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1. Introduction

The conformal invariant two-point function for the fields with arbitrary spin and scale dimensions are given in papers /1-4/. To analyse the positivity condition it is convenient to decompose the two-point function with respect to the relativistic spin /5-7/. To achieve this goal the formalism of homogeneous functions (polynomials) of the complex spinor /5,7/ or isotopic vector is used /6/. Independently of its compactness the formalism of the homogeneous polynomials has some difficulty: there is no convenient scalar product /8/. This difficulty also remains when one goes to the Euclidean space-time /6,9,10/. In this case we have the same difficulties as in the case when we have a sum over the repeated tensor indices. The transition between the formalism of homogeneous polynomials and the formalism of polynomials on sphere (not only unit) in the Euclidean space is given in paper /11/.

In the present paper, to avoid these difficulties, the tensor indices are replaced by the continuous variables on the unit sphere in the four-dimensional Euclidean spin space. In this space the scalar product is given as an integral of the Gegenbauere and Legandre polynomials. In the paper we restrict ourselves to the consideration of only the fundamental tensors. The scale dimensions may be arbitrary.

In chapter II the general form of the relativistic invariant Euclidean two-point function is given as a decomposition according to the relativistic spin (the second Casimir operator of the corresponding Poincare group).

In chapter III the general form of the scale invariant two-point function for tensor fields with rank n is given. This function depends on $n+1$ arbitrary constants.

The conformal invariant two-point function for the fundamental tensor fields is found in chapter IV. This function depends only on one normalized constant.

Canonical basis vectors of the unitary representation of $SO(4)$ and a scalar product with respect to which they are normalized are given in Appendix A. In Appendices B and C the projection operators on the subspaces with definite spin are given and their orthonormality and completeness conditions are checked.

II. Decomposition of the Relativistic Invariant Euclidean Two-Point Function with Respect to Spin

Let us consider the Euclidean two-point function $F(x_1, z_1, n_1, d_1; x_2, z_2, n_2, d_2)$ for tensor fields with rank and scale dimensions n_1, d_1 and n_2, d_2 respectively, i.e., the two-point function of fields which transform according to irreducible representation $\chi_1 = (n_1, d_1)$ and $\chi_2 = (n_2, d_2)$ of the conformal group. Conformal invariance implies that

$$F(\Lambda x_1, \Lambda z_1, \chi_1; \Lambda x_2, \Lambda z_2, \chi_2) = F(x_1, z_1, \chi_1; x_2, z_2, \chi_2), \tag{2.1}$$

where $\Lambda \subset SO(5,1)$.

It is well known, if $n_a \neq 0$ $(a=1,2)$, the function (2.1) describes the propagation of particles with spin $s = 0, 1, \ldots, \min(n_1, n_2)$. In this case it is necessary to decompose the two-point function not only with respect to the total mass but also with respect to the spin variable (i.e., with respect to the second Casimir operator of the Poincare group). /7/. To achieve this, it is convenient to go to the momentum space. The translational invariance allows us to write down:

$$F(x_1 - x_2; z_1, \chi_1; z_2 \chi_2) = \int d^4p\, e^{-ip(x_1 - x_2)} \tilde{F}(p; z_1, \chi_1, z_2 \chi_2). \tag{2.2}$$

From conformal invariance condition (2.1), we have the corresponding condition for the kernel of the Fourier transformation (2.2)

$$\tilde{F}(\Lambda p; \Lambda z_1, \chi_1, \Lambda z_2, \chi_2) = \tilde{F}(p; z_1, \chi_1, z_2, \chi_2); \tag{2.3}$$

where $\Lambda \subset SO(5,1)$.

The relativistic invariance requires that the kernel of the two-point function be the function of only the relativistic invariants, which can be formed by the 4-vectors p, z_1 and z_2, i.e., we have:

$$\tilde{F}(p; z_1, \chi_1, z_2, \chi_2) = \tilde{F}^{[\chi_1, \chi_2]}(p^2, pz_1, pz_2, z_1 z_2), \quad (z_1^2 = z_2^2 = 1). \tag{2.4}$$

