ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ

ДУБНА

E2 - 8213

D.B.Ion

SATURATION OF ISOSPIN BOUNDS AND CONSTRAINTS ON EXPERIMENTAL DATA AND AMPLITUDE ANALYSIS IN π N-SCATTERING

1974

ЛАБОРАТОРИЯ
ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

E2 - 8213

D.B.Ion*

SATURATION OF ISOSPIN BOUNDS AND CONSTRAINTS ON EXPERIMENTAL DATA AND AMPLITUDE ANALYSIS IN π N-SCATTERING

Submitted to *Annals of Physics*

* On leave of absence from Institute for Atomic Physics, Bucharest, Romania.

1. Introduction

Recently [1,2], we have investigated the most stringent isospin bounds on H (see the definitions (2b) and (3a)) as well as the constraints on polarization parameters, in the pion nucleon scattering, using a set of bilinear forms which can be constructed from the scattering amplitudes. Thus we have obtained that the most stringent bounds on H are exactly saturated on the zeros-trajectories of these bilinear forms. The exact saturation of these bounds was, recently [2] investigated using the CERN-phase shift solutions [3] for the pion-nucleoin scattering. Also, the upper bound on H, derived recently by Doncel et al. [4] was analyzed by Tornqvist [5]. A systematic comparison with the experimental data of the isospin inequalities on unpolarized differential cross sections was given in ref. [6] in connection with the isospin polarization parameter introduced in ref. [7]. Next, defining $\bar{\Sigma}^n$ integrated cross sections and using classical Minkowski's and Hölder's inequalities we have obtained a large class of isospin inequalities on (unpolarized and polarized) integrated cross sections.

If the complete experimental data are available then a study of the saturations of the most stringent isospin bounds on H [1,2] is useful in order to obtain the strong constraints on the amplitude analysis ($\mathrm{Im}\, Z_{ij}^{(0)} = 0$, $\mathrm{Im}\, Z_{ij}^{(n)} = 0$ see the definitions (1a, 1b)). If some experimental data are lacking, as is usually the case, it would be interesting to see if these constraints can be obtained directly from the experimental data or not. In this paper, in Sect. 2, we improve the isospin bounds (10) on $-\lambda(\sigma_+, \sigma_-, 2\sigma_{CE})$- function which satisfy the condition that their saturation is strictly connected with the zeros-trajectories of $\operatorname{Im} Z^{(0)}_{ij}$ and $\operatorname{Im} Z^{(n)}_{ij}$, $n = 1,2,3$. Also, in Sect. 2, we discuss the constraints imposed on the experimental data and amplitude analysis when the isospin bounds are saturated or degenerated. The exact saturation of the isospin bounds (10), using the CERN-phase shift [3] for the pion-nucleon scattering, are investigated in Sect. 3. The "integrated" analogues of the bounds (10) as well as the stringent bounds on $\Sigma^{(n)}$ -integrated cross sections are obtained in Sect. 4, Some extensions of the isospin inequalities, based on Young's inequality are suggested in Sects. 2 and 4. The constraints (8a,b) on the experimental data and amplitude analysis obtained in this paper are relevant and very useful for a phenomenological description of the scattering processes.

2. Isospin Bounds and Constraints on Experimental Data and Amplitude Analysis

In order to discuss the isospin bounds and the constraints imposed on the experimental data and amplitude analysis, when the isospin bounds are saturated or degenerated, in a systematical way, we start with the following definitions.

Let f_i and g_i be the spin-non-flip and spin-flip pion-nucleon scattering amplitudes and let $K^{(\pm)}_i = f_i \pm i g_i$ and $H^{(\pm)}_i = f_i \pm g_i$. The indices $i = +, -, CE, 2I_s, 2I_t, 2I_u$ refer to the charge or (I_s, I_t, I_u) -isospin channels. We define

$$M^{(\pm 1)}_{ij} \equiv [K^{(\pm)}_i]^* K^{(\pm)}_j \; ; \; M^{(\pm 2)}_{ij} \equiv [H^{(\pm)}_i]^* H^{(\pm)}_j ; \; M^{(+3)}_{ij} \equiv 2 f^*_i f_i \; ;$$

$$M^{(-3)}_{ij} \equiv 2 g^*_i g_j \; ; \tag{1a}$$

$$Z^{(0)}_{ij} \equiv \frac{1}{2}[M^{(+n)}_{ij} + M^{(-n)}_{ij}] ; \; Z^{(n)}_{ij} \equiv \frac{1}{2}[M^{(+n)}_{ij} - M^{(-n)}_{ij}] ; \tag{1b}$$

$$M^{(\pm n)}_{kk} = (1 \pm X_k)\sigma_k \; ; \; Z^{(0)}_{kk} = \sigma_k \; ; \; Z^{(n)}_{kk} = X_k \sigma_k \, , \tag{1c}$$

where $X_k \equiv (P_k, T_k, S_k)$ for $n=1,2,3$ respectively, and

$$\lambda(z,y,z) \equiv x^2 + y^2 + z^2 - 2xy - 2xz - 2yz \, , \tag{2a}$$

$$H_{ij} \equiv \frac{1}{2}[1 - \vec{P}_i \cdot \vec{P}_j]\sigma_i \sigma_j \, , \quad \vec{P}_i \cdot \vec{P}_j = P_i P_j + T_i T_j + S_i S_j , \tag{2b}$$

$$C_{+-} = \frac{1}{2} C_{+CE} = \frac{1}{2} C_{-CE} = \frac{9}{4} C_{13s} = \frac{1}{4} C_{02t} = \frac{9}{4} C_{13u} = 1 \, . \tag{2c}$$

Then the isospin invariance alone (see refs. [1,8]) implies

$$C_{ij} H_{ij} = H \geq 0 \, , \tag{3a}$$

$$C_{ij}[\operatorname{Im} M^{(\pm n)}_{ij}]^2 = -\frac{1}{4}\lambda[M^{(\pm n)}_{++}, M^{(\pm n)}_{--}, 2M^{(\pm n)}_{CECE}] \equiv -\frac{1}{4}\lambda^{(\pm)}_n , \tag{3b}$$

$$C_{ij}[\operatorname{Im} Z^{(0)}_{ij}]^2 = -H - \frac{1}{4}\lambda(\sigma_+, \sigma_-, 2\sigma_{CE}) \geq 0 \, , \tag{3c}$$

$$C_{ij}[\operatorname{Im} Z^{(n)}_{ij}]^2 = H - \frac{1}{4}\lambda[Z^{(n)}_{++}, Z^{(n)}_{--}, 2Z^{(n)}_{CECE}] , \tag{3d}$$

and

$$C_{ij}\{[\operatorname{Re} N_{ij}]^2 - N_{ii} N_{jj}\} = \frac{1}{4}\lambda[N_{++}, N_{--}, 2N_{CECE}] , \tag{3e}$$

for any $N_{ij} \equiv M^{(\pm n)}_{ij}, Z^{(0)}_{ij}, Z^{(n)}_{ij}$, $n = 1,2,3$ and any $(ij) = (+-), (+CE), (-CE), (13s), (13u), (02t)$.

Therefore, the positivity condition

$$[\operatorname{Re} N_{ij}]^2 \geq 0 \, , \quad [\operatorname{Im} N_{ij}]^2 \geq 0 \, ,$$

implies the following isospin bounds

$$0 \leq -\lambda^{(+)}_n \leq 4 \cdot \min_{(ij)} \{C_{ij} M^{(+n)}_{ii} M^{(+n)}_{jj}\} , \tag{4a}$$

$$0 \leq -\lambda^{(-)}_n \leq 4 \cdot \min_{(ij)} \{C_{ij} M^{(-n)}_{ii} M^{(-n)}_{jj}\} , \tag{4b}$$

$$-4 \cdot \max_{(ij)} \{C_{ij} Z^{(n)}_{ii} Z^{(n)}_{jj}\} \leq \lambda[Z^{(n)}_{++}, Z^{(n)}_{--}, 2Z^{(n)}_{CECE}] \leq 4H \, , \tag{4c}$$

$$4H \le -\lambda(\sigma_+, \sigma_-, 2\sigma_{CE}) \le 4.\min_{(ij)} \{C_{ij}\sigma_i\ \sigma_j\},\tag{4d}$$

$$\max\{0, \frac{1}{4}\lambda[Z^{(n)}_{++}, Z^{(n)}_{--}, 2Z^{(n)}_{CECE}]\} \le H \le -\frac{1}{4}\lambda(\sigma_+, \sigma_-, 2\sigma_{CE}),\tag{4e}$$

for any $n = 1,2,3$ and $(ij) = (+-), (+CE), (-CE), (13s), (02t), (13u)$.

