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QUANTUM MECHANICS AND HOPF FIBRE BUNDLES

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Spinors are of a growing importance in quantum and classical theories (supersymmetries, twistor formalism). There is suggested and has been proposed by many people /1/ overall transition from Cartesian coordinates to the spinors. Here we discuss classical and quantum mechanics in a spinor representation, namely, after the change of variables x)

$$x_m = \bar{\xi}\sigma_m\xi \quad (m=1,2,3) \qquad r = \bar{\xi}\xi, \tag{1}$$

where ξ is the usual 2-component complex spinor ("spinor coordinates"), and σ_m are the Pauli matrices. If one represents the spinor as follows

$$\xi = \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} = \begin{pmatrix} a_0 + i a_1 \\ b_0 + i b_1 \end{pmatrix} = \begin{pmatrix} u_1 + i u_2 \\ u_3 + i u_4 \end{pmatrix} \left(= \begin{pmatrix} u_1 + i u_4 \\ u_3 + i u_2 \end{pmatrix} \right) = \begin{pmatrix} \sqrt{r}\cos\frac{\theta}{2} e^{\frac{i}{2}(\alpha - \varphi)} \\ \sqrt{r}\sin\frac{\theta}{2} e^{\frac{i}{2}(\alpha + \varphi)} \end{pmatrix} \tag{2}$$

$$0 \le r \le \infty, \quad 0 \le \theta \le \pi, \quad 0 \le \varphi \le 2\pi, \quad 0 \le \alpha \le 4\pi.$$

then

$$\begin{aligned} x_1 &= \bar{\xi}\sigma_1\xi = 2(a_0 b_0 + a_1 b_1) = r\sin\theta\cos\varphi, \\ x_2 &= \bar{\xi}\sigma_2\xi = 2(a_0 b_1 - a_1 b_0) = r\sin\theta\sin\varphi, \\ x_3 &= \bar{\xi}\sigma_3\xi = a_0^2 + a_1^2 - b_0^2 - b_1^2 = r\cos\theta, \\ r &= \bar{\xi}\xi = a_0^2 + a_1^2 + b_0^2 + b_1^2 = u_\mu u_\mu. \end{aligned} \tag{3}$$

In eq. (2) in the brackets a correspondence is given with variables of Kustaanheimo and Stiefel /2/. They introduced these variables for regularization of celestial mechanics equations (to eliminate the Newton potential singularity). Their approach and technique seem somewhat cumbersome and unusual to physicists. We wish to stress that the use of spinors ξ is most suitable for both relevant spaces R_3 (see,

x) We can define space reflections of ξ as follows (cf. L.D.Landau /3/)

$\xi \rightarrow$	$\sigma_3\bar{\xi}$	$\bar{\xi}$	$\sigma_1\bar{\xi}$	$\sigma_2\bar{\xi}$	$\sigma_1\xi$	$\sigma_2\xi$	$\sigma_3\xi$
signs acquired by x_1, x_2, x_3	−++	+−+	++−	−−−	+−−	−+−	−−+

A standard reflection of ξ is not preferable from a point of view of the space R_4 of the variables u_μ.

e.g., the above significant expressions (1)) and R_4, and because of the well-known advanced machinery of Fierz identities to deal with the spinors ξ . These identities follow from the completeness relation for the σ-matrices

$$\sum_{\mu=0}^{3}(\sigma_\mu)_{\alpha\beta}(\sigma_\mu)_{\gamma\delta} = 2\delta_{\alpha\delta}\delta_{\gamma\beta} \qquad \left(\sigma_0 = I = \begin{pmatrix}1&0\\0&1\end{pmatrix}\right). \tag{4}$$

In particular, the simplest Fierz identity

$$\sum_{m=1}^{3}(\bar\xi\sigma_m\xi)^2 = (\bar\xi\xi)^2 \tag{5}$$

demonstrates that $\bar\xi\xi = r$ is not an independent quantity. As to $\psi = (u_1,u_2,u_3,u_4)$ it is in fact a real 4-component Majorana spinor, for which the Fierz identity $\sum_{\mu=1}^{4}(\bar\psi\gamma_\mu\psi)^2=0$ is equivalent to eq. (5), and $x_m = i\bar\psi\gamma_m\psi$, $r=\bar\psi\gamma_4\psi$. However, the above 2-component spinor formalism seems to be simpler. For other relevant Fierz identities see Appendix A.

The change of variables from ξ (or u) to x_m according to eq. (3) is in fact the Hopf map $S^3 \to S^2$ /2,3,5/ at each fixed r (r and $\sqrt{r}$ being radii of spheres S^2 and S^3, respectively), i.e., the fibre bundle of S^3 with the base space S^2, coordinates of which are written in a concise form (1), and the fibre S^1 ($e^{i\lambda}\xi$). In other words, this is the treatment of classical and quantum mechanics as a CP^1-theory(model).

Let us put eqs. (1) directly into the Kepler problem Lagrangian, thus obtaining

$$L = \frac{m}{2}\dot{\vec x}\dot{\vec x} + \frac{e^2}{r} = 2m(\bar\xi\xi)(\dot{\bar\xi}\dot\xi) + \frac{m}{2}(\dot{\bar\xi}\xi-\bar\xi\dot\xi)^2 + \frac{e^2}{\bar\xi\xi} \tag{6}$$

with the use of the Fierz identities, and now the derivation of equations of motion becomes routine. One can substitute any other potential $V(r)=V(\bar\xi\xi)$ or $V(\vec x)=V(\bar\xi\vec\sigma\xi)$ for the Newtonian one. The Lagrangian is invariant under gauge transformations $\xi(t)\to e^{i\lambda(t)}\xi(t)$, $\lambda(t)$ being an arbitrary function. This means that the equations fail to define one of four unknown functions. It is just $\alpha(t)$ that remains an arbitrary function of t. We can omit the term with $\dot{\bar\xi}\xi-\bar\xi\dot\xi$ and adopt the new Lagrangian

$$\tilde L = 2m(\bar\xi\xi)(\dot{\bar\xi}\dot\xi) + \frac{e^2}{\bar\xi\xi} \qquad \text{(or with any other V)} \tag{7}$$