The kernel (2.4) is not an arbitrary function of the "spin variables" z_1 and z_2. The two-point function (2.1) and its kernel (2.4) are transformed, as one direct product of two irreducible tensors according to the Lorentz group $SO(4)$. It is well known, that in terms of tensor indices irreducibility is equivalent to the symmetry and fraceless properties with respect to this indices. In our case when the tensor indices are replaced by the continuous variables z_1 and z_2, the irreducibility conditions are given with the following equations:

$$\left[\frac{1}{2} \Sigma^{(a)}_{\alpha\beta} \Sigma^{(a)}_{\alpha\beta} - n^{(a)}(n^{(a)}+2)\right] \tilde{F}^{[\chi_1, \chi_2]}(p; z_1, z_2) = 0, \tag{2.5}$$
$$(a = 1, 2), \quad (\alpha, \beta = 1, \ldots, 4),$$

where $\Sigma_{\alpha\beta}$ are generators of the $SO(4)$ - they are given in App. A, $1/2\, \Sigma_{\alpha\beta} \Sigma_{\alpha\beta}$ is the Casimir operator of the $SO(4)$. The second Casimir operator for tensor representations is known to be equal to zero. The kernel $\tilde{F}^{[\chi_1 \chi_2]}(p; z_1, z_2)$ may be written down as

$$\tilde{F}^{[\chi_1, \chi_2]}(p; z_1, z_2) = \tilde{F}^{\alpha_1, \ldots, \alpha_{n_1}; \beta_1, \ldots, \beta_{n_2}}(p)\, z'_{\alpha_1} \ldots z'_{\alpha_{n_1}} z_{\beta_1} \ldots z_{\beta_{n_2}}, \tag{2.6}$$

where the tensor $\tilde{F}^{\{\alpha\}\{\beta\}}(p)$ is symmetric and traceless with respect to the indices $\{\alpha\}$ and $\{\beta\}$ separately. The formula (2.6) gives the transition rule between the conventional tensor formalism and our formalism of function on unit sphere.

As we said before, the two-point function (2.1) describes the propagation of particles with spin $s = 0, 1, \ldots, \min(n_1, n_2)$. In the case, when $\min(n_1, n_2) \neq 0$ it is necessary to decompose the two-point function with respect to the spin, i.e., we have:

$$\tilde{F}^{[\chi_1, \chi_2]}(p; z_1, z_2) = \sum_{s=0}^{\min(n_1,n_2)} \tilde{F}_s^{[\chi_1, \chi_2]}(p; z_1, z_2). \quad (2.7)$$

Here, the kernels $\tilde{F}_s^{[\chi_1, \chi_2]}(p; z_1, z_2)$ are eigenfunctions of the spin operator and, consequently, they satisfy the following equations:

$$[S^2_{(a)} - s(s+1)]\tilde{F}_s^{[\chi_1, \chi_2]}(p; z_1, z_2) = 0, \; [s = 0, \ldots, \min(n_1, n_2)] \; (a = 1, 2), \quad (2.8)$$

where $S^2_{(a)}$ are relativistic spin operators

$$S^2_a = \frac{1}{2}\Sigma^{(a)}_{\alpha\beta}\Sigma^{(a)}_{\alpha\beta} - \frac{1}{p^2}\Sigma^{(a)}_{\alpha\gamma}\Sigma^{(a)}_{\beta\gamma}p^\alpha p^\beta. \quad (2.9)$$

Any terms of the decomposition (2.7) are necessary to satisfy the equations (2.5) and (2.8). As the first step, let us consider the equations (2.8). As far as we are looking for Lorentz-invariant solutions of eqs. (2.8) it is convenient to go to the rest frame $(\vec{p} = 0)$, in which

$$S^2_{(a)}(\vec{p}=0) = \sum_{j=0}^{3} M_j^2 = 2z_j^{(a)}\frac{\partial}{\partial z_j^a} + z_j^{(a)} z_k^{(a)}\frac{\partial^2}{\partial z_j^{(a)}\partial z_k^{(a)}} - \vec{z}_a^{\,2}\frac{\partial^2}{\partial z_j^{(a)}\partial z_j^{(a)}}. \quad (2.10)$$

If we substitute (2.10) in eqs. (2.8) we have:

$$[(1-w^2)\frac{\partial^2}{\partial w^2} - 2w\frac{\partial}{\partial w} + s(s+1)]\tilde{F}_s^{[\chi_1, \chi_2]}(p; z_1, z_2) = 0, \quad (2.11)$$

where

$$\cos\theta = w = \frac{p^2 z_1 z_2 - pz_1 pz_2}{\sqrt{(p^2 - pz_1^2)(p^2 - pz_2^2)}}. \quad (2.12)$$

In the rest frame $\cos\theta(\vec{p}=0) = \frac{\vec{z}^1 \cdot \vec{z}^2}{|\vec{z}^1||\vec{z}^2|}$, i.e., $\theta(\vec{p}=0)$ is the angle between the space parts of the 4-vectors z_1 and z_2.

The solutions of eqs. (2.11), which are regular in $w^2 = 1$ are as follows:

$$\tilde{F}_s^{[\chi_1, \chi_2]}(p; z_1, z_2) = h_s^{[\chi_1, \chi_2]}(p^2, pz_1, pz_2)P_s(w), \quad (2.13)$$

where $P_s(w)$ are the Legandre polynomials and $h_s^{[\chi_1, \chi_2]}(p^2, pz_1, pz_2)$ are arbitrary functions. The dependence of these functions on pz_1, pz_2 is found from eqs. (2.5).