Therefore, when $\lambda[Z^{(n)}_{++}, Z^{(n)}_{--}, 2Z^{(n)}_{CECE}] < 0$ for any $n = 1,2,3$, the lower bound (4e) on H can be appreciably weaker. In this case the best isospin bounds on H can be improved using the inequalities:

$$2|\mathrm{Re}\, Z^{(0)}_{ij}| \le 2|Z^{(0)}_{ij}| \le |M^{(+n)}_{ij}| + |M^{(-n)}_{ij}|,\tag{5a}$$

$$2|\mathrm{Re}\, Z^{(n)}_{ij}| \le 2|Z^{(n)}_{ij}| \le |M^{(+n)}_{ij}| + |M^{(-n)}_{ij}|,\tag{5b}$$

from which we obtain

$$\max_{(ij)(X)}\{C_{ij}\sigma_i\ \sigma_j\ [(X_i - X_j)^2 + \xi^{(-)}_{ij}]\} \le 4H \le -\lambda(\sigma_+, \sigma_-, 2\sigma_{CE}),\tag{6a}$$

$$\max_{(n)}\{0, \lambda[Z^{(n)}_{++}, Z^{(n)}_{--}, 2Z^{(n)}_{CECE}]\} \le 4H \le \min_{(X)(ij)}\{C_{ij}\sigma_i\ \sigma_j\ [(X_i - X_j)^2 + \xi^{(+)}_{ij}]\},\tag{6b}$$

respectively for any $X_k \equiv (P_k, T_k, S_k)$, where

$$\xi^{(\pm)}_{ij} = 2 - X_i^2 - X_j^2 \pm 2[(1 - X_i^2)(1 - X_j^2)]^{1/2}.\tag{6c}$$

The isospin bounds (4a,b,c,d,e) and (10a,b,c) are sufficient for the study of any experimental situation at all energies and any scattering angles.

The following consequences of the isospin invariance are of great interest for an amplitude analysis:

(i) If $-C_{ij}Z^{(n)}_{ii}Z^{(n)}_{jj} = C_{k\ell}\sigma_k\ \sigma_\ell$ then the isospin bounds (4c), (4d), (4e) are degenerated and the lower isospin bounds (4a) and (4b) are saturated. The constraints on the scattering amplitudes are $|Z^{(0)}_{k\ell}| = 0$, $|Z^{(n)}_{ij}| = 0$.

(ii) If $\lambda(\sigma_+, \sigma_-, 2\sigma_{CF}) = C_{ij}Z^{(n)}_{ii}Z^{(n)}_{jj}$, then the isospin bounds (4c) and (4e) are degenerated and the lower bounds (4a), (4b) are saturated. The constraints on the scattering amplitudes are $\mathrm{Im}\, Z^{(0)}_{ij} = 0$ and $|Z^{(n)}_{ij}| = 0$.

(iii) If $\lambda[Z^{(n)}_{++}, Z^{(n)}_{--}, 2Z^{(n)}_{CE}] = C_{k\ell}\sigma_k\ \sigma_\ell$, then the isospin bounds (4d), (4e) are degenerated and the lower bounds (4a), (4b) are saturated. The constraints on the scattering amplitudes are $\mathrm{Im}\, Z^{(n)}_{ij} = 0$ and $|Z^{(0)}_{k\ell}| = 0$.

(iv) If $-\lambda(\sigma_+, \sigma_-, 2\sigma_{CE}) = \lambda(Z^{(n)}_{++}, Z^{(n)}_{--}, 2Z^{(n)}_{CECE})$, then the lower isospin bounds (4a), (4b) are saturated while the bounds (4e) are degenerated. The constraints on the scattering amplitudes are $\mathrm{Im}\, Z^{(0)}_{ij} = \mathrm{Im}\, Z^{(n)}_{ij} = 0$.

The constraints on the experimental data when $\mathrm{Re} N_{ij} = 0$, $N_{ij} \equiv M^{(+n)}_{ij}, M^{(-n)}_{ij}, Z^{(0)}_{ij}, Z^{(n)}_{ij}$, are given in table 1 (we have used the relations (19a,b,c,d,e,f) from ref. /1/). Now, in order to obtain the constraints on the experimental data when $\mathrm{Im}\, Z^{(0)}_{ij} = 0$ or $\mathrm{Im}\, Z^{(n)}_{ij} = 0$ we observe that, using (1b) and (3b), $\mathrm{Im}\, Z^{(0)}_{ij}$ and $\mathrm{Im}\, Z^{(n)}_{ij}$ can be written in the form:

$$\begin{aligned} C_{ij}\ [\mathrm{Im}\, Z^{(0)}_{ij}]^2 &= \frac{1}{16}[\sqrt{-\lambda^{(+)}_1} + \epsilon_1 \sqrt{-\lambda^{(-)}_1}]^2 \\ &= \frac{1}{16}[\sqrt{-\lambda^{(+)}_2} + \epsilon_2 \sqrt{-\lambda^{(-)}_2}]^2 \\ &= \frac{1}{16}[\sqrt{-\lambda^{(+)}_3} + \epsilon_3 \sqrt{-\lambda^{(-)}_3}]^2, \end{aligned}\tag{7a}$$

where

$$C_{ij}[\mathrm{Im} Z^{(n)}_{ij}]^2 = \frac{1}{16}[\sqrt{-\lambda^{(+)}_n} - \epsilon_n \sqrt{-\lambda^{(-)}_n}]^2,\tag{7b}$$

$$\begin{aligned} \epsilon_n &\equiv \mathrm{sign}[\mathrm{Im} M^{(+n)}_{ij} \cdot \mathrm{Im} M^{(-n)}_{ij}] = \mathrm{sign}\{[\mathrm{Im}\, Z^{(0)}_{ij}]^2 - [\mathrm{Im} Z^{(n)}_{ij}]^2\} = \\ &= \mathrm{sign}\{-2H - \frac{1}{4}\lambda(\sigma_+, \sigma_-, 2\sigma_{CE}) + \frac{1}{4}\lambda[Z^{(n)}_{++}, Z^{(n)}_{--}, 2Z^{(n)}_{CECE}\}. \end{aligned}\tag{7c}$$

Therefore, if $\mathrm{Im}\, Z^{(0)}_{ij} = 0$, $\epsilon_1 = \epsilon_2 = \epsilon_3 = -1$, then we obtain

$$2Z^{(n)}_{CECE}\ (\sigma_+ + \sigma_- - 2\sigma_{CE}) = Z^{(n)}_{++}(\sigma_+ - \sigma_- - 2\sigma_{CE}) - \tag{8a}$$
$$-Z^{(n)}_{--}(\sigma_+ - \sigma_- + 2\sigma_{CE}),$$

or equivalently

$$2\sigma_{CE}[Z^{(n)}_{++} + Z^{(n)}_{--} - 2Z^{(n)}_{CECE}] = \sigma_+ [Z^{(n)}_{++} - Z^{(n)}_{--} - 2Z^{(n)}_{CECE}] -$$
$$-\sigma_- [Z^{(n)}_{++} - Z^{(n)}_{--} + 2Z^{(n)}_{CECE}], \tag{8b}$$

valid for any $n = 1, 2, 3$. The constraints (8a) or (8b) are also obtained when $\mathrm{Im}\, Z^{(n)}_{ij} = 0$ only for that n for which $\mathrm{Im}\, Z^{(n)}_{ij} = 0$. Equations (8a) or (8b), table I and table II, are sufficient for the study of all contraints imposed on the experimental data and amplitude analysis when the isospin bounds are degenerated or exactly saturated. For example, if one knows that the bounds (4d) are degenerated then $H = C_{ij}\sigma_i \sigma_j$ and $|Z^{(0)}_{ij}| = 0$. Therefore, from definition (3a) of H or using Eqs. (8a,b) and table I we obtain: $\vec{P}_i = -\vec{P}_j$. In this case the angles $\theta_{k\ell} = \cos^{-1}(\vec{P}_k \cdot \vec{P}_\ell)$ are all known since $H_{k\ell} = C_{ij}\sigma_i \sigma_j / C_{k\ell}$ (see table II). In a similar way, if the isospin bounds (4c) are degenerated ($|Z^{(n)}_{ij}|^2 = 0$), from Eqs. (8a,b) and table I we obtain $Z^{(n)}_{ii}/\sigma_i = -Z^{(n)}_{jj}/\sigma_j$ and $Z^{(n')}_{ii}/\sigma_i = Z^{(n')}_{jj}/\sigma_j$ for any $n' \neq n$. Also in this case $\theta_{k\ell}$ can be determined from the relation $H_{k\ell} = -C_{ij} Z^{(n)}_{ii} Z^{(n)}_{jj} / C_{k\ell}$ (see table II) for any $(k\ell) = (+-), (+CE), (-CE), (13s), (02t), (13u)$.