(like do in electrodynamics). It is invariant under above transformations but with constant λ . This leads to the conservation law

$$(\bar\xi\xi)(\dot{\bar\xi}\xi-\bar\xi\dot\xi) = \text{const}(t). \tag{8}$$

If we choose const=0, and thus, impose the subsidiary condition (SC)

$$\dot{\bar\xi}\xi-\bar\xi\dot\xi=0 \quad \text{(a), or} \quad u_1\dot u_2-u_2\dot u_1+u_3\dot u_4-u_4\dot u_3=0 \quad \text{(b),}$$
$$\text{or} \quad r(\dot\alpha-\cos\theta\,\dot\varphi)=0 \quad \text{(c),} \tag{9}$$

the new theory becomes equivalent to the original one (equivalent equations, conservation laws, etc.), but is preferable for quantization, since it defines all four degrees of freedom. Lagrangian (7) (being constructed out of combinations of the type $\bar\xi\xi = u_\mu u_\mu$) has O_4-symmetry important in what follows. However, the combination $\dot{\bar\xi}\xi-\bar\xi\dot\xi$, Lagrangian (6), and SC are only O_3 symmetric.

Lagrangian (7) yields the equation of motion

$$2m(\bar\xi\xi)\ddot\xi + 2m(\bar\xi\xi)^{\cdot}\dot\xi - \left[2m(\dot{\bar\xi}\dot\xi) - \frac{e^2}{(\bar\xi\xi)^2}\right]\xi = 0. \tag{10}$$

It results in (with the use of SC and the Fierz identities, see Appendix A) the Newton equation $m(\bar\xi\sigma_m\xi)^{\cdot\cdot} + \frac{e^2(\bar\xi\sigma_m\xi)}{(\bar\xi\xi)^3}=0$, the conservation law of energy $\dot H=0$, where $H=2m(\bar\xi\xi)(\dot{\bar\xi}\dot\xi)-\frac{e^2}{\bar\xi\xi}$ is the Hamiltonian, etc. The equation can be written as

$$2m(\bar\xi\xi)^2\ddot\xi + 2m(\bar\xi\xi)(\bar\xi\xi)^{\cdot}\dot\xi - H\xi = 0 \tag{10'}$$

and upon replacing of t by the new parameter s: $ds = \frac{dt}{r} = \frac{dt}{\bar\xi\xi}$ as

$$2m\overset{\circ\circ}{\xi} - H\xi = 0 \qquad \left(\overset{\circ}{\xi} = \frac{d\xi}{ds}\right). \tag{10''}$$

When the energy H is fixed, eq. (10″) is linear, and for H < 0 it is an equation for a 4-dimensional oscillator. This is in accord with ref. /2/.

<u>Quantum mechanics. Path integral</u> depends on the classical Lagrangian and suggests itself as a simple way to incorporate the spinor variables in quantum mechanics. In fact, this was done by Duru and Kleinert /6/ when attempted to calculate the path integral for the Coulomb Green function in terms of the K.-S. variables. They have gone close to a correct Schwinger result /9/, but with some distinction. Their reasoning that, nevertheless, the result is correct seems, however, to be questionable. We can interpret the situation as follows. To reduce L to $\tilde L$, let us take into account SC(9.c), inserting

$$1=\int_{-\infty}^{\infty}d\alpha_1\cdots\int_{-\infty}^{\infty}d\alpha_N\prod_{n=1}^{N}\delta\left(\alpha_n-\alpha_{n-1}-\cos\theta_n(\varphi_n-\varphi_{n-1})\right)=$$

$$=\int_{-\infty}^{\infty}d\alpha_1\cdots\int_{-\infty}^{\infty}d\alpha_N\prod_{n=1}^{N}\sqrt{\frac{m\tau_n\tau_{n-1}}{2\pi\hbar i\Delta t}}\;e^{i\frac{m}{2\hbar\Delta t}\tau_n\tau_{n-1}\left[\alpha_n-\alpha_{n-1}-\cos\theta_n(\varphi_n-\varphi_{n-1})\right]^2} \qquad (11)$$

under the path integral. This indeed transforms L into $\tilde{L}$, as desired, but with an incorrect range of α_N : $[-\infty,\infty]$ instead of $[0,4\pi]$. It is known that for angular variables (unlike for the Cartesian ones) approximations like $\sin(\alpha_n-\alpha_{n-1})\approx\alpha_n-\alpha_{n-1}$ may fail to work /10/ . In any case, the path integral calculation seems to be ambiguous.

Heisenberg picture. Canonical quantization of the theory with Lagrangian (7) leads to the commutation relations (see Appendix B)

$$[\xi_\alpha,\xi_\beta]=[\xi_\alpha,\bar{\xi}_\beta]=[\xi_\alpha,\dot{\xi}_\beta]=[\bar{\xi}_\alpha,\dot{\bar{\xi}}_\beta]=0,$$

$$[\xi_\alpha,\dot{\bar{\xi}}_\beta]=[\bar{\xi}_\alpha,\dot{\xi}_\beta]=\frac{i\hbar}{2m\bar{\xi}\xi}\delta_{\alpha\beta},$$

$$[\dot{\xi}_\alpha,\dot{\xi}_\beta]=\frac{i\hbar}{2m(\bar{\xi}\xi)^2}(\xi_\alpha\dot{\xi}_\beta-\dot{\xi}_\alpha\xi_\beta),\quad [\dot{\bar{\xi}}_\alpha,\dot{\bar{\xi}}_\beta]=\frac{i\hbar}{2m(\bar{\xi}\xi)^2}(\bar{\xi}_\alpha\dot{\bar{\xi}}_\beta-\dot{\bar{\xi}}_\alpha\bar{\xi}_\beta),$$

$$[\dot{\xi}_\alpha,\dot{\bar{\xi}}_\beta]=\frac{i\hbar}{4m(\bar{\xi}\xi)^2}(\xi_\alpha\dot{\bar{\xi}}_\beta+\dot{\bar{\xi}}_\beta\xi_\alpha-\dot{\xi}_\alpha\bar{\xi}_\beta-\bar{\xi}_\beta\dot{\xi}_\alpha). \qquad (12)$$