If we substitute (2.13) in eqs. (2.5), we have

$$[(1-u_a^2)\frac{\partial^2}{\partial u_a^2} - 3u_a\frac{\partial}{\partial u_a} + n_a(n_a+2) - \frac{s(s+1)}{1-u_a^2}] \times F_s^{[\chi_1, \chi_2]}(p; z_1, z_2) = 0, \quad (2.14)$$

where

$$u_{(a)} = \frac{pz_{(a)}}{\sqrt{p^2}}, \quad (a = 1, 2), \quad (2.15)$$

in the rest frame $u_{(a)}(\vec{p}=0) = z_1^{(a)}$.

The general solution of eq. (2.14), regular for $u^2_{(a)} = 1$, is

$$\tilde{F}_s^{[\chi_1, \chi_2]}(p; z_1, z_2) = \sigma_s^{[\chi_1, \chi_2]}(p^2)\, t^s_{n_1}(u_1)\, t^s_{n_2}(u_2) P_s(w), \quad (2.16)$$

where $\sigma_s^{[\chi_1, \chi_2]}(p^2)$ are arbitrary functions of p^2,

$$t_n^s(u) = (1-u^2)^{s/2}\, C_{n-s}^{s+1}(u)\, \frac{2^s\, s!\,(n-s)!}{(n+s+1)!} \tag{2.17}$$

and $C_k^m(u)$ are Gegenbauere polynomials.

From (2.17) and (2.16) we have the general form of the Euclidean relativistic invariant kernel of the two-pôint function for tensor fields

$$\tilde{F}^{[\chi_1,\chi_2]}(p;z_1,z_2) = \sum_{s=0}^{\min(n_1,n_2)} \sigma_s^{[\chi_1,\chi_2]}(p)\, t_{n_1}^s(u_1)\, t_{n_2}^s(u_2) \times$$
$$\times P_s(w). \tag{2.18}$$

III. Dilatational Invariant Two-Point Kernel

We shall have the dilatational invariant kernel, if the kernel (2.18) is invariant according to scale transformations. In infinitesimal form, we have

$$D\tilde{F}^{[\chi_1,\chi_2]}(p;z_1,z_2) = 0, \tag{3.1}$$

where D is the generator of the dilatations. In our case this generator has the form /5/

$$D = -i(d_1, d_2 - 4 - p_\alpha \frac{\partial}{\partial p_\alpha}). \tag{3.2}$$

From (2.18), (3.2) and (3.1), we have

$$[d_1 + d_2 - 4 - 2p^2 \frac{\partial}{\partial p^2}]\tilde{F}^{[\chi_1,\chi_2]}(p^2, u_1, u_2, w) = 0 . \tag{3.3}$$

This equation acts only on the functions $\sigma_s^{[\chi_1,\chi_2]}(p^2)$, because the variables u_1, u_2 and w, which are given with (2.12) and (2.15), are scale invariant. The solution of eq. (3.3) may be written as:

$$\sigma_s^{[\chi_1,\chi_2]}(p^2) = a_s^{[\chi_1,\chi_2]}(p^2)^{\frac{d_1+d_2}{2}-2}, \tag{3.4}$$

where $a_s^{[\chi_1,\chi_2]}$ are arbitrary constants.

From (2.18) and (3.4) we can write down the general form of the dilatational invariant kernel of the two-point function:

$$\tilde{F}^{[\chi_1,\chi_2]}(p;z_1,z_2) = \sum_{s=0}^{\min(n_1,n_2)} a_s^{[\chi_1,\chi_2]}(p^2)^{\frac{d_1+d_2}{2}-2}\, t_{n_1}^s(u_1) \times$$
$$\times t_{n_2}^s(u_2)\, P_s(w). \tag{3.5}$$

It may be proved that the kernel (3.5) is invariant not only according to infinitesimal scale transformations, but also according to any finite scale transformations, i.e.,

$$\tilde{F}^{[\chi_1,\chi_2]}(\rho^{-1}p;\rho z_1,\rho z_2) = \rho^{2-\frac{d_1+d_2}{2}}\, \tilde{F}^{[\chi_1,\chi_2]}(p;z_1,z_2), \tag{3.6}$$

where $\rho > 0$. From (3.5) it follows that this requirement is satisfied.