In general, when $\sigma_+, \sigma_-, \sigma_{CE}$ and also $Z^{(n)}_{++}$, $Z^{(n)}_{--}$, $Z^{(n)}_{CECE}$ are known from experimental data for a given n then the true solution for H is one of the values obtained from the relation

$$H = -\frac{1}{8}\lambda(\sigma_+, \sigma_-, 2\sigma_{CE}) + \frac{1}{8}\lambda[Z^{(n)}_{++}, Z^{(n)}_{--}, 2Z^{(n)}_{CECE}] -$$
$$-\frac{\epsilon_n}{8}\sqrt{-\lambda^{(+)}_n}\sqrt{-\lambda^{(-)}_n}, \tag{9}$$

where $\epsilon_n = \pm 1$.

In order to choose the correct solution for H one tries to use the isospin bounds (6a,b) or an additional theoretical input. If furthermore H is known from the experimental data then the relation (9) can be used for a fundamental test of the isospin invariance in the pion-nucleon scattering.

Next, from the isospin bounds (4e) and relation (9) we obtain the bounds

$$\lambda[Z^{(n)}_{++}, Z^{(n)}_{--}, 2Z^{(n)}_{CECE}] + \sqrt{-\lambda^{(+)}_n}\sqrt{-\lambda^{(-)}_n} \leq -\lambda(\sigma_+, \sigma_-, \sigma_{CE}), \tag{10}$$

valid at any energy and scattering angle for any $n = 1, 2, 3$, as well as the equalities:

$$\lambda[Z^{(n)}_{++}, Z^{(n)}_{--}, 2Z^{(n)}_{CECE}] - \epsilon_n\sqrt{-\lambda^{(+)}}\sqrt{-\lambda^{(-)}} =$$
$$= \lambda[Z^{(n')}_{++}, Z^{(n')}_{--}, 2Z^{(n')}_{CE}] - \epsilon_{n'}\sqrt{-\lambda^{(+)}_{n'}}\sqrt{-\lambda^{(-)}_{n'}} \equiv I \tag{11}$$

for any $n, n' = 1, 2, 3$.

The quantity I is invariant under rotations of the spin reference frame. The equalities (11) can also be used for a fundamental test of the isospin invariance when two components of the polarization vectors $\vec{P}_+, \vec{P}_-, \vec{P}_{CE}$ are known from the experimental data.

The isospin bounds (10) are particular cases of the inequalities (we have used the inequalities (14.7) from ref. /9/)

$$[-\lambda^{(+)}_n]^{1/p}\ [-\lambda^{(-)}_n]^{1/q} \leq -\frac{\lambda^{(+)}_n}{p} - \frac{\lambda^{(-)}_n}{q}, \tag{12a}$$

$$[-\lambda^{(+)}_n]^{1/p}\ [-\lambda^{(-)}_n]^{1/q} \geq -\frac{\lambda^{(+)}_n}{p} - \frac{\lambda^{(-)}_n}{q}, \tag{12b}$$

according as $p > 0$ or $0 < p < 1$. The sign of equality holds in the inequalities (10) and (12a,b) if and only if $\lambda^{(+)}_n = \lambda^{(-)}_n$. Therefore the bounds (12a,b) and the lower bounds (10) on $-\lambda(\sigma_+, \sigma_-, 2\sigma_{CE})$ are exactly saturated on the zeros trajectories of $\mathrm{Im}\, Z^{(0)}_{ij}$ and $\mathrm{Im}\, Z^{(n)}_{ij}$.

Therefore in this case the constraints on the experimental data are given by Eqs. (8a,b). The bounds (10) as well as the bounds (12a,b) are more stringent than the isospin bound $4H \le -\lambda(\sigma, \sigma, 2\sigma_{CE})$ derived by Doncel et al. [4] and are the best possible ones since giving only $\sigma_+, \sigma_-, \sigma_{CE}$ and X_+, X_-, X_{CE} we can obtain the strong constraints on the data and amplitude analysis as well as a fine test of the isospin invariance in the pion-nucleon scattering.

In a similar way we obtain the inequalities

$$\{\frac{1}{4}\lambda[Z^{(n)}_{++}, Z^{(n)}_{--}, 2Z_{CECE}]\}^{\alpha_1}\{-\frac{1}{4}\lambda(\sigma_+, \sigma_-, 2\sigma_{CE})\}^{\alpha_2} \le$$

$$\le \frac{\alpha_1}{4}\lambda[Z^{(n)}_{++}, Z^{(n)}_{--}, 2Z^{(n)}_{CECE}] - \frac{\alpha_2}{4}\lambda(\sigma_+, \sigma_-, 2\sigma_{CE}), \tag{13a}$$

$$\{-\frac{1}{4}\lambda(\sigma_+, \sigma_-, 2\sigma_{CE})\}^{\alpha_1} H^{\alpha_2} \le -\frac{\alpha_1}{4}\lambda(\sigma_+, \sigma_-, 2\sigma_{CE}) + \alpha_2 H, \tag{13b}$$

$$\{\frac{1}{4}\lambda[Z^{(n)}_{++}, Z^{(n)}_{--}, 2Z^{(n)}_{CECE}]\}^{\alpha_1} H^{\alpha_2} \le \frac{\alpha_1}{4}\lambda[Z^{(n)}_{++}, Z^{(n)}_{--}, 2Z^{(n)}_{CECE}] +$$

$$+\alpha_2 H, \tag{13c}$$

for any $\alpha_1 > 0, \alpha_2 > 0, \alpha_1 + \alpha_2 = 1$, and $\lambda[Z^{(n)}_{++}, Z^{(n)}_{--}, 2Z^{(n)}_{CECE}] \ge 0$, and

$$H^{\alpha_1}[-\frac{1}{4}\lambda(\sigma_+, \sigma_-, 2\sigma_{CE})]^{\alpha_2}\{\frac{1}{4}\lambda[Z^{(n)}_{++}, Z^{(n)}_{--}, 2Z^{(n)}_{CECE}]\}^{\alpha_3} \le \tag{13d}$$

$$\le \alpha_1 H - \frac{\alpha_2}{4}\lambda(\sigma_+, \sigma_-, 2\sigma_{CE}) + \frac{1}{4}\lambda[Z^{(n)}_{++}, Z^{(n)}_{--}, 2Z^{(n)}_{CECE}],$$

for $\alpha_1 > 0$, $\alpha_2 > 0$, $\alpha_3 > 0$, $\alpha_1 + \alpha_2 + \alpha_3 = 1$, and

$$\lambda[Z^{(n)}_{++}, Z^{(n)}_{--}, 2Z^{(n)}_{CECE}] \ge 0.$$

The sign of equality holds in (13a,b,c,d) if and only if

$$\lambda[Z^{(n)}_{++}, Z^{(n)}_{--}, 2Z^{(n)}_{CECE}] = -\lambda(\sigma_+, \sigma_-, 2\sigma_{CE}), \tag{14a}$$

$$4H = -\lambda(\sigma_+, \sigma_-, 2\sigma_{CE}), \tag{14b}$$

$$4H = \lambda[Z^{(n)}_{++}, Z^{(n)}_{--}, 2Z^{(n)}_{CFCE}], \tag{14c}$$

$$4H = -\lambda(\sigma_+, \sigma_-, 2\sigma_{CE}) = \lambda[Z^{(n)}_{++}, Z^{(n)}_{--}, 2Z^{(n)}_{CECE}], \tag{14d}$$

respectively. Therefore, the isospin bounds (13 b,c) are exactly saturated on the zeros-trajectories of $\mathrm{Im}\, Z^{(0)}_{ij}$ and $\mathrm{Im}\, Z^{(n)}_{ij}$ respectively, while the bounds (14a,d) are saturated when both the $\mathrm{Im}\, Z^{(0)}_{ij}$ and $\mathrm{Im}\, Z^{(n)}_{ij}$ are zero.