Now SC must be imposed on state vectors

$$(\bar{\xi}\xi)(\dot{\bar{\xi}}\xi-\bar{\xi}\dot{\xi})\,|\;\rangle=0. \qquad (13)$$

The operator $(\bar{\xi}\xi)(\dot{\bar{\xi}}\xi-\bar{\xi}\dot{\xi})$ commutes with the physical quantities $x_m=\bar{\xi}\sigma_m\xi$, $\tau=\bar{\xi}\xi$, $\dot{x}_m=(\bar{\xi}\sigma_m\xi)^{\cdot}$ (like $\partial_\mu A_\mu$ with field strengths in quantum electrodynamics). Now one cannot replace t by s in the equation of motion (10). There is an essential difficulty with ordering of operators in the Lagrangian, equations of motion, etc. When finding Green functions it is expedient to turn to the Schrödinger picture.

Schrödinger picture. Let us transform the Schrödinger equation into spinor variables. To transform the Laplace operator in terms of variables ξ or u_μ we need to solve the overdetermined set of equations

$$\frac{\partial}{\partial u_\mu}=\frac{\partial x_m}{\partial u_\mu}\frac{\partial}{\partial x_m} \qquad (14)$$

with respect to $\partial/\partial x_m$. This is possible, and, in particular, we get

$$iL^{(3)}_{mn}=x_m\frac{\partial}{\partial x_n}-x_n\frac{\partial}{\partial x_m}=-\frac{1}{2}\left[u_m\frac{\partial}{\partial u_n}-u_n\frac{\partial}{\partial u_m}-\varepsilon_{mn\ell}\left(u_\ell\frac{\partial}{\partial u_4}-u_4\frac{\partial}{\partial u_\ell}\right)\right]=-\frac{i}{2}\left(L^{(4)}_{mn}-\varepsilon_{mn\ell}L^{(4)}_{\ell4}\right) \qquad (15)$$

(here $\varepsilon_{\ell mn}$ is the totally antisymmetric tensor $\varepsilon_{123}=1$), i.e., the O_3 generators are expressed via the O_3-subalgebra of the O_4 algebra. As a consistency condition of set (14) there arises the SC

$$\frac{1}{2}\left[u_1\frac{\partial}{\partial u_2}-u_2\frac{\partial}{\partial u_1}+u_3\frac{\partial}{\partial u_4}-u_4\frac{\partial}{\partial u_3}\right]=\frac{i}{2}\left(L^{(4)}_{12}+L^{(4)}_{34}\right)=\frac{\partial}{\partial\alpha}=0 \qquad (16)$$

applied to functions of interest. Eq. (16) is a natural quantum counterpart of SC (9). It includes the second O_3-subalgebra generator of the O_4 algebra. If spherical harmonics of O_4 x) are labelled by eigenvalues l,m,n of the operators $L^{(4)2}$, $\frac{1}{2}(L^{(4)}_{12}-L^{(4)}_{34})$ and $\frac{1}{2}(L^{(4)}_{12}+L^{(4)}_{34})$, then eq. (16) allows only functions with n=0. Therefore functions of interest are representable as

$$f(\vec{x})=\frac{1}{4\pi}\int_0^{4\pi}d\alpha\,\tilde{f}(\xi). \qquad (17)$$

This is integration over the fiber S^1, or integration over the subgroup O_2. It eliminates the second O_3-subalgebra of O_4. Note that this O_4 works differently than O_4 in the famous Fock approach /8,9/. From eq. (15) it follows that

$$L^{(3)2}=\frac{1}{4}L^{(4)2}\qquad\left(L^2=-\frac{1}{2}\left(x_\mu\frac{\partial}{\partial x_\nu}-x_\nu\frac{\partial}{\partial x_\mu}\right)^2\right) \qquad (18)$$

and further that

$$\Delta^{(3)}=\frac{1}{4\tau}\Delta^{(4)}\equiv\frac{1}{4\tau}\frac{\partial}{\partial u_\mu}\frac{\partial}{\partial u_\mu}\equiv\frac{1}{\bar{\xi}\xi}\frac{\partial}{\partial\bar{\xi}_\alpha}\frac{\partial}{\partial\xi_\alpha}, \qquad (19)$$

where $\Delta^{(n)}$ are the Laplace operators $\Delta^{(n)}=\frac{1}{\tau^{n-1}}\frac{\partial}{\partial\tau}\tau^{n-1}\frac{\partial}{\partial\tau}-\frac{1}{\tau^2}L^{(n)2}$ and in $\Delta^{(4)}$ u_μ and $\rho=\sqrt{\tau}=\sqrt{\bar{\xi}\xi}$ serve as x_μ and τ . Let us consider the Green functions

$$G_{\substack{ret\\adv}}(\vec{x},\vec{x}_0,t)=\pm\theta(\pm t)\langle\vec{x}|e^{-i\hbar^{-1}\hat{H}t}|\vec{x}_0\rangle=\frac{1}{2\pi}\int dE\,e^{-i\hbar^{-1}Et}G_{\substack{ret\\adv}}(\vec{x},\vec{x}_0,E) \qquad (20)$$

(hence $D^{(-)}(\vec{x},\vec{x}_0,t)=\langle\vec{x}|e^{-i\hbar^{-1}\hat{H}t}|\vec{x}_0\rangle=G_{ret}-G_{adv}$).