IV. Conformal Invariant Two-Point Kernel

We shall have the conformal invariant two-point kernel, if we require invariance of (3.5), according to the special conformal transformations. In infinitesimal form we have

$$K_\alpha \tilde{F}^{[\chi_1,\chi_2]}(p;z_1,z_2) = 0, \quad (\alpha = 1, \ldots, 4), \tag{4.1}$$

where the generators K_α of special conformal transformations, acting on the kernel of the two-point function, are given as /5/ :

$$K_\alpha = i\{(p_\alpha y_\beta - y_\alpha p_\beta - \delta_{\alpha\beta}\, p y)\frac{\partial}{\partial p^\beta} - 2i[\delta_{\alpha\beta}(d_1+d_2-4) +$$
$$+ \Sigma_{\alpha\beta}^{(1)} + \Sigma_{\alpha\beta}^{(2)}]y_\beta - 2i[(d_1-d_2)\,\delta_{\alpha\beta} + \Sigma_{\alpha\beta}^{(1)} - \Sigma_{\alpha\beta}^{(2)}]\frac{\partial}{\partial p^\beta}\}, \tag{4.2}$$

where $y = x_1 + x_2$.

It is shown /12,13/ that we have a non-vanishing conformal invariant two-point function, if

$\chi_1 = \chi_2$, i.e., $n_1 = n_2$ and $d_1 = d_2$ (4.3)

If we go to the variables p^2, u_1, u_2 and w in eq. (4.1) we have

$$K_\alpha \tilde{F}(p;z_1,z_2) = i\{y_\alpha D + p_\alpha A + z^1_\alpha B + z^2_\alpha C\}\tilde{F}(p;z_1,z_2) = 0, \qquad (4.4)$$

where D is the generator of dilatations (3.2) and operators A , B and C are:

$$A = \frac{1}{p^2}\{(1-u^2_{(1)})\frac{\partial^2}{\partial u^2_{(1)}} - u_{(1)}\frac{\partial}{\partial u_{(1)}} + 2p^2\frac{\partial^2}{\partial p^2 \partial u_{(1)}} -$$
$$-(1-u^2_{(2)})\frac{\partial^2}{\partial u^2_{(2)}} + u_{(2)}\frac{\partial}{\partial u_{(2)}} - \qquad (4.5)$$
$$-2p^2 u_{(2)}\frac{\partial^2}{\partial p^2 \partial u_{(2)}} + (\frac{1}{1-u^2_{(1)}} - \frac{1}{1-u^2_{(2)}}) \times$$
$$\times[(1-w^2)\frac{\partial^2}{\partial w^2} - 2w\frac{\partial}{\partial w} - 2p^2 w\frac{\partial^2}{\partial p^2 \partial w}] +$$
$$+2[u_{(1)}w - u_{(2)}\frac{\sqrt{1-u^2_{(1)}}}{\sqrt{1-u^2_{(2)}}}]\frac{\partial^2}{\partial u_{(1)}\partial w} +$$
$$+2[u_{(1)}\frac{\sqrt{1-u^2_{(1)}}}{\sqrt{1-u^2_{(1)}}} - u_{(2)} w]\frac{\partial^2}{\partial u_{(2)}\partial w}\},$$

$$B = \frac{2}{\sqrt{p^2(1-u_1^2)}}\{-\sqrt{1-u_1^2}\frac{\partial}{\partial u_1} - p^2\sqrt{1-u_1^2}\frac{\partial^2}{\partial p^2 \partial u_1} +$$
$$+\frac{u_1}{\sqrt{1-u_1^2}}[(1-w^2)\frac{\partial^2}{\partial w^2} - 2w\frac{\partial}{\partial w}] + \qquad (4.6)$$
$$-p^2(\frac{u_2}{\sqrt{1-u_2^2}} + \frac{u_1}{\sqrt{1-u_1^2}}w)\frac{\partial^2}{\partial p^2 \partial w} -$$
$$-w\sqrt{1-u_1^2}\frac{\partial^2}{\partial u_1 \partial w} - \sqrt{1-u_2^2}\frac{\partial^2}{\partial u_2 \partial w}\},$$

$$C = -B(u_1 \to u_2). \qquad (4.7)$$

Equations (4.4) are equivalent to the following system (p , z_1 and z_2 are nonvanishing):

$$A\tilde{F}^{[\chi_1,\chi_2]}(p;z_1,z_2) = 0, \qquad (4.8a)$$
$$B\tilde{F}^{[\chi_1,\chi_2]}(p;z_1,z_2) = 0, \qquad (4.8b)$$
$$C\tilde{F}^{[\chi_1,\chi_2]}(p;z_1,z_2) = 0. \qquad (4.8c)$$

It is convenient to replace eq. (4.8a) by the equation

$$p_\alpha K_\alpha \tilde{F}^{[\chi_1,\chi_2]}(p;z_1,z_2) = (p^2 A + pz_1 B \cdot pz_2 C) \times$$
$$\times \tilde{F}^{[\chi_1,\chi_2]}(p;z_1,z_2) = 0. \qquad (4.9)$$

If in the latter equation we insert (4.5), (4.6) and (4.7) we have

$$\{(1-u_1^2)\frac{\partial^2}{\partial u_1^2} - 3u_1\frac{\partial}{\partial u_1} - (1-u_2^2)\frac{\partial^2}{\partial u_2^2} + 3u_2\frac{\partial}{\partial u_2} +$$
$$+(\frac{1}{1-u_1^2} - \frac{1}{1-u_2^2})[(1-w^2)\frac{\partial^2}{\partial w^2} - 2w\frac{\partial}{\partial w}]\}\times \qquad (4.10)$$
$$\times F^{[\chi_1,\chi_2]}(p;z_1,z_2) = 0.$$

From (2.14) and (4.3) it follows that (4.10) is satisfied identically.