Finally, we remark that a large class of isospin inequalities can be derived using the Young inequality [9]

$$ab \le \int_0^a \phi(x)\,dx + \int_0^b \phi^{-1}(y)\,dy, \tag{15}$$

where a and b are any combinations of H, $\lambda^{(+)}_n$, $\lambda^{(-)}_n$ such that $a \ge 0$, $b \ge 0$, $y = \varphi(x)$ is any continuous strictly increasing function of x for $x \ge 0$ with $\phi(0) = 0$ and $\phi^{-1}(y)$ is the function inverse to $\phi(x)$. The sign of equality in (15) holds if and only if $b = \phi(a)$. Therefore, specializing $\phi(x)$, a and b in (15) we can obtain a number of interesting isospin inequalities. In particular with the substitution $a \to a^{1/p}$, $b \to b^{1/q}$, $\frac{1}{p} + \frac{1}{q} = 1$ and with $y = x^{p-1}$ (15) yields

$$a^{1/p} b^{1/q} \le \frac{a}{p} + \frac{b}{q} \text{ if } p > 1 \text{ and } a^{1/p} b^{1/q} \ge \frac{a}{p} + \frac{b}{q} \text{ if } 0 < p < 1$$

$(p \ne 0)$ from which we have derived the inequalities (12a,b) and (13a,b,c) specializing a and b.

3. Exact Saturation of Isospin Bounds (10)

For the study of saturation of isospin bounds (10) we have used the CERN-phase shift solutions [3] in order to calculate the zeros-trajectories of $\mathrm{Im}\, Z^{(0)}_{ij}$ and $\mathrm{Im}\, Z^{(n)}_{ij}$ (or $\mathrm{Im}\, Z'^{(n)}_{ij}$ in the helicity reference frame), $n = 1, 2, 3$ in the $(p_{LAB}, \cos\theta)$ -plane, where θ is the scattering angle in the centre-of-mass reference frame, so that the zeros-trajectories of $\mathrm{Im}\, Z_{ij}^{(0)}$ (solid lines) and zeros-trajectories of $\mathrm{Im}\, Z_{ij}^{(n)}$ ($\mathrm{Im}\, Z_{ij}^{\prime(n)}$) (dashed lines) for different $n = 1, 2, 3$ are presented in Figs. 1-5, respectively. In order to determine the regions from ($p_{LAB}, \cos\theta$)-plane where the isospin bounds (10) are nearly saturated (the unhatched regions from Figs. 1-5), we have used the quantities $F(F')$, $F^{(0)}(F'^{(0)})$ and $F^{(n)}(F'^{(n)})$ defined by the relations (7a,b,c) from ref. /2/, such that the hatched regions shown in Figs. 1-5 are obtained according to $F - F^{(0)} \geq 0.1$ or $F^{(n)} - F \geq 0.1$ (or $F'^{(n)} - F' \geq 0.1$ respectively), since it was pointed out by Törnqvist /5/ that these differences are known, through the phase shift solutions, with an accuracy of 0.1-0.3. Therefore, we find that the isospin bounds (10) are saturated within the experimental error limits in the entire $\cos\theta$-region below one pion production threshold for all $n = 1, 2, 3$ (see Figs. 1-5), this result being in agreement with the saturation of the isospin bounds on unpolarized integrated cross sections observed by Roy /10/ in the same energy region.

As can be seen from Figs. 1-5, the isospin bounds (10) are exactly saturated along certain lines in the ($p_{LAB}, \cos\theta$)-plane. These lines are independent of $n = 1, 2, 3$ and (ij) when correspond to the zeros-trajectories of $\mathrm{Im}\, Z_{ij}^{(0)}$ (solid lines) and are dependent on $n = 1, 2, 3$ and independent of (ij) when correspond to the zeros-trajectories of $\mathrm{Im}\, Z_{ij}^{(n)}$ (dashed lines) and impose the strong constraints (8a) on the experimental data.

Therefore, if σ_+, σ_-, σ_{CE} and also P_+, P_-, P_{CE} are known from the experimental data our results presented in Fig. 1 can be used for a fine test of change independence since we know the regions from (p_{LAB}, $\cos\theta$) -plane where the isospin bound (10) for $n = 1$ is almost saturated (the unhatched regions) according to the phase shift analysis. For such a test it is sufficient to verify the relation

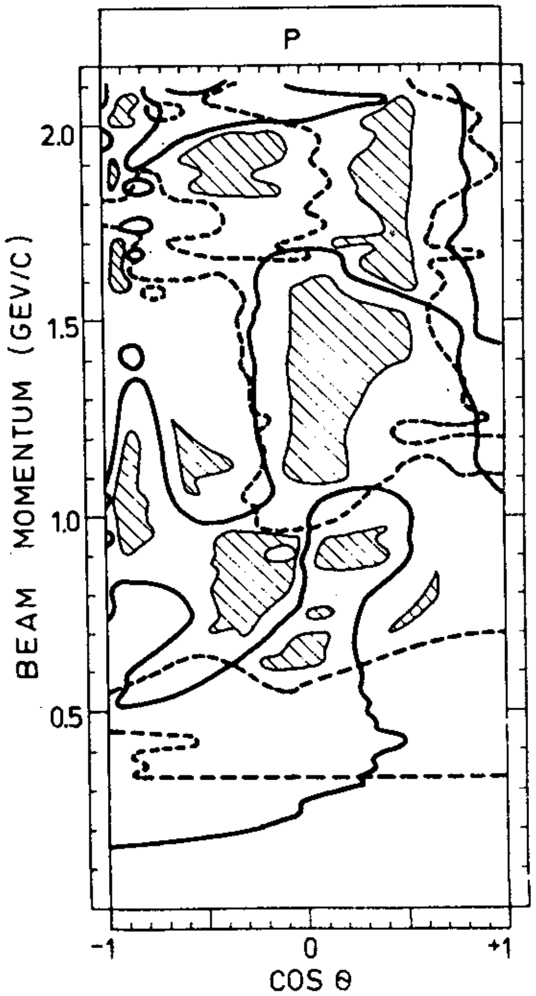

Fig. 1. The saturation of the isospin bound (10) for $n = 1, (P), (Z_{ii}^{(1)} \equiv P_i\, \sigma_i)$. The solid lines correspond to exact saturation of the bound (10) for $n = 1$ due to $\mathrm{Im}\, Z_{ij}^{(0)} = 0$ while the dashed lines correspond to exact saturation of bound (10) due to $\mathrm{Im}\, Z_{ij}^{(1)} = 0$. Unhatched regions: the bound (10) for $n = 1$ is saturated within the experimental errors ($F - F^{(0)} \leq 0.1$ or $F^{(1)} - F \leq 0.1$).

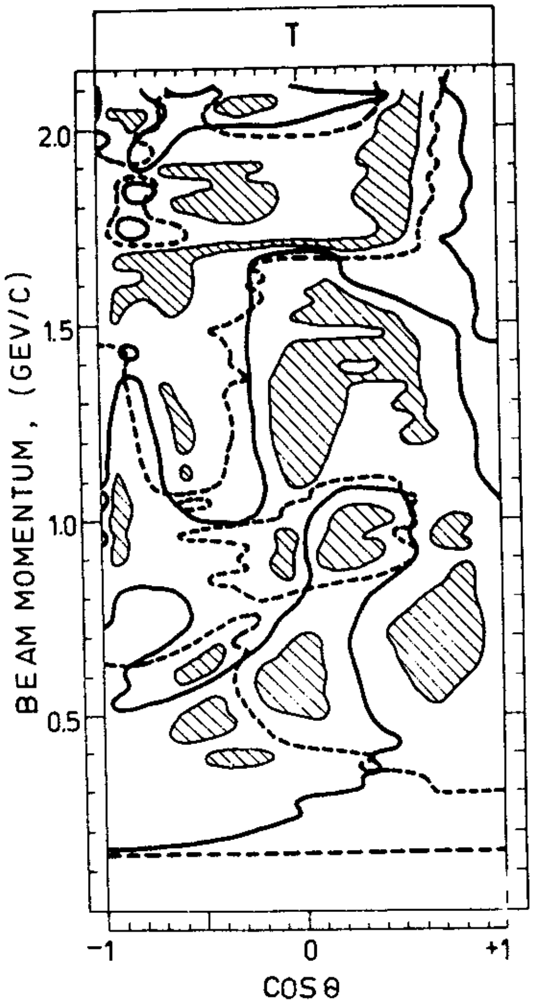

Fig. 2. The saturation of the isospin bound (10) for $n = 2$, (T), ($Z^{(2)}_{ij} \equiv T_i \sigma_j$). The solid lines correspond to exact saturation of the bound (10) for $n = 2$ due to $\mathrm{Im}\, Z^{(0)}_{ij} = 0$ while the dashed lines correspond to exact saturation due to $\mathrm{Im}\, Z^{(2)}_{ij} = 0$. Unhatched regions: the bound (10) for $n = 2$ is saturated within the experimental errors ($F - F^{(0)} \leq 0.1$ or $F^{(2)} - F \leq 0.1$).