Then the Schrödinger equation

$$\left[-\hbar^2\frac{\Delta^{(3)}}{2m}-\frac{e^2}{\tau}+W(\tau)-E\right]G_{\substack{ret\\adv}}(\vec{x},\vec{x}_0,E)=-i\delta(\vec{x}-\vec{x}_0) \qquad (21)$$

is reduced to the Schrödinger equation in R_4

$$\left[-\hbar^2\frac{\Delta^{(4)}}{8m}-e^2+W(\rho^2)\rho^2-E\rho^2\right]\tilde{G}_{\substack{ret\\adv}}(\xi,\xi_0,e^2)=-i\frac{\pi}{4}\delta^4(u-u_0) \qquad (22)$$

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x) $Y_\ell^m(\theta,\varphi)\,e^{in\frac{\alpha}{2}}$.

upon taking into account

$$\delta(\vec{x}-\vec{x}_0)=\frac{1}{16r}\int_0^{4\pi}d\alpha\,\delta^4(u-u_0)$$

$$\left(d^4u=\frac{1}{16r}d^3x\,d\alpha\right). \qquad (23)$$

In eq. (22) e^2 and E exchange roles: E enters now into an oscillator -type potential (but with E of both signs), and e^2 serves as if a new energy variables.

Hence we obtain the relation

$$G_{\substack{ret\\adv}}(\vec{x},\vec{x}_0,E)=\frac{1}{16\hbar 4\pi}\int_0^{4\pi}d\alpha\int_0^{4\pi}d\alpha_0\int_{-\infty}^{\infty}ds\; e^{i\hbar^{-1}e^2 s}\,\tilde{G}_{\substack{ret\\adv}}(\xi,\xi_0,s) \qquad (24)$$

between the Green functions, where s is a parameter of the type of time, and the Green functions $\tilde{G}_{\substack{ret\\adv}}(\xi,\xi_0,s)$ obey the Schrödinger equation

$$i\hbar\frac{\partial}{\partial s}\tilde{G}_{\substack{ret\\adv}}(\xi,\xi_0,s)=\left[-\hbar^2\frac{\Delta^{(4)}}{8m}+W(\bar\xi\xi)\bar\xi\xi-E\bar\xi\xi\right]\tilde{G}_{\substack{ret\\adv}}(\xi,\xi_0,s)+i\hbar\delta^4(u-u_0)\delta(s). \qquad (25)$$

In the Coulomb case (W=0) and in the free case (e^2=0) $\tilde{G}_{\substack{ret\\adv}}(\xi,\xi_0,s)$ are functions of the type of Green functions for a 4-dimensional oscillator

$$\tilde{G}_{\substack{ret\\adv}}(\xi,\xi_0,s)=\pm\theta(\pm s)\left(\frac{M\omega}{2\pi\hbar i\sin\omega s}\right)^2 e^{i\frac{M\omega}{2\hbar\sin\omega s}\left[(\bar\xi\xi+\bar\xi_0\xi_0)\cos\omega s-(\bar\xi\xi_0+\bar\xi_0\xi)\right]}, \qquad (26)$$

where M=4m, $\omega=\sqrt{\frac{-2E}{M}}$ $\left(\bar\xi\xi=r=u_\mu u_\mu,\ \bar\xi_0\xi_0=r_0=u_{0\mu}u_{0\mu},\ \bar\xi\xi_0+\bar\xi_0\xi=2u_\mu u_{0\mu}\right)$.

Thus obtained $G_{ret}(\vec{x},\vec{x}_0,E)$ is in accord with the expression of ref./6/, which, in turn, is fitted to the Schwinger Coulomb Green function/9/. One can write the expressions for the Coulomb Green functions in terms of the spinor variables as follows

$$G_{\substack{ret\\adv}}(\vec{x},\vec{x}_0,E)=\begin{cases}\pm\dfrac{ikq}{(4\pi)^2}\displaystyle\int_0^{4\pi}d\alpha\int_0^{\infty}\frac{d\eta}{\mathrm{sh}^2\eta}\,e^{\pm\frac{ik}{\mathrm{sh}\eta}\left[(\bar\xi\xi+\bar\xi_0\xi_0)\mathrm{ch}\eta-(\bar\xi\xi_0+\bar\xi_0\xi)\right]}e^{\pm 2i\nu\eta} & \text{for } E>0\\[2ex] -\dfrac{\varkappa q}{(4\pi)^2}\displaystyle\int_0^{4\pi}d\alpha\int_0^{\infty}\frac{d\eta}{\mathrm{sh}^2\eta}\,e^{-\frac{\varkappa}{\mathrm{sh}\eta}\left[(\bar\xi\xi+\bar\xi_0\xi_0)\mathrm{ch}\eta-(\bar\xi\xi_0+\bar\xi_0\xi)\right]}e^{2\nu\eta} & \text{for } E<0,\end{cases}$$

$$(q=2im\hbar^{-2}) \qquad (27)$$

where $k=\hbar^{-1}\sqrt{2mE}$, $\varkappa=\hbar^{-1}\sqrt{-2mE}$, $\nu=\frac{e^2m}{\hbar\sqrt{2m|E|}}$. The last expression of eq. (27) is valid for $\nu<1$. For removing this restriction (in terms of the variable $\rho=e^{-2\eta}$) see ref./9/. For some other expressions of these Green functions see Appendix C.