Let us now insert (3.5) into eqs. (4.8b) and (4.8c). If we use the following identities

$$\sqrt{1-x^2}\,\frac{dt_n^s(x)}{dx} = \mu_1(n,s)t_n^{s+1}(x) + \mu_2(n,s)t_n^{s-1}(x),$$
$$\frac{x}{\sqrt{1-x^2}}\,t_n^s(x) = \nu_1(n,s)\,t_n^{s+1}(x) + \nu_2(n,s)\,t_n^{s-1}(x), \tag{4.11}$$

where

$$\mu_1(n,s) = \frac{(n+s+2)(n-s)(s+1)}{2s+1}, \quad \mu_2(n,s) = -\frac{s}{2s+1},$$
$$\nu_1(n,s) = \frac{(n+s+2)(n-s)}{2s+1}, \qquad \nu_2(n,s) = \frac{1}{2s+1}, \tag{4.12}$$

from eq. (3.8b) we have

$$\sum_{s=0}^{n} a_s^{[n,d]}\{(d_1-1)[\mu_1(n,s)t_n^{s+1}(u_1) + \mu_2(u,s)t_n^{s-1}(u_1)]t_n^s(u_2)P_s(w) +$$
$$+ s(s+1)[\nu_1(n,s)t_n^{s+1}(u_1) + \mu_1(n,s)t_n^{s-1}(u_1)]t_n^s(u_2)P_s(w) +$$
$$+(d-2)[\nu_1(n,s)t_n^{s+1}(u_1) + \mu_2(n,s)t_n^{s-1}(u_1)]\,t_n^s(u_2)\times$$
$$\times[sP_s(w) + \sum_{k=0}^{\inf[\frac{s-2}{2},\frac{s-3}{2}]}(2s-4k-3)P_{s-2k-2}(w)] + \tag{4.13}$$
$$+(d-2)\,t_n^s(u_1)[\nu_1(n,s)t_n^{s+1}(u_2) + \mu_2(n,s)t_n^{s-1}(u_2)]\times$$
$$\times\sum_{k=0}^{\inf(\frac{s-1}{2},\frac{s-2}{2})}(2s-4k-1)P_{s-2k-1}(w) +$$
$$+[\mu_1(n,s)t_n^{s+1}(u_1) + \mu_2(n,s)t_n^{s-1}(u_1)]t_n^s(u_2)[\,sP_s(w) +$$
$$+\sum_{k=0}^{\inf[\frac{s-2}{2},\frac{s-3}{2}]}(2s-4k-3)P_{s-2k-2}(w)] +$$
$$+ t_n^s(u_1)[\mu_1(n,s)t_n^{s+1}(u_2) + \mu_2(n,s)t_n^{s-1}(u_2)]\times$$
$$\times\sum_{k=0}^{\inf[\frac{s-1}{2},\frac{s-2}{2}]}(2s-4k-1)P_{s-2k-1}(w)\} = 0,$$

where the sum over k is taken up to one of the two numbers $[\alpha,\beta]$ which is integer.

From (4.8c) we can write down, a second algebraic equation which is completely equivalent to eq. (4.13) due to consistency condition (4.3).

If $d=n+2$, i.e., we deal with canonical dimensions, from (4.13) it follows that in the decomposition (3.5) the term with max. spin value $s=n$ alone is present. This is the case of conserved tensor currents.

In the general case, when $d \neq n+2$, in the decomposition of the conformal invariant kernel all spin values $s=0,\dots,n$ are present. In this case from eq. (2.12) we have the following relations:

$$a_{s+1}^{[n,d]} = -\frac{(n+s+2)(n-s)(d+s-1)(2s+1)}{(2s+1)(d-s-3)}\,a_s^{[n,d]}. \tag{4.14}$$

These relations are satisfied if we have:

$$a_s^{[n,d]} = (-1)^s N_n^d\,\frac{(2s+1)(n+s+1)!\,\Gamma(d+s-1)\Gamma(d-s-2)}{(n-s)!}, \tag{4.15}$$

where N_n^d is one normalized constant depending only on the rank and scale dimension of tensor fields.

