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BÄCKLUND TRANSFORMATION FOR THE LIOUVILLE EQUATION AND GAUGE CONDITIONS IN THE RELATIVISTIC STRING THEORY

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1. INTRODUCTION

Recently new formulations of the relativistic string model have been proposed[1-7]. In papers[2-5] the geometric approach to the relativistic string theory was developed by describing string dynamics in terms of the differential forms defined on its world sheet. This approach is based on the consideration of the moving frame on the world sheet of the string. Changing this frame in motion of its origin along the string world sheet is given in terms of the differential forms by linear equations. The integrability conditions of these equations are considered as dynamical equations in this approach. In differential geometry these conditions are known as the embedding equations of Gauss, Peterson-Codazzi and Ricci. If on the world sheet of the string the conformally-flat coordinate system is chosen, then, as is shown in papers[3-5], the embedding equations for the string world sheet are reduced to one nonlinear Liouville equation. In the three-dimensional space-time this equation defines a real function and in the four-dimensional Minkowski space a complex-valued function*.

In describing the string world surface in terms of the differential forms there is the gauge freedom connected with the SO(1,1)xSO(n-2) rotations of the moving frame at each point of the world surface, in the tangent plane the SO(1,1)-group acting; and in the normal space, the SO(n-2)-group, where n is the dimension of space-time in which the string is moving. In the Kamimura paper[7] this freedom has been used for imposing special gauge conditions on the differential forms of the world sheet of the string. As a result these forms are defined in[7] by the solution of the D'Alembert equation instead of the solution of the nonlinear Liouville equation.

In this note we show that the Kamimura gauge is a direct consequence of the Backlund transformation relating the solution of the Liouville equation with that of the D'Alembert equation. The rotation angle of the moving frame on the world sheet of the string which enables transition to Kamimura's gauge is defined by the solution of the D'Alembert equation, and the Liouville equation describes the metric on the string world sheet in the usual orthogonal gauge.

Another interpretation of the Kamimura gauge is possible also. It will be shown here that this gauge can be treated as a consequence of the conformal invariance in the relativistic string theory moving in three-dimensional space-time.

The paper is arranged as follows. Section 2 is devoted to the formulation of the relativistic string model using the method of the moving frame in the surface theory. In section 3 the Kamimura gauge is obtained as a consequence of the Bäcklund transformation for the Liouville and D'Alembert equations. In section 4 the same gauge is considered as a consequence of the conformal invariance in the string theory. Section 5 is devoted to the purely geometric derivation of the Bäcklund transformation relating solutions of the Liouville and D'Alembert equations.

2. MOVING FRAME IN THE RELATIVISTIC STRING MODEL

Let $x^\mu(u^1,u^2)$, $\mu = 0,1,\dots,n-1$ be the parametric representation of the world sheet of the string moving in an n-dimensional pseudo-Euclidean space-time with the metric signature $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,\dots)$. The string coordinates obey the following equations[11,12]

$$x^\mu_{,11} - x^\mu_{,22} = 0, \tag{2.1}$$

$$g_{11} = (x^\mu_{,1})^2 = -g_{22} = -(x^\mu_{,2})^2 = a^2(u^1,u^2),\quad g_{12} = g_{21} = x^\mu_{,1}\, x_{\mu,2} = 0, \tag{2.2}$$

where $g_{ij}(u^1,u^2)$, $i,j = 1,2$ is the induced metric on the string world sheet and the index with comma denotes the partial derivative with respect to the corresponding parameter u^1 or u^2.

At any point of the world surface of the string we construct the moving orthonormal frame using two tangent vectors e^μ_1 and e^μ_2 and $(n-2)$ unit normals e^μ_a, $a = 3,4,\dots,n$

$$e^\mu_a e_{\mu b} = \eta_{ab}, \quad a,b = 1,\dots,n, \qquad \eta_{ab} = \mathrm{diag}(1,-1,-1,\dots). \tag{2.3}$$

Alteration of the basis $\{e^\mu_a\}$ by the motion of its origin $x^\mu(u^1,u^2)$ along the surface is given by the following equations[13]

$$dx^\mu = \omega^i e^\mu_i, \tag{2.4}$$

$$de^\mu_a = \Omega_{a\cdot}^{\cdot b}\, e^\mu_b, \tag{2.5}$$

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* The embedding equations for the string world sheet in 5- and 6-dimensional pseudo-Euclidean spaces were obtained in papers[8-10]. Ibid their general solutions were constructed.