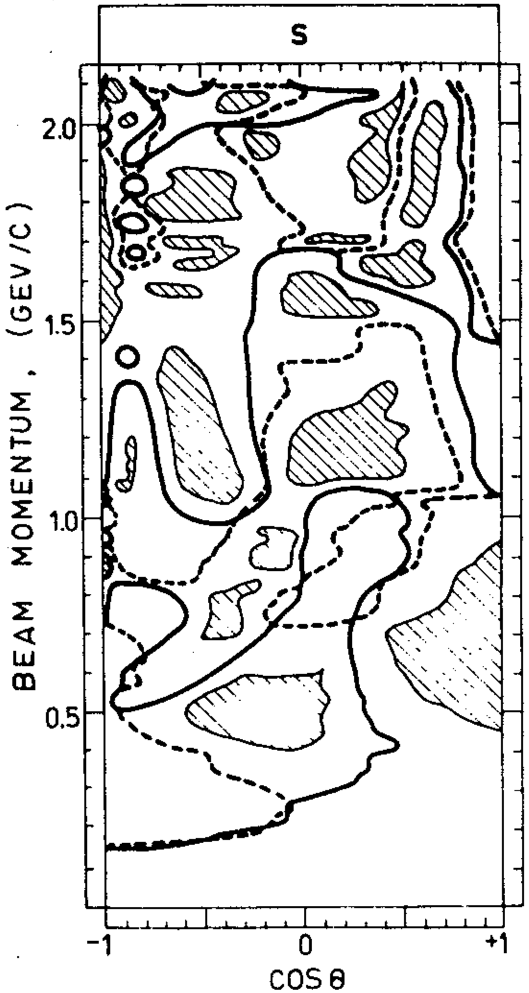

Fig. 3. The saturation of the isospin bound (10) for $n = 3$, (S), ($Z^{(3)}_{ij} \equiv S_i \sigma_j$). The solid lines correspond to exact saturation of the bound (10) for $n = 3$ due to $\mathrm{Im}\, Z^{(0)}_{ij} = 0$ while the dashed lines correspond to exact saturation due to $\mathrm{Im}\, Z^{(3)}_{ij} = 0$. Unhatched regions: the bound (10) for $n = 2$ is saturated within the experimental errors ($F - F^{(0)} \leq 0.1$ or $F^{(3)} - F \leq 0.1$).

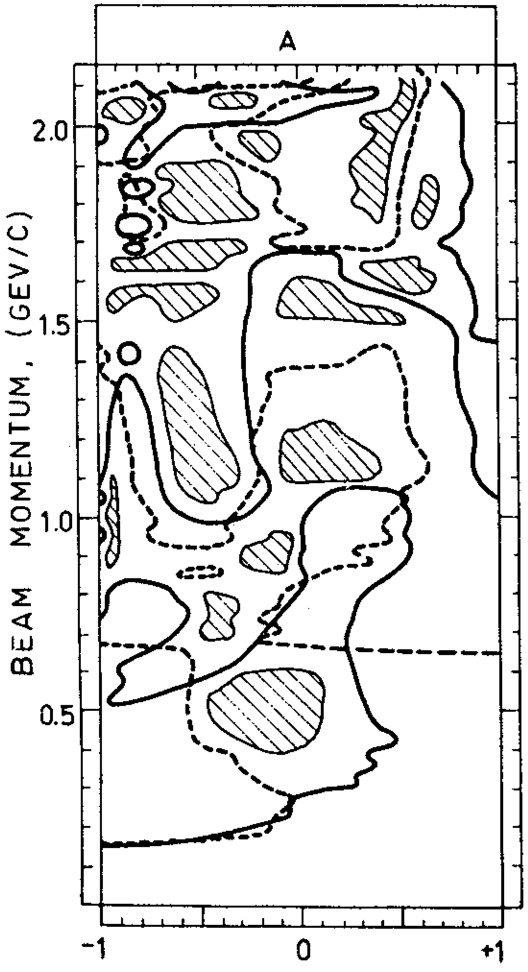

Fig. 4. The saturation of the isospin bound (10) for $n = 2$, (A), ($Z'^{(2)}_{ii} = -A_i\,\sigma_i$). The solid lines correspond to exact saturation of the bound (10) for $n = 2$, due to $\mathrm{Im}\,Z'^{(0)}_{ij} = 0$ while the dashed lines correspond to exact saturation due to $\mathrm{Im}\,Z'^{(2)}_{ij} = 0$. Unhatched regions: the bound (10) for $n = 2$ is saturated within the experimental errors ($F' - F'^{(0)} \leq 0.1$ or $F'^{(2)} - F' \leq 0.1$).

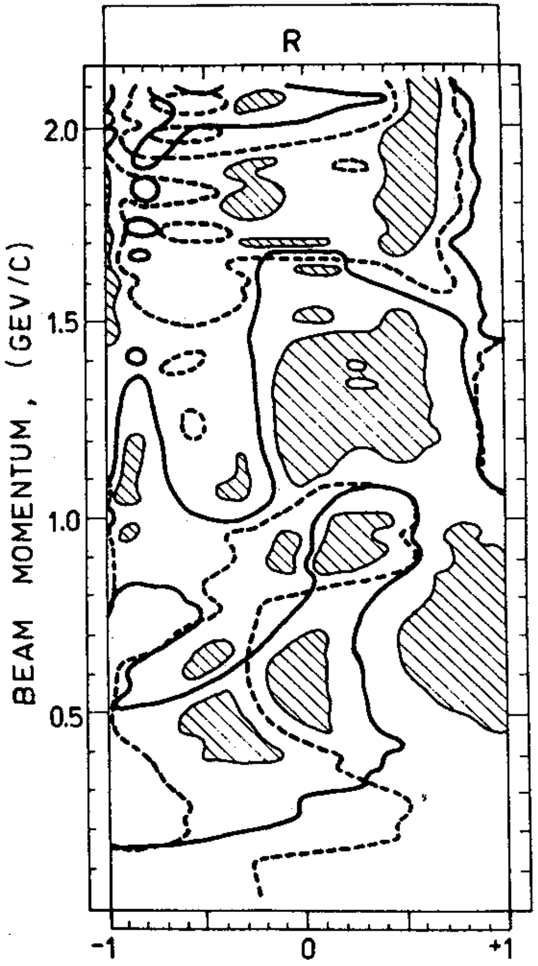

Fig. 5. The saturation of the isospin bound (10) for $n = 3$, (R), ($Z'^{(3)}_{ii} = R_i\,\sigma_i$). The solid lines correspond to exact saturation of the bound (10) for $n = 3$ due to $\mathrm{Im}\,Z'^{(0)}_{ij} = 0$ while the dashed lines correspond to exact saturation due to $\mathrm{Im}\,Z'^{(3)} = 0$. Unhatched regions: the bound (10) for $n = 3$ is saturated within the experimental errors ($F' - F'^{(0)} \leq 0.1$ or $F'^{(3)} - F' \leq 0.1$).

$$P_{CE} = \frac{P_+\sigma_+(\sigma_+ - \sigma_- - 2\sigma_{CE}) - P_-\sigma_-(\sigma_+ - \sigma_+ + 2\sigma_{CE})}{2\sigma_{CE}(\sigma_+ + \sigma_- - 2\sigma_{CE})}, \quad (16)$$

in the unhatched regions. This relation seems to be verified in the entire $\cos\theta$ -region for $P_{LAB}<0.6$ GeV/c, for all values of P_{LAB} up to 2.1 GeV/c when $\cos\theta \geq 0.6$, and in the entire backward hemisphere for $1.2<p_{LAB}<1.6$ GeV/c. Furthermore, this relation is also well verified in the near-forward region for $5<p_{LAB}<13.3$ GeV/c since we know that the isospin bounds (6a) for $X=P$ are degenerated within experimental errors (see ref. [11]). Also, since the degeneration of the isospin bounds (6a) implies $\mathrm{Im}\, Z^{(0)}_{ij}=0$ then we expert that the constraints (16) are also valid for the spin rotation parameters in this energy and transfer momentum region.