From (3.5) and (4.15) we can write the general form of the conformal invariant kernel of a two-point function

$$\tilde{F}^{[\chi]}(p;z_1,z_2) = N_n^d(p^2)^{d-2}\sum_{s=0}^{n}\beta_s^{[n,d]}\,\Pi_s(u_1,u_2,w), \tag{4.16}$$

where Π^n_s are the spin projection operators, which are

$$\Pi^n_s(u_1,u_2,w)=\kappa_s(1-u_1^2)^{s/2}(1-u_2^2)^{s/2}C^{s+1}_{n-s}(u_1)C^{s+1}_{n-s}(u_2)P_s(w) \quad (4.17)$$

here

$$\kappa_s=\frac{(n+1)(2s+1)(s!)^2(n-s)!}{2^{2(n-s)}(n+s+1)!} \quad (4.18)$$

are normalized coefficients.

From (2.17), (4.15), (4.16) and (4.17) we have:

$$\beta_s=(-1)^s\frac{2^{2n}\Gamma(d+s-1)\Gamma(d-s-2)}{n+1}. \quad (4.19)$$

The positivity and locality conditions of the Euclidean two-point function are analyzed in paper /6/.

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Appendix A

The generators of the group SO(4), in terms of continuous variables ξ can be written down:

$$\Sigma_{\alpha\beta}=i(\xi_\alpha\frac{\partial}{\partial\xi_\beta}-\xi_\beta\frac{\partial}{\partial\xi_\alpha}). \quad (A.1)$$

To construct the basis functions of the irreducible unitary representatuons of SO(4), it is convenient to go to the spherical coordinates in 4-dimensional space:

$$\xi_1=\rho\sin\chi\sin\theta\sin\phi,$$

$$\xi_2=\rho\sin\chi\sin\theta\cos\phi,$$

$$\xi_3=\rho\sin\chi\cos\theta,$$

$$\xi_4=\rho\cos\chi, \quad (A.2)$$

where $\rho>0$, $0\le\phi\le 2\pi$ and $0\le\theta,\chi\le\pi$. (A.2) gives the general case of spherical coordinates in Euclidean 4-dimensional space. In the case, when ξ are on the unit sphere, we have $\rho=1$, i.e.,

$$z_\alpha=\frac{\xi_\alpha}{\rho}=\frac{\xi_\alpha}{\sqrt{\xi^2}}. \quad (A.3)$$

The generators (A.2), in terms of variables χ, θ and ϕ can be written down

$$M_\pm=e^{\mp i\phi}(-i\frac{\partial}{\partial\theta}\mp\frac{1}{tg\,\theta}\frac{\partial}{\partial\phi})$$

$$M_3=-i\frac{\partial}{\partial\phi} \quad (A.4)$$

$$N_\pm=e^{\mp i\phi}(\mp\sin\theta\frac{\partial}{\partial\chi}\mp\frac{\cos\theta\cos\phi}{ctg\chi}\frac{\partial}{\partial\theta}+\frac{i\,tg\chi}{\sin\theta}\frac{\partial}{\partial\phi})$$

$$N_s=i(\cos\theta\frac{\partial}{\partial\chi}-\frac{\sin\theta}{ctg\,\chi}\frac{\partial}{\partial\theta}),$$

where $M_j=\frac{1}{2}\epsilon_{jk\ell}\Sigma_{k\ell}$, $M_\pm=M_1\pm iM_2$, $N_j=\Sigma_{4j}$ and $N_\pm=N_1\pm iN_2$.

The Casimir operators of group SO(4) and its subgroup SO(3) are:

$$\frac{1}{2}\Sigma\quad\Sigma\quad=M+N=(1-\cos\chi)\frac{\partial}{\partial(\cos\chi)}-3\cos\chi\frac{\partial}{\partial\cos\chi}+\frac{1}{1-\cos\chi}M, \quad (A.5)$$

$$\frac{1}{2}\Sigma_{\alpha\beta}\Sigma_{\alpha\beta}=\vec{M}^2+\vec{N}^2=(1-\cos^2\chi)\frac{\partial^2}{\partial(\cos\chi)^2}-3\cos\chi\frac{\partial}{\partial\cos\chi}+$$

$$+\frac{1}{1-\cos^2\chi}\vec{M}^2\,, \qquad (A.6)$$

$$\vec{M}^2=(1-\cos^2\theta)\frac{\partial^2}{\partial(\cos\theta)^2}-2\cos\theta\frac{\partial}{\partial\cos\theta}-\frac{1}{1-\cos^2\theta}\frac{\partial^2}{\partial\phi^2}\,.$$