The relation (24) seems to be important not only for the Coulomb case. According to eq. (24) 3-dimensional space Green functions G for potentials $V=-\frac{e^2}{r}+W(r)$ are expressed via 4-dimensional space Green functions $\tilde{G}$ for potentials $\tilde{V}=W(\rho^2)\rho^2-E\rho^2$, e.g.,

$$V=-\frac{e^2}{r}+gr^n\to\tilde{V}=g\rho^{2n+2}-E\rho^2,\quad V=-\frac{e^2}{r}+\frac{g}{r^2}\to\tilde{V}=\frac{g}{\rho^2}-E\rho^2$$

$$V=-\frac{e^2}{r}+\frac{g}{r^{1+\nu}}\to\tilde{V}=\frac{g}{\rho^{2\nu}}-E\rho^2,\quad V=-\frac{e^2}{r}+\frac{g}{r^\nu}\to\tilde{V}=g\rho^{2-2\nu}-E\rho^2$$

$$V=-\frac{e^2}{r}+\frac{g}{r^{3/2}}\to\tilde{V}=\frac{g}{\rho}-E\rho^2,\quad V=-\frac{e^2}{r}+\frac{g}{\sqrt{r}}\to\tilde{V}=g\rho-E\rho^2$$

so that for $\nu<1$ singularities become weaker.

It is worthwhile to note that this spinor representation of quantum mechanics is of interest also as a model for investigation of some problems of non-linear and gauge theories, like the role of some gauges and treatment of SC's in path integrals.

One can analogously use another Hopf map $S^7\to S^4$.

Appendix A. The completeness relation (4) leads to the (Fierz) identities

$$\pm\sum_{m=1}^{3}(\bar\eta\sigma_m\xi\pm\bar\xi\sigma_m\eta)^2\mp(\bar\eta\xi-\bar\xi\eta)^2=4(\bar\xi\xi)(\bar\eta\eta) \qquad (A.1)$$

$$\sum_{m=1}^{3}(\bar\eta\sigma_m\xi)(\bar\eta\sigma_m\xi)=(\bar\eta\xi)(\bar\eta\xi) \qquad (A.2)$$

$$\sum_{m=1}^{3}(\bar\eta\sigma_m\xi)(\bar\xi\sigma_m\eta)=2(\bar\eta\eta)(\bar\xi\xi)-(\bar\eta\xi)(\bar\xi\eta) \qquad (A.3)$$

$$\sum_{m=1}^{3}(\bar\xi\sigma_m\eta)(\bar\eta\sigma_i\sigma_m\xi)=2(\bar\xi\xi)(\bar\eta\sigma_i\eta)-(\bar\xi\eta)(\bar\eta\sigma_i\xi) \qquad (A.4)$$

$$\sum_{m=1}^{3}(\bar\xi\sigma_m\eta)(\bar\eta\sigma_m\sigma_i\xi)=2(\bar\eta\eta)(\bar\xi\sigma_i\xi)-(\bar\xi\eta)(\bar\eta\sigma_i\xi) \qquad (A.5)$$

$$(\bar\xi\xi)(\bar\eta\sigma_i\eta)+(\bar\eta\eta)(\bar\xi\sigma_i\xi)=(\bar\xi\eta)(\bar\eta\sigma_i\xi)+(\bar\eta\xi)(\bar\xi\sigma_i\eta). \qquad (A.6)$$

The first of them with upper sign yields with $\eta=\dot\xi$ the identity

$$\sum_{m=1}^{3}(\bar\xi\sigma_m\xi)^{\cdot 2}-(\dot{\bar\xi}\xi-\bar\xi\dot\xi)^2=4(\bar\xi\xi)(\dot{\bar\xi}\dot\xi), \qquad (A.7.a)$$

or

$$\dot{\vec{x}}\dot{\vec{x}} + r^2(\dot{\alpha}-\cos\theta\,\dot{\varphi})^2 = 4r\,\dot{u}_\mu\dot{u}_\mu\,, \qquad (A.7.b)$$

which is used for the transformation of the Lagrangian (6), and with substitutions $\xi \to \xi_n+\xi_{n-1}$, $\eta \to \xi_n-\xi_{n-1}$ it yields finite-difference generalization of eqs. (A.7)

$$\sum_{m=1}^{3}(\bar{\xi}_n\sigma_m\xi_n-\bar{\xi}_{n-1}\sigma_m\xi_{n-1})^2-(\bar{\xi}_n\xi_{n-1}-\bar{\xi}_{n-1}\xi_n)^2=$$
$$=((\bar{\xi}_n+\bar{\xi}_{n-1})(\xi_n+\xi_{n-1}))((\bar{\xi}_n-\bar{\xi}_{n-1})(\xi_n-\xi_{n-1}))=(u_n+u_{n-1})^2(u_n-u_{n-1})^2,$$

$$(\vec{x}_n-\vec{x}_{n-1})^2+4r_nr_{n-1}\Big[\sin\frac{\alpha_n-\alpha_{n-1}}{2}\cos\frac{\theta_n-\theta_{n-1}}{2}\cos\frac{\varphi_n-\varphi_{n-1}}{2}- \qquad (A.8.a)$$
$$-\cos\frac{\alpha_n-\alpha_{n-1}}{2}\cos\frac{\theta_n+\theta_{n-1}}{2}\sin\frac{\varphi_n-\varphi_{n-1}}{2}\Big]^2=(u_n+u_{n-1})^2(u_n-u_{n-1})^2. \qquad (A.8.b)$$

Note that both eqs. (A.1) are merely a "four-quadrate identity".

Equation (A.6) follows from eqs. (A.4) and (A.5) and it is used for derivation of the Newton equations.