Next, since the true structure of the zeros trajectories of $\mathrm{Im}\, Z^{(0)}_{ij}$ and $\mathrm{Im}\, Z^{(n)}_{ij}$ is expected to be much simpler and smoother than that observed, the small details in Figs. 1-5 cannot be taken seriously because of the uncertainties of phase shifts. Thus, the exact positions of the lines where the isospin bounds (10) or (12 a,b) are exactly saturated are not well known; if these quantities are calculated from phase shifts, the result depends mainly on the theoretical assumptions. In such a situation, it will be interesting to obtain these lines (where the isospin bounds (10) are exactly saturated) from other phase shift analyses (e.g., Saclay[12] phase shift) and to compare these results with our results presented in Figs. 1-5 in order to study in more detail the ambiguities present in the phase shift analysis. On the other hand, any suggestion for a simple zeros-line pattern should be useful in order to construct different theoretical models. Also we note that the continuations of Figs. 1-5 to higher energies are of great interest for an amplitude analysis and a phenomenological study of the pion-nucleon scattering. On the other hand, our results presented in Figs. 1-5 will be useful for the localization and a detailed investigation of the isospin breaking effects [13] (e.g., the indirect effects due to mass (and width) differences, mixing between π^0 and η or between formed Δ's and N^*'s, differences in coupling constants). It would be interesting to perform a phase shift analysis of $\pi N \to \pi N$ at beam momenta up to 2 GeV/c, in a model-independent way, relaxing the isospin invariance or tolerating isospin breaking of order 10%[14], and to use the relations (3a) in order to estimate the breaking effects. Since the new solutions must lie near the isospin invariant ones, if the actually isospin invariant phase shift solutions are unique (see ref. [15]), then the breaking phenomena are expected to be present in the unhatched regions from Figs. 1-5. Finally, we note that the equalities (3a), (9) and (11) are sufficient to determine quantitatively the breaking phenomena when accurate experimental data will be available.

4. The Isospin Bounds on Integrated Cross Sections and Average Polarizations

Let $\bar\Sigma^{(n)}$ be the integrated cross sections defined as

$$\bar\Sigma^{(n)} = [\int_D \Sigma^n \, d\mu]^{1/n}, \quad \frac{1}{2} < n < +\infty, \quad (17)$$

where $\Sigma_i \equiv \sigma_i$, $(1 \pm X_i)\sigma_i$, $X_i \equiv P_i, T_i, S_i$ or P_i, A_i, R_i, D is a region from the physical domain, and μ is a positive measure defined on the physical domain, $i = +, -, CE$. Then, the isospin invariance alone implies (see ref. [8]), that the $\bar\Sigma^{(n)}_+, \bar\Sigma^{(n)}_-, \bar\Sigma^{(n)}_{CE}$-integrated cross sections satisfy the inequality

$$-\lambda[\bar\Sigma^{(n)}_+, \bar\Sigma^{(n)}_-, 2\bar\Sigma^{(n)}_{CE}] \geq 0. \quad (18)$$

Next, let $\lambda^{(+n)}_{\bar X}$ and $\lambda^{(-n)}_{\bar X}$ be defined as

$$\lambda^{(\pm n)}_{\bar X} \equiv \lambda[\bar\Sigma^{(n)}_+, \bar\Sigma^{(n)}_-, 2\bar\Sigma^{(n)}_{CE}]. \quad (19)$$

for $\Sigma_i = (1+X_i)\sigma_i$ and $\Sigma_i = (1-X_i)\sigma_i$ respectively. Then, since $-\lambda^{(\pm n)}_{\bar X} \geq 0$ we can write the following inequalities

$$[-\lambda^{(+n)}_{\bar X}]^{1/p}\ [-\lambda^{(-n)}_{\bar X}]^{1/q} \leq -\frac{\lambda^{(+n)}_{\bar X}}{p} - \frac{\lambda^{(-n)}_{\bar X}}{q}, \tag{20}$$

for any $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. The inequality is reversed for $0 < p < 1$. The sign of equality holds in (20) if and only if $\lambda^{(+n)}_{\bar X} = \lambda^{(-n)}_{\bar X}$. Now, specializing (20) for different n, p, q we obtain a number of interesting results. For example, if $n = 1$, $p = q = 2$, we obtain the inequality

$$[-\lambda^{(+)}_{\bar X}]^{1/2}\ [-\lambda^{(-)}_{\bar X}]^{1/2} \leq -\lambda(\bar\sigma_+, \bar\sigma_-, 2\bar\sigma_{CE}) - \tag{21a}$$

$$-\lambda[\bar X_+\bar\sigma_+, \bar X_-\bar\sigma_-, 2\bar X_{CE}\bar\sigma_{CE}],$$

where

$$\lambda^{(\pm)}_{\bar X} \equiv \lambda[(1 \pm \bar X_+)\bar\sigma_+, (1 \pm \bar X_-)\bar\sigma_-, 2(1 \pm \bar X_{CE})\bar\sigma_{CE}], \tag{21b}$$

$\bar\sigma \equiv \bar\sigma^{(1)}$ are the usual integral cross sections and $\bar X_i = (\bar P_i, \bar T_i, \bar S_i), (\bar P_i, \bar A_i, \bar R_i), \ldots$ are the average values of the polarization components for a given kinematical region

$$\bar X_i = \frac{1}{\bar\sigma_i} \int_{\Omega_0} X_i\, \sigma_i\, d\Omega,\quad \bar\sigma_i = \int_{\Omega_0} \sigma_i\, d\Omega,\ \Omega_0 \leq 4\pi. \tag{21c}$$

The isospin bounds (21) are equivalent to

$$[\bar\sigma_- - \bar\sigma_+]^2 + [\bar X_-\bar\sigma_- - \bar X_+\bar\sigma_+]^2 \leq 4(1 + \bar X_{0t}\bar X_{CE}) \times$$

$$\times\bar\sigma_{CE}(\bar\sigma_+ + \bar\sigma_- - \bar\sigma_{CE}) - [-\lambda^{(+)}_{\bar X}]^{1/2}\ [-\lambda^{(-)}_{\bar X}]^{1/2}, \tag{22}$$

$$\bar X_{0t} = \frac{\bar X_+\bar\sigma_+ + \bar X_-\bar\sigma_- - \bar X_{CE}\bar\sigma_{CE}}{\bar\sigma_+ + \bar\sigma_- - \bar\sigma_{CE}}. \tag{23}$$

The sign of equality holds in (20) (for $n = 1$), (21a) and (22), if and only if $\lambda^{(+)}_{\bar X} = \lambda^{(-)}_{\bar X}$. This condition is equivalent to

$$2\bar X_{CE}\bar\sigma_{CE}(\bar\sigma_+ + \bar\sigma_- - 2\bar\sigma_{CE}) = \bar X_+\bar\sigma_+(\bar\sigma_+ - \bar\sigma_- - 2\bar\sigma_{CE}) - \tag{24a}$$

$$-\bar X_-\bar\sigma_-(\bar\sigma_+ - \bar\sigma_- + 2\bar\sigma_{CE}),$$

$$2\bar\sigma_{CE}(\bar X_+\bar\sigma_+ + \bar X_-\bar\sigma_- - 2\bar X_{CE}\bar\sigma_{CE}) = \bar\sigma_+(\bar X_+\bar\sigma_+ - \bar X_-\bar\sigma_- - 2\bar X_{CE}\bar\sigma_{CE}) -$$

$$-\bar\sigma_-(\bar X_+\bar\sigma_+ - \bar X_-\bar\sigma_- + 2\bar X_{CE}\bar\sigma_{CE}). \tag{24b}$$

The isospin bounds (22) enable us to understand the small differences between elastic integrated cross sections at high energies in terms of small charge exchange integrated cross sections. These bounds require that if

$$\bar\sigma_{CE}(\bar\sigma_+ + \bar\sigma_-) \xrightarrow[s\to\infty]{} 0, \tag{25a}$$

then

$$\bar\sigma_- - \bar\sigma_+ \xrightarrow[s\to\infty]{} 0,\quad \bar X_- - \bar X_+ \xrightarrow[s\to\infty]{} 0, \tag{25b}$$

and

$$[-\lambda^{(+)}_{\bar X}]^{1/2}\ [-\lambda^{(-)}_{\bar X}]^{1/2} \xrightarrow[s\to\infty]{} 0, \tag{25c}$$

for any average polarization component in any spin reference frame, and, conversely the πN - charge exchange integrated cross sections cannot vanish for $s \to \infty$ if one of the above relations (25b,c) does not hold when $s \to \infty$. On the other hand, from the isospin bounds (22), which are more stringent than the isospin bounds (36) from ref. /8/ we obtain the Pomeranchuk-type theorem (25b) assuming that

$$4[1 + \bar X_{0t}\bar X_{CE}]\bar\sigma_{CE}(\bar\sigma_+ + \bar\sigma_- - \bar\sigma_{CE}) - [-\lambda^{(+)}_{\bar X}]^{1/2}\ [-\lambda^{(-)}_{\bar X}]^{1/2} \xrightarrow[s\to\infty]{} 0. \tag{26}$$