The basic vectors of the tensor representations of the group SO(4) are given as the eigenfunctions of the following three commuting operators $\vec{M}^2+\vec{N}^2$, $\vec{M}^2$ and M_3 , i.e.,

$$(M_s-\zeta)f^n_{s\zeta}(\chi,\theta,\phi)=0,$$

$$[\vec{M}^2-s(s+1)]f^n_{s\zeta}(\chi,\theta,\phi)=0, \qquad (A.7)$$

$$[\vec{M}^2+\vec{N}^2-n(n+2)]f^n_{s\zeta}(\chi,\theta,\phi)=0.$$

The solutions of the system (A.7), may be written down

$$f^n_{s\zeta}(\chi,\theta,\phi)=A^n_{s\zeta}\sin^3\chi\, C^{s+1}_{n-s}(\cos\chi)P^{\zeta}_s(\cos\theta)e^{-i\zeta\phi}\,, \qquad (A.8)$$

where $A^n_{s\zeta}$ are normalized constants, C^m_n are Gegenbauere polynomials, P^{ζ}_s are spherical functions.

For the functions $f(\xi)$, we have the following scalar product

$$(f,g)=2^{2n}\int\bar{f}(\xi)g(\xi)\delta(\xi^2-1)d^4\xi\,. \qquad (A.9)$$

In the spherical coordinates (A.9) may be written down

$$(f,g)=\frac{2^{2n-1}}{\pi^2}\int_{-1}^{1}\sin\chi\, d\cos\chi\int_{-1}^{1}cd\cos\theta\int_0^{2\pi}d\phi\,\bar{f}(\chi,\theta,\phi)g(\chi,\theta,\phi)\,. \qquad (A.10)$$

The orthonormalized conditions for the functions (A.8) are

$$(f^n_{s\zeta},f^{n'}_{s'\zeta'})=\delta_{nn'}\delta_{ss'}\delta_{\zeta\zeta'}\,. \qquad (A.11)$$

From (A.8), (A.9) and (A.11), we have:

$$A^n_{s\zeta}=\frac{s!}{2^{n-s}}\left[\frac{(n+1)(2s+1)(n-s)!(s-\zeta)!}{(n+s+1)!(s+\zeta)!}\right]^{1/2}. \qquad (A.12)$$

The coefficient in the scalar product (A.9) is taken to have a direct correspondence with the tensor component. For instance:

$$f^1_{00}(\chi,\theta,\phi)=\cos\chi=z_4\,,$$

$$f^1_{10}(\chi,\theta,\phi)=\sin\chi\, P^{\circ}_1(\cos\theta)=\sin\chi\cos\theta=z_3\,.$$

Appendix B

The projection operators (4.17) may be decomposed

$$\Pi^{(n)}_s(u_1,u_2,w)=\sum_{\zeta=-s}^{s}u^n_{s\zeta}(p;z_1)\overline{u^n_{s\zeta}(p;z_2)}, \qquad (B.1)$$

where

$$u^n_{s\zeta}(p;z)=f^n_{s\zeta}(L_p z). \qquad (B.2)$$

Here L_p are boost transformations, i.e., transformations which transform the momentum 4-vector p from rest frame to any arbitrary frame, i.e.,

$$p_a = (L_p)_{a4}\, \overset{\circ}{p}_4 \;,\; \overset{\circ}{p} = (\vec{0}, \overset{\circ}{p}_4). \tag{B.3}$$

To prove (B.1) we take into account the relativistic invariance of (4.17). If we go to the rest frame for the momentum p from (4.17) we have:

$$\Pi^n_s(\vec{p}=0, z_1, z_2) = \kappa_s \sin^s\chi_1 \sin^s\chi_2 C^{s+1}_{n-s}(\cos\chi_1)\times$$

$$\times C^{s+1}_{n-s}(\cos\chi_2) P_s(w) = \tag{B.4}$$

$$= \kappa_s \sin^s\chi_1 \sin^s\chi_2 C^{s+1}_{n-s}(\cos\chi_1) C^{s+1}_{n-s}(\cos\chi_2)[P_s(\cos\theta_1)P_s(\cos\theta_2) +$$

$$+ 2\sum_{h=1}^{s} \frac{(s-\zeta)!}{(s+Z\zeta)!} P^{+\zeta}_s(\cos\theta_1) P^{-\zeta}_s(\cos\theta_2)\, e^{-ih(\phi_1-\phi_2)}] =$$

$$= \sum_{h=-s}^{s} f^n_{s\zeta}(z_1)\overline{f^n_{s\zeta}(z_2)},$$

where

$$w(\vec{p}=0) = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos(\phi_1-\phi_2). \tag{B.5}$$

Here we take into account the summing formula for the Legendre polynomials.

