e.g., the geometrical model. In this case

$$\langle n\rangle \frac{\sigma_n}{\sigma} \xrightarrow[\langle n\rangle\to\infty]{} e^{-x}, \quad x = \frac{n}{\langle n\rangle} \qquad \text{(A7)}$$

where

$$\frac{\sigma_n}{\sigma} = \frac{\langle n\rangle^n}{(\langle n\rangle+1)^{n+1}}. \qquad \text{(A8)}$$

From relation (A3) one obtains

$$p(x) \approx \frac{1}{\langle n\rangle}(x-1). \qquad \text{(A9)}$$

On the other hand, using the corresponding generating function

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*More precisely,

$$p(x) = (x-1)\,\theta(x-1) \qquad \text{(A6')}$$

$$Q(z,y) = \sum_{n=0}^{\infty} z^n \frac{\sigma_n}{\sigma} = \frac{1}{1+\langle n\rangle(1-z)} \qquad \text{(A10)}$$

and relation (A2) one has

$$p(z,y) = -\frac{1}{y}\ln[1+\langle n\rangle(1-z)]. \qquad \text{(A11)}$$

The convergence of $Q(z,y)$ requires

$$\frac{1}{\langle n\rangle} + 1 > z \geq 0. \qquad \text{(A12)}$$

Therefore in the limit $\langle n\rangle \to \infty$, $p(x) \approx \frac{1}{\langle n\rangle}(x-1)$. It is interesting to note that the same result may be obtained using formally the Bander relation. In conclusion one may argue that there is no inconsistentcy between KNO scaling and the scaling relation proposed by Bander. However, the former relation is more general and always implies the later one.

An interesting question would be the following. Starting with the Bander relation what can we say about the KNO scaling? One may observe that this question is connected with the behaviour of the number density.

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