Next, from (5a,b) and Hölder's inequalities (see refs /8, 9/) we obtain:

$$C_{ij}\bar\sigma_i\bar\sigma_j[(\bar X_i-\bar X_j)^2+\bar\xi^{(-)}_{ij}]\le-\lambda(\bar\sigma_+,\bar\sigma_-,2\sigma_{CE}), \tag{27a}$$

$$\lambda[\bar X_+\bar\sigma_+,\bar X_-\bar\sigma_-,2\bar X_{CE}\bar\sigma_{CE}]\le C_{ij}\bar\sigma_i\bar\sigma_j[(\bar X_i-\bar X_j)^2+\bar\xi^{(+)}_{ij}], \tag{27b}$$

valid at an energy for any $\bar X$ -average polarization component in any spin reference frame and for any $(ij)=(+-),(+CE),(-CE),(13s),(02t),(13u)$ where

$$\bar\xi^{(\pm)}_{ij}=2-\bar X_i^2-\bar X_j^2\pm2[(1-\bar X_i^2)(1-\bar X_j)]^{1/2} \tag{27c}$$

and C_{ij} are given by (2c).

Next, if we define

$$\int_{\Omega_0}\mathrm{Re}N_{ij}\,d\Omega\equiv\overline{\mathrm{Re}N_{ij}},\quad N_{ij}\equiv M^{(+n)}_{ij},M^{(-n)}_{ij},Z^{(0)}_{ij},Z^{(n)}_{ij}, \tag{28a}$$

$$\int_{\Omega_0}N_{ij}\,d\Omega\equiv\bar N_{ij}, \tag{28b}$$

then it is easy to see that

$$C_{ij}\{\bar N_{ii}\bar N_{jj}-[\overline{\mathrm{Re}N_{ij}}]^2\}=-\frac{1}{4}\lambda[\bar N_{++},\bar N_{--},2\bar N_{CECE}]. \tag{29}$$

Therefore, from (9), (29) and (21a) we obtain:

$$0\le-\lambda^{(+)}_{\bar X}\le4\cdot\min_{(ij)}\{C_{ij}(1+\bar X_i)\bar\sigma_i(1+\bar X_j)\sigma_j\}, \tag{30a}$$

$$0\le-\lambda^{(-)}\le4\cdot\min_{(ij)}\{C_{ij}(1-\bar X_i)\bar\sigma_i(1-\bar X_j)\bar\sigma_j\}, \tag{30b}$$

$$-4\cdot\max_{(ij)}\{C_{ij}\bar X_i\bar X_j\bar\sigma_i\bar\sigma_j\}\le\lambda(\bar X_+\bar\sigma_+,\bar X_-\bar\sigma_-,2\bar X_{CE}\bar\sigma_{CE})\le$$
$$\le-\lambda(\bar\sigma_+,\bar\sigma_-,2\bar\sigma_{CE})-[-\lambda^{(+)}_{\bar X}]^{1/2}[-\lambda^{(-)}_{\bar X}]^{1/2}, \tag{30c}$$

$$\lambda(\bar X_+\bar\sigma_+,\bar X_-\bar\sigma_-,2\bar X_{CE}\bar\sigma_{CE})+[-\lambda^{(+)}_{\bar X}]^{1/2}[-\lambda^{(-)}_{\bar X}]^{1/2}\le$$
$$\le-\lambda(\bar\sigma_+,\bar\sigma_-,2\bar\sigma_{CE})\le4\cdot\min_{(ij)}\{C_{ij}\bar\sigma_i\bar\sigma_j\}. \tag{30d}$$

We remark that the isospin bounds (30a), (30b), (21a), the lower bounds (30c) and the upper bounds (30d) are the "integrated" analogues of the isospin bounds (4a), (4b), (10), the lower bounds (4c), and the upper bounds (4d), respectively.

Next, if we define $\bar H_{ij}$ by

$$\bar H_{ij}=\frac{1}{2}[1-\vec{\bar P}_i\cdot\vec{\bar P}_j]\bar\sigma_i\bar\sigma_j;\quad \vec{\bar P}_i\cdot\vec{\bar P}_j=\bar P_i\bar P_j+\bar T_i\bar T_j+\bar S_i\bar S_j \tag{31}$$

Then it would be interesting to obtain the conditions for which the equalities (3a) and (9) as well as the isospin bounds (4c) have an "integrated" analogue. For this we observe that if we define the scattering amplitudes $\bar f_i$ and $\bar g_i$ by the relations:

$$\bar\sigma_i=|\bar f_i|^2+|\bar g_i|^2;\quad\bar\sigma_i\bar P_i=2\,\mathrm{Im}(\bar f_i\bar g_i^{*}),\ \bar\sigma_i\bar T_i=2\,\mathrm{Re}(\bar f_i\bar g_i^{*})$$
$$\bar\sigma_i\bar S_i=|\bar f_i|^2-|\bar g_i|^2 \tag{32}$$

and also we define the bilinear forms $\bar M^{(\pm n)}_{ij}$, $\bar Z^{(0)}_{ij}$, and $\bar Z_{ij}$, $n=1,2,3$, by the relations (1a,b,c) using the substitution $f_k\to\bar f_k$, $g_k\to\bar g_k$, $K^{(\pm)}_k\to\bar K^{(\pm)}_k=\bar f_k\pm i\bar g_k$, $H^{(\pm)}\to\bar H^{(\pm)}=\bar f\pm\bar g$ then it is sufficient to assume that the scattering amplitudes $\bar f_i$ and $\bar g_i$ satisfy the same isospin sum rules as f_i and g_i ones, in order to obtain for each equality or inequality from Sect. 2 an "integrated" analogue. It would be interesting to examine in more detail this hypothesis and to test it from the experimental data or available amplitude analysis.

Finally we remark that other interesting results can be obtained if we use the Young inequality (15) with a and b as functions $\lambda^{(+)}_{\bar X}$, $\lambda^{(-)}_{\bar X}$, $\bar H_{ij}$ and $-\lambda(\sigma_+,\sigma_-,2\sigma_{CE})$ such that $a\ge0$, $b\ge0$.

5. Conclusions

In this paper we have obtained that the isospin invariance alone implies the relations (3a,b,c,d,e), (7a,b),

(9) and (11) as well as the isospin bounds (4a,b,c,d,e), (6a,b,c) and (10). These relations are sufficient for the study of all the constraints imposed on the experimental data and amplitude analysis when the isospin bounds are saturated or degenerated (see the tables I, II and relations (8a,b)). On the other hand, since the isospin bounds (10) are equivalent to

$$[\sigma_- - \sigma_+]^2 + [X_-\sigma_- - X_{CE}\sigma_{CE}]^2 + [-\lambda_n^{(+)}]^{1/2}[-\lambda_n^{(-)}]^{1/2} \leq$$

$$\leq 4(1 + X_{0t}X_{CE})\,\sigma_{CE}(\sigma_+ + \sigma_- - \sigma_{CE}), \tag{33}$$

these bounds enable us to understand the small differences between elastic differential cross sections at high energies and fixed momentum transfer, in terms of the small charge-exchange differential cross sections and to obtain Pomeranchuk-type theorems:

$$\sigma_+ - \sigma_- \xrightarrow[s\to\infty,\ t-\text{fixed}]{} 0, \quad X_+ - X_- \xrightarrow[s\to\infty,\ t-\text{fixed}]{} 0, \tag{34a}$$

if

$$4(1 + X_{0t}X_{CE})\,\sigma_{CE}(\sigma_+ + \sigma_- - \sigma_{CE}) - \tag{34b}$$

$$-[-\lambda_n^{(+)}]^{1/2}[-\lambda_n^{(-)}]^{1/2} \xrightarrow[s\to\infty,\ t-\text{fixed}]{}.$$

The bounds (10) (as well as the bounds (12a,b)) are more stringent than the isospin bound $4H \leq -\lambda(\sigma_+, \sigma_-, 2\sigma_{CE})$ derived by Doncel et al. /4/ and are the best possible ones since giving only $\sigma_+, \sigma_-, 2\sigma_{CE}$ and X_+, X_-, X_{CE} we can obtain the strong constraints on the data and amplitude analysis. The saturations of the isospin bounds (10) are investigated in Sect. 2 using the CERN-phase shift solutions for the pion-nucleon scattering. We have found that the isospin bounds (10) are exactly saturated along certain lines (the solid and dashed lines in Figs. 1-5) in the $(p_{LAB}, \cos\theta)$ -plane. On these lines the strong constraints on experimental data and scattering amplitudes are imposed, so that the solid lines from Figs. 1-5

Table I

The constraints on N_{ii} when $\mathrm{Re}N_{ij} = 0$, $N_{ij} \equiv M_{ij}^{(+n)}$, $M_{ij}^{(-n)}$, $Z_{ij}^{(0)}$, $Z_{ij}^{(n)}$, $n = 1,2,3$.