Appendix C

Here we shall prove the projection properties of the operators (4.17), i.e.:

$$\Pi^n_s \Pi^{n'}_{s'} = \delta_{nn'}\delta_{ss'}\Pi^n_s, \tag{C.1}$$

$$\sum_{s=0}^{n} \Pi^n_s = I, \tag{C.2}$$

where I is the unit operator in the representation space, i.e.,

$$(If)(z) = f(z). \tag{C.3}$$

From (B.1), (A.10) and (C.2), we have:

$$(\Pi^n_s \Pi^{n'}_{s'})(u_1, u_3, w') = \frac{2^{2n-1}}{\pi^2}\kappa^n_s \kappa^{n'}_{s'} (1-u_1^2)^{s/2}(1-u_3\;)^{s'/2}\times$$

$$\times C^{s+1}_{n-s}(u_1) C^{s'+1}_{n'-s'}(u_3)\int_{-1}^{1}(1-u_2^2)^{\frac{s+s'}{2}} C^{s+1}_{n-s}(u_2)\times$$

$$\times C^{s'+1}_{n'-s'}(u_2)\,du_2\times \tag{C.4}$$

$$\times\int_{-1}^{1} d\cos\theta_2 \int_0^{2\pi} d\phi_2 \,\{P_s(\cos\theta_1)P_s(\cos\theta_2) + 2\sum_{h=1}^{s} P^{+h}_s(\cos\theta_1)P^{-h}_s(\cos\theta_2)$$

$$\times \cos\zeta(\phi_2-\phi_1)\}\{P_{s'}(\cos\theta_2)P_{s'}(\cos\theta_3) + 2\sum_{h'=1}^{s} P^{h'}_{s'}(\cos\theta_2)\times$$

$$\times P^{-h'}_{s'}(\cos\theta_3)\cos\zeta'(\phi_3-\phi_2)\} =$$

$$= \delta_{nn'}\delta_{ss'}\kappa^n_s \frac{2^{2(n-s)}(n+s+1)!}{(n+1)(2s+1)(s\,!)^2(n-s)!}\Pi^n_s(u_1, u_3, w'),$$

where

$$w' = \cos\Theta' = \cos\theta_1\cos\theta_3 + \sin\theta_1\sin\theta_3\cos(\phi_1-\phi_3).$$

Equation (C.4) gives the constants κ_3 (4.18).

The completeness condition (C.2) may be proved if we put (4.17) in (C.2):

$$\sum_{s=0}^{n}\Pi_s(u_1, u_2, w) = \frac{n+1}{2^{2n}}\sum_{s=0}^{n}\frac{2^{2s}(s!)^2(n-s)!}{(n+s+1)!}(2s+1)\times$$

$$\times (1-u_1^2)^{s/2} (1-u_2^2)^{s/2} C_{n-s}^{s+1}(u_1) C_{n-s}^{s+1}(u_2) P_s(w) =$$

$$= \frac{n+1}{2^{2n}} C_n^1 [u_1 u_2 + (1-u_1^2)^{1/2} (1-u_2^2)^{1/2} w] = \qquad (C.5)$$

$$= \frac{n+1}{2^{2n}} C_n^1 (z_1 z_2).$$

Here we take into account the summing formula for the Gegenbauere polynomials [15]:

$$C_n^{\lambda}(\cos\psi \cos\phi + \sin\psi \sin\phi \cos\theta) = \frac{\Gamma(2\lambda+1)}{[\Gamma(\lambda)]^2} \times$$

$$\times \sum_{k=0}^{n} \frac{2^k (n-k)! [\Gamma(\lambda+k)]}{\Gamma(2\lambda+n+k)} \qquad (C.6)$$

$$\times (2\lambda+2k-1) \sin^k\psi \sin^k\phi \, C_{n-k}^{\lambda+k}(\cos\psi) C_{n-k}^{\lambda+k}(\cos\phi) C_k^{\lambda-1/2}(\cos\theta),$$

for

$$\lambda = 1 - C_k^{1/2}(x) = P_k(x).$$

The right-hand side of (C.5) gives the unit operator, i.e.,

$$I(z_1 z_2) = \frac{n+1}{2^{2n}} C_n^1 (z_1 z_2). \qquad (C.7)$$

This may be proved if we take into account (C.6).

From (4.17), for $n = 1$, we have:

$$\Pi_0^1 = u_1 u_2 = \Pi_0^{\alpha\beta}(p) z_\alpha^1 z_\beta^2,$$

$$\Pi_1^1 = (1-u_1^2)^{1/2} (1-u_2^2)^{1/2} w = \Pi_1^{\alpha\beta}(p) z_\alpha^1 z_\beta^2,$$

where

$$\Pi_0^{\alpha\beta} = \frac{p^\alpha p^\beta}{p^2}, \quad \Pi_1^{\alpha\beta} = \delta^{\alpha\beta} - \frac{p^\alpha p^\beta}{p^2}$$

are the well known projection operators for 4-vector fields.

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