In the Majorana representation we oan choose the 4x4 γ-matrioes as follows

$$\gamma_1=\begin{pmatrix}0&\sigma_1\\ \sigma_1&0\end{pmatrix},\ \gamma_2=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\ \gamma_3=\begin{pmatrix}0&\sigma_3\\ \sigma_3&0\end{pmatrix},\ \gamma_4=\begin{pmatrix}0&\sigma_2\\ \sigma_2&0\end{pmatrix},\ \gamma_5=\begin{pmatrix}0&i\\-i&0\end{pmatrix},$$
$$\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu=2\delta_{\mu\nu}\quad(\mu,\nu=1,2,3,4,5), \qquad (A.9)$$

the entries being the 2x2 matrices

$$\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\ \sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\ \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\ 1=\begin{pmatrix}1&0\\0&1\end{pmatrix},\ i=i\begin{pmatrix}1&0\\0&1\end{pmatrix}. \qquad (A.10)$$

These γ_1, γ_2, γ_3 are pure real matrices, and γ_4 and γ_5 pure imaginary. Because ψ is a pure real four-component column, $\bar{\psi}\equiv\psi^T\gamma_4$ is a pure imaginary row (T means transposed). Now the four-quadrate identity oan be written in the following forms

$$-\sum_{m=1}^{3}(\bar{\psi}'\gamma_m\psi)^2-(\bar{\psi}'\gamma_4\gamma_5\psi)^2=(\bar{\psi}'\gamma_4\psi')(\bar{\psi}\gamma_4\psi), \qquad (A.11)$$

$$\sum_{m=1}^{3}(\bar{\psi}'\gamma_4\gamma_m\psi)^2+(\bar{\psi}'\gamma_5\psi)^2=(\bar{\psi}'\gamma_4\psi')(\bar{\psi}\gamma_4\psi), \qquad (A.12)$$

$$\frac{1}{2}\sum_{m=1}^{3}(\bar{\psi}'\sigma_{mn}\psi)^2-(\bar{\psi}'\psi)^2=(\bar{\psi}'\gamma_4\psi')(\bar{\psi}\gamma_4\psi), \qquad (A.13)$$

$$\sum_{m=1}^{3}(\bar{\psi}'\gamma_m\gamma_5\psi)^2+(\bar{\psi}'\gamma_4\psi)^2=(\bar{\psi}'\gamma_4\psi')(\bar{\psi}\gamma_4\psi). \qquad (A.14)$$

Since

$$\bar{\psi}'\gamma_\mu\psi=\frac{1}{2}(\bar{\psi}'\gamma_\mu\psi+\bar{\psi}\gamma_\mu\psi'),\quad \bar{\psi}'\gamma_\mu\gamma_5\psi=\frac{1}{2}(\bar{\psi}'\gamma_\mu\gamma_5\psi-\bar{\psi}\gamma_\mu\gamma_5\psi'),$$
$$\bar{\psi}'\sigma_{\mu\nu}\psi=\frac{1}{2}(\bar{\psi}'\sigma_{\mu\nu}\psi+\bar{\psi}\sigma_{\mu\nu}\psi'),\quad \bar{\psi}'\gamma_5\psi=\frac{1}{2}(\bar{\psi}'\gamma_5\psi-\bar{\psi}\gamma_5\psi') \qquad (A.15)$$

due the symmetry of γ_m (m = 1,2,3) and antisymmetry of γ_4 and γ_5, it is clear how to obtain identities (A.7) and (A.8) in terms of the Majorana spinors.

However, it is not easy to obtain identities (A.11)-(A.14) from the completeness relation of the γ-matrices

$$\sum_{A=1}^{16}(\gamma_A)_{\alpha\beta}(\gamma_A)_{\gamma\delta}=4\delta_{\alpha\delta}\delta_{\gamma\beta}$$
$$(\gamma_A:\ 1,\ \gamma_\mu,\ \sigma_{\mu\nu}=-i(\gamma_\mu\gamma_\nu-\delta_{\mu\nu}),\ i\gamma_\mu\gamma_5,\ \gamma_5). \qquad (A.16)$$

Now the subsidiary condition (9) can be written in the forms

$$\bar{\psi}'\gamma_4\gamma_5\psi=0\ (a),\ \text{or}\ \ \bar{\psi}'\gamma_5\psi=0\ (b),\ \ \text{or}\ \ \bar{\psi}'\psi=0\ (c) \qquad (A.17)$$

Note that from eqs. (A.11)-(A.15) there follow

$$\sum_{\mu=1}^{4}(\bar{\psi}'\gamma_\mu\psi)^2-\sum_{\mu=1}^{4}(\bar{\psi}'i\gamma_\mu\gamma_5\psi)^2=0, \qquad (A.18)$$

$$\frac{1}{2}\sum_{\mu,\nu=1}^{4}(\bar{\psi}'\sigma_{\mu\nu}\psi)^2-(\bar{\psi}'\psi)^2-(\bar{\psi}'\gamma_5\psi)^2=0, \qquad (A.19)$$

$$\bar{\psi}\psi=\bar{\psi}\gamma_5\psi=\bar{\psi}\gamma_\mu\gamma_5\psi=0, \qquad (A.20)$$

$$\sum_{\mu=1}^{4}(\bar{\psi}\gamma_\mu\psi)^2=0,\qquad \sum_{\mu,\nu=1}^{4}(\bar{\psi}\sigma_{\mu\nu}\psi)^2=0. \qquad (A.21)$$