Zeros of	Costraints on N_{ii}
$\mathrm{Re}N_{+-}$ *)	$N_{CECE} = \frac{1}{2}[N_{++} + N_{--}]$
$\mathrm{Re}N_{+CE}$	$N_{CECE} = \frac{1}{2}[N_{--} - N_{++}]$
$\mathrm{Re}N_{-CE}$	$N_{CECE} = \frac{1}{2}[N_{++} - N_{--}]$
$\mathrm{Re}N_{13s}$	$N_{CECE} = \frac{1}{6}[3N_{--} + N_{++}]$
$\mathrm{Re}N_{02t}$	$N_{++} = N_{--}$
$\mathrm{Re}N_{13u}$	$N_{CECE} = \frac{1}{6}[3N_{++} + N_{--}]$

*) $N_{ij} \equiv M_{ij}^{(+n)}, M_{ij}^{(-n)}, Z_{ij}^{(0)}, Z_{ij}^{(n)}$, $n = 1,2,3$.

Table II
The constraints on H when the isospin bounds are degenerated or saturated

Zeros of	H
$\lambda(\sigma_+, \sigma_-, 2\sigma_{CE})$	0
$\lvert Z_{ij}^{(0)} \rvert$	$C_{ij}\sigma_i\sigma_j$
$\lvert Z_{ij}^{(n)} \rvert$	$-C_{ij} Z_{ii}^{(n)} Z_{jj}^{(n)}$
$\mathrm{Im}\, Z_{ij}^{(0)}$	$-\frac{1}{4}\lambda(\sigma_+, \sigma_-, 2\sigma_{CE})$
$\mathrm{Im}\, Z_{ij}^{(n)}$	$\frac{1}{4}\lambda(Z_{++}^{(n)}, Z_{--}^{(n)}, 2Z_{CECE}^{(n)})$
$\mathrm{Im}\, M_{ij}^{(+n)}$	$-\frac{1}{8}\lambda(\sigma_+, \sigma_-, 2\sigma_{CE}) + \frac{1}{8}\lambda(Z_{++}^{(n)}, Z_{--}^{(n)}, 2Z_{CECE}^{(n)})$
$\mathrm{Im}\, M_{ij}^{(-n)}$	$-\frac{1}{8}\lambda(\sigma_+, \sigma_-, 2\sigma_{CE}) + \frac{1}{8}\lambda(Z_{++}^{(n)}, Z_{--}^{(n)}, 2Z_{CECE}^{(n)})$

correspond to the zeros-trajectories of $\mathrm{Im}\, Z_{ij}^{(0)}$ (see the definitions (la,b,c)) and to constraints:

$$2\vec{P}_{CE}\sigma_{CE}(\sigma_+ + \sigma_- - 2\sigma_{CE}) = \vec{P}_+\sigma_+(\sigma_+ - \sigma_- - 2\sigma_{CE}) - \tag{35}$$
$$-\vec{P}_-\sigma_-(\sigma_+ - \sigma_- + 2\sigma_{CE})$$

in any spin reference frames and, the dashed lines correspond to the zeros-trajectories of $\mathrm{Im}\, Z_{ij}^{(n)}$ and to constraints (8a) or (8b) valud only for that n for which the isospin bound (10) is exactly saturated. Also we have found that the isospin bounds (10) (or)(12a,b) are systematically saturated, within the experimental error limits, in the entire $\cos\theta$ -region below one-pion production threshold and also in the forward and backward regions, at all beam momenta considered here, especially for P, T, A, -polarization parameters. These statements hold also for the bounds (10) on S, R parameters only for $p_{LAB} > 1$ GeV/c in the forward region and for $p_{LAB} < 1.5$ GeV/c in the backward region. The continuation of Figs. 1-5 to higher energies are of great interest for an amplitude analysis and for a phenomenological study of the pion-nucleon scattering. An interesting feature of the pion-nucleon scattering at high energies and low transfer momenta, is the saturation of the isospin bounds (10) since we know from ref./11/ that the bound (6a) on $X \equiv P$ (where $\xi_{+-}^{(-)}$ have been neglected) are nearly degenerated in the near forward region. Therefore, we expect, that at high energies in the diffractive region, the relation (35) can be good approximations in order to determine $\vec{P}_{CE}$ when $\sigma_+, \sigma_-, \sigma_{CE}$ and also $\vec{P}_+$, $\vec{P}_-$ are known from the experimental data. Also, we expect that the true structure of the zeros trajectories of $\mathrm{Im}\, Z_{ij}^{(0)}$ and $\mathrm{Im}\, Z_{ij}^{(n)}$, $n = 1, 2, 3$, determined from the experimental data using the isospin bounds (10) or directly from the constraints (35) or (8a) in the entire (p_{LAB}, $\cos\theta$) plane, will be relevant for a phenomenological description of the scattering processes. On the other hand, our results presented in Figs. 1-5, will be useful for a fundamental test of the isospin invariance as well as for a localization and a detailed investigation of the breaking isospin effects [13]. We note that the equalities (3a), (9) and (11) are sufficiently in order to determine quantitatively, directly from the experimental data, the breaking effects when accurate experimental data will be available.

Finally we note that the isospin bounds (30a,b) and (21a) obtained in Sect. 4, are the "integrated" analogues of the isospin bounds (4a,b) and (10) respectively. The isospin bounds (21a), which are more stringent as familiar triangle inequalities require the validity of the Pomeranchuk like theorems (25b) on the integrated cross sections and average polarization parameters if the conditions (26) hold in the high energy limit. Some extensions of the isospin bounds, based on the Young inequality, are suggested in Sects. 2 and 4.

I would like to thank Drs. C.A.Gheorghe, S.Berceanu, S.Holan and F.G.Nichitiu for useful suggestions and discussions.

References

1. D.B.Ion. JINR Preprint, E2-7732, Dubna, 1974.
2. D.B.Ion, F.G.Nichitiu, A.C.Mihul. JINR Preprint, E2-7961, Dubna, 1974.
3. S.Almehed, C.Lovelace. CERN, TH-1403 and Nucl. Phys., B40, 157 (1972).
4. M.G.Doncel, L.Michel, P.Minneart. Phys.Lett., B38, 42 (1972).
5. N.A.Tőrnqvist. Helsinki. Preprint 26-73 (1973).
6. D.B.Ion and A.C.Mihul. JINR Preprint, E2-7733, Dubna, 1974.
7. D.B.Ion and A.C.Mihul. JINR Preprint, E2-7734, Dubna (1974).
8. D.B.Ion. JINR Preprint, E2-7868, Dubna, 1974.
9. E.F.Beckenbach, R.Bellman. Inequalities, Springer-Verlag, Berlin, Heidelberg, New-York, 1971, p. 15.
10. S.M.Roy. Nucl.Phys., 34B, 528 (1971).
11. G.V.Dass, J.Froyland, F.Halzen, A.Martin, C.Michel, S.M.Roy. Phys.Lett., 36B, 339 (1971).
12. R.Ayed, P.Bareyre, G.Villet. Phys.Lett., 31B, 598 (1970).
13. N.Tőrnqvist. Phys. Lett., 40B, 109 (1972) and references contained here.
14. P.Laurikainen, N.A.Tornquist. CERN preprint TH-1594 (1972).
15. G.Höhler, F.Kaiser, F.Wagner. Lett Nuovo Cim., 8, 232 (1973).

Received by Publishing Department
on August 15, 1974.