<u>Appendix B.</u> Canonical quantization means the commutation relations (CR's)

$$[\xi_\alpha,\xi_\beta]=[\xi_\alpha,\bar{\xi}_\beta]=[\xi_\alpha,\pi_\beta]=[\bar{\xi}_\alpha,\bar{\pi}_\beta]=[\pi_\alpha,\pi_\beta]=[\bar{\pi}_\alpha,\bar{\pi}_\beta]=[\pi_\alpha,\bar{\pi}_\beta]=0,$$
$$[\xi_\alpha,\bar{\pi}_\beta]=i\hbar\delta_{\alpha\beta},\quad [\bar{\xi}_\alpha,\pi_\beta]=i\hbar\delta_{\alpha\beta}, \qquad (B.1)$$

where

$$\pi_\alpha=m((\bar{\xi}\xi)\dot{\xi}_\alpha+\dot{\xi}_\alpha(\bar{\xi}\xi)),\quad \bar{\pi}_\alpha=m((\bar{\xi}\xi)\dot{\bar{\xi}}_\alpha+\dot{\bar{\xi}}_\alpha(\bar{\xi}\xi)) \qquad (B.2)$$

are canonically conjugate momenta. We reduce these CR's to eqs. (12), supposing that unknown commutators $[\xi_\alpha,\dot{\xi}_\beta]$, ... $[\dot{\xi}_\alpha,\dot{\bar{\xi}}_\beta]$ commute with $\bar\xi\xi$, and the right hand sides of eqs. (12) turn out to be in accord with this assumption.

One may expect that CR's (12) lead to the usual commutation relations

$$[x_m,x_n]=0,\qquad [x_m,p_n]=i\hbar\delta_{mn},\qquad [p_m,p_n]=0. \tag{B.3}$$

The first two of them are easily obtained from eqs. (12). However, instead of the latter we find (an anomaly)

$$[p_m,p_n]=[m(\bar\xi\sigma_m\xi)^\cdot,\ m(\bar\xi\sigma_m\xi)^\cdot]=$$

$$=\frac{i\hbar}{2(\bar\xi\xi)^3}\Big(m(\bar\xi\xi)(\dot{\bar\xi}\xi-\bar\xi\dot\xi)+i\hbar\Big)\Big((\bar\xi\sigma_m\sigma_n\xi)-(\bar\xi\sigma_n\sigma_m\xi)\Big)=$$

$$=-\frac{\hbar}{(\bar\xi\xi)^3}\Big(m(\bar\xi\xi)(\dot{\bar\xi}\xi-\bar\xi\dot\xi)+i\hbar\Big)\varepsilon_{mnl}\,\bar\xi\sigma_l\xi \tag{B.4}$$

(the latter line is due to the relation $\sigma_m\sigma_n=\delta_{mn}+i\varepsilon_{mnl}\sigma_l$). Possibly, a more correct SC is

$$\Big(m(\bar\xi\xi)(\dot{\bar\xi}\xi-\bar\xi\dot\xi)+i\hbar\Big)|\ \rangle=0 \tag{B.5}$$

rather than eq. (13). Then CR's (B.3) are effectively fulfilled in a physical subspace.

One can easily check that the operator in SC commutes with $\vec x=\bar\xi\vec\sigma\xi$, $r=\bar\xi\xi$. Then its commutation with $\dot x_m=(\bar\xi\sigma_m\xi)^\cdot$ can be checked as follows

$$[(\bar\xi\xi)(\dot{\bar\xi}\xi-\bar\xi\dot\xi),\dot{\vec x}]=[(\bar\xi\xi)(\dot{\bar\xi}\xi-\bar\xi\dot\xi),\vec x]^\cdot-[((\bar\xi\xi)(\dot{\bar\xi}\xi-\bar\xi\dot\xi))^\cdot,\vec x]=0, \tag{B.6}$$

where in the r.h.s. the second term vanishes due to conservation of $(\bar\xi\xi)(\dot{\bar\xi}\xi-\bar\xi\dot\xi)$.

In derivation of eq. (B.5) we make the use of the identities

$$(\bar\xi\xi)(\dot{\bar\xi}\sigma_m\sigma_n\xi)=(\dot{\bar\xi}\xi)(\bar\xi\sigma_m\sigma_n\xi)+(\dot{\bar\xi}\sigma_m\xi)(\bar\xi\sigma_n\xi)-(\bar\xi\sigma_m\xi)(\dot{\bar\xi}\sigma_n\xi)+\frac{i\hbar}{m(\bar\xi\xi)}(\bar\xi\sigma_m\sigma_n\xi), \tag{B.7}$$

$$(\bar\xi\xi)(\bar\xi\sigma_m\sigma_n\dot\xi)=(\bar\xi\dot\xi)(\bar\xi\sigma_m\sigma_n\xi)+(\bar\xi\sigma_n\dot\xi)(\bar\xi\sigma_m\xi)-(\bar\xi\sigma_n\xi)(\bar\xi\sigma_m\dot\xi)+\frac{i\hbar}{m}\delta_{mn}, \tag{B.8}$$

which follow from the Fierz identities

$$\sum_{\mu=0}^{3}(\bar\xi\sigma_m\sigma_\mu\xi)(\dot{\bar\xi}\sigma_\mu\sigma_n\xi)=2(\dot{\bar\xi}\xi)(\bar\xi\sigma_m\sigma_n\xi)+\frac{2i\hbar}{m(\bar\xi\xi)}(\bar\xi\sigma_m\sigma_n\xi), \tag{B.9}$$

$$\sum_{\mu=0}^{3}(\bar\xi\sigma_\mu\sigma_m\xi)(\dot{\bar\xi}\sigma_\mu\sigma_n\xi)=2(\dot{\bar\xi}\sigma_m\xi)(\bar\xi\sigma_n\xi), \tag{B.10}$$

$$\sum_{\mu=0}^{3}(\bar\xi\sigma_\mu\sigma_m\dot\xi)(\bar\xi\sigma_n\sigma_\mu\xi)=2(\bar\xi\xi)(\bar\xi\sigma_n\sigma_m\dot\xi)-\frac{2i\hbar}{m}\delta_{mn}, \tag{B.11}$$

$$\sum_{\mu=0}^{3}(\bar\xi\sigma_m\sigma_\mu\dot\xi)(\bar\xi\sigma_n\sigma_\mu\xi)=2(\bar\xi\sigma_m\xi)(\bar\xi\sigma_n\dot\xi) \tag{B.12}$$

by summing them pairwise. An operator nature of ξ is now taken into account.

<u>Appendix C.</u> The Coulomb Green functions can be represented as Hankel transformations

$$G_{\substack{ret\\adv}}(\vec x,\vec x_0,E)=\frac{2im}{\hbar^2}$$

$$\begin{cases}\pm\dfrac{i}{4\pi}\displaystyle\int_0^\infty\frac{w\,dw}{\sqrt{w^2+k^2}}J_0\big(w\sqrt{2(rr_0+\vec x\vec x_0)}\big)\,e^{\pm i(r+r_0)\sqrt{w^2+k^2}}\rho^{\mp i\nu} & \text{for } E>0\\[2ex] -\dfrac{1}{4\pi}\displaystyle\int_0^\infty\frac{w\,dw}{\sqrt{w^2+\varkappa^2}}I_0\big(w\sqrt{2(rr_0+\vec x\vec x_0)}\big)\,e^{-(r+r_0)\sqrt{w^2+k^2}}\rho^{-\nu} & \text{for } E<0,\end{cases} \tag{C.1}$$

where $k=\hbar^{-1}\sqrt{2mE}$, $\varkappa=\hbar^{-1}\sqrt{-2mE}$, and $a=r+r_0$ and $b=\sqrt{2(rr_0+\vec x\vec x_0)}$ are constituents of $(\vec x-\vec x_0)^2=a^2-b^2$;

$$\rho=\frac{\sqrt{w^2+k^2}-k}{\sqrt{w^2+k^2}+k}$$, and with $k\to\varkappa$ for $E<0$, and $\nu=\dfrac{e^2m}{\hbar\sqrt{2m|E|}}$.

In the free case ($\nu=0$)

$$G^0_{\substack{ret\\adv}}(\vec x,\vec x_0,E)=\frac{2im}{\hbar^2}\begin{cases}-\dfrac{e^{\pm ik|\vec x-\vec x_0|}}{4\pi|\vec x-\vec x_0|} & \text{for } E>0\\[2ex] -\dfrac{e^{-\varkappa|\vec x-\vec x_0|}}{4\pi|\vec x-\vec x_0|} & \text{for } E<0\end{cases}. \tag{C.2}$$

The representation (C.1) of $G^0_{ret}(\vec x,\vec x_0,E)$ for $E>0$ is well-known, but for $E<0$ another representation is usually used. Let us put for the Bessel functions

$$J_0(\sqrt{c^2+d^2})=\frac{1}{4\pi}\int_0^{4\pi}d\alpha\, e^{ic\cos\frac{\alpha-\alpha_0}{2}+id\sin\frac{\alpha-\alpha_0}{2}}=\frac{1}{4\pi}\int_0^{4\pi}d\alpha\, e^{-iw(\bar\xi\xi_0+\bar\xi_0\xi)}$$

$$I_0(\sqrt{c^2+d^2})=\frac{1}{4\pi}\int_0^{4\pi}d\alpha\, e^{-c\cos\frac{\alpha-\alpha_0}{2}-d\sin\frac{\alpha-\alpha_0}{2}}=\frac{1}{4\pi}\int_0^{4\pi}d\alpha\, e^{-w(\bar\xi\xi_0+\bar\xi_0\xi)}, \tag{C.3}$$

where $c^2+d^2=w^2 2\,(rr_0+\vec{x}\vec{x}_0)$, and one can find c and d separately, if one writes $\bar{\xi}\xi_0+\bar{\xi}_0\xi$ in terms of r,θ,φ,α and $r_0,\theta_0,\varphi_0,\alpha_0$. Now

$$
G_{\substack{\mathrm{ret}\\ \mathrm{adv}}}(\vec{x},\vec{x}_0,E)=\frac{2im}{\hbar^2}\cdot\begin{cases}\pm\dfrac{i}{(4\pi)^2}\displaystyle\int_0^{4\pi}d\alpha\int_0^{\infty}\frac{w\,dw}{\sqrt{w^2+k^2}}\,e^{\pm i[(\bar{\xi}\xi+\bar{\xi}_0\xi_0)\sqrt{w^2+\varkappa^2}-w(\bar{\xi}\xi_0+\bar{\xi}_0\xi)]}\,\rho^{\mp i\nu}\\[2ex] -\dfrac{1}{(4\pi)^2}\displaystyle\int_0^{4\pi}d\alpha\int_0^{\infty}\frac{w\,dw}{\sqrt{w^2+\varkappa^2}}\,e^{-[(\bar{\xi}\xi+\bar{\xi}_0\xi_0)\sqrt{w^2+k^2}-w(\bar{\xi}\xi_0+\bar{\xi}_0\xi)]}\,\rho^{-\nu}\end{cases}\tag{C.4}
$$

Hence, after the change $w=k/\mathrm{sh}\,\eta$ or $\varkappa/\mathrm{sh}\,\eta$ there follow eqs. (27).

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Квантовая механика и расслоения Хопфа

Исследуется квантовая механика в спинорном представлении /как CP^1-модель/. В шредингеровской картине показано, что функции Грина в трехмерном пространстве для того или иного потенциала выражаются через функции Грина в четырехмерном пространстве для другого потенциала. Кратко обсуждены также подходы, основанные на интеграле по путям и гейзенберговской картине.

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Quantum Mechanics and Hopf Fibre Bundles

Quantum mechanics is treated in a spinor representation (like CP^1-model). It is shown in the Schrödinger picture that 3-dimensional Green functions for some potential are expressed via 4-dimensional Green functions for other potential. Path-integral and Heisenberg picture approaches are also shortly discussed.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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