$$\Omega_{ab} + \Omega_{ba} = 0, \qquad i,j,k,\ldots = 1,2, \quad a,b,c,\ldots = 1,\ldots,n, \tag{2.6}$$

where ω^i and $\Omega_{a\cdot}^{\cdot b}$ are linear differential forms[14] $\omega^i = \omega^i_j du^j$, $\Omega_{a\cdot}^{\cdot b} = \Omega_{a\cdot|j}^{\cdot b} du^j$. The integrability conditions for eqs. (2.4) and (2.5)

$$d^2 x^\mu = d \wedge dx^\mu = 0, \qquad d^2 e^\mu_a = 0 \tag{2.7}$$

give rise to the basic equations in the surface theory

$$\omega^i \wedge \Omega_{i\cdot}^{\cdot\alpha} = 0, \tag{2.8}$$

$$d\omega^i = \omega^j \wedge \Omega_{j\cdot}^{\cdot i}, \tag{2.9}$$

$$d\Omega_{a\cdot}^{\cdot b} = \Omega_{a\cdot}^{\cdot c} \wedge \Omega_{c\cdot}^{\cdot b}, \tag{2.10}$$

$$i,j = 1,2, \quad a,b,c = 1,\ldots,n, \quad \alpha = 3,4,\ldots,n.$$

The solutions of eqs. (2.8)-(2.10) ω^i and $\Omega_{a\cdot}^{\cdot b}$ determine the surface $x^\mu(u^1,u^2)$ up to its motion as the whole in space.

Eqs. (2.8)-(2.10) are the well-known embedding equations of Gauss, Peterson-Codazzi and Ricci in the classical surface theory[15]. They are written here in terms of linear differential forms of the surface.

In addition to the linear forms ω^i and $\Omega_{a\cdot}^{\cdot b}$ in the surface theory the quadratic differential forms g_{ij}, $b_{\alpha|ij}$ and the torsion vectors $\nu_{\alpha\beta|i}$ are used also. The latter forms define the motion along the surface of the basis $\{x^\mu_{,1}, x^\mu_{,2}, e^\mu_\alpha\}$ according to the formulas

$$\nabla_i x^\mu_{,j} = \sum_{\alpha=3}^{n} \epsilon_\alpha b_{\alpha|ij} e^\mu_\alpha, \tag{2.11}$$

$$e^\mu_{\alpha,i} = -b_{\alpha|ij} g^{jk} x^\mu_{,k} + \sum_\beta \epsilon_\beta \nu_{\beta\alpha|i} e^\mu_\beta, \tag{2.12}$$

$$i,j,k,\ldots = 1,2, \quad \alpha,\beta,\gamma,\ldots = 3,\ldots,n,$$

$$\eta_{ab} = \mathrm{diag}\{\epsilon_1, \epsilon_2, \ldots, \epsilon_n\}.$$

Here ∇_i denotes the covariant differentiation with respect to the metric $g_{ij} = x^\mu_{,i} x_{\mu,j}$.

The quadratic and linear differential forms of the surface are connected in the following way

$$g_{ij} = \sum_{k,\ell=1}^{2} \eta_{k\ell}\, \omega^k_i\, \omega^\ell_j,$$

$$b_{\alpha|ij} = -\sum_{k,\ell=1}^{2} \eta_{k\ell}\, \omega^k_i\, \Omega_{\alpha\cdot|j}^{\cdot\ell}, \tag{2.13}$$

$$\nu_{\gamma\alpha|i} = \sum_{\beta=3}^{n} \eta_{\gamma\beta}\, \Omega_{\alpha\cdot|i}^{\cdot\beta},$$

$$i,j,k,\ldots = 1,2, \quad \alpha,\beta,\gamma,\ldots = 3,\ldots,n.$$

The condition that the world sheet of the string be a minimal surface (eqs. (2.1) and (2.2)) is expressed in terms of the quadratic forms as follows

$$b_{\alpha|ij}\, g^{ij} = 0, \qquad \alpha = 3,\ldots,n. \tag{2.14}$$

The co-moving frame e^μ_a, $a = 1,\ldots,n$ at each point of the surface is defined nonuniquely. We can go from the basis $\{e^\mu_a\}$ to a new one $\{\bar{e}^\mu_a\}$

$$e^\mu_a = g_{a\cdot}^{\cdot b}\, \bar{e}^\mu_b \tag{2.15}$$

with the matrix g from the SO(1,1)xSO(n-2) group. Under this transformation the tangent space and the normal space at each point of the world sheet of the string are not mixed. The embedding equations (2.8)-(2.10) determining the surface by differential forms are covariant under this transformation of the co-moving frame (2.15).

3. THE ROTATION OF THE CO-MOVING FRAME ON THE WORLD SHEET OF THE STRING

First we shall consider the relativistic string theory in the three-dimensional space-time where it is described by one nonlinear Liouville equation for a real function[3-5]. Taking into account eqs. (2.6), (2.8), (2.9) and the minimality condition of the world sheet of the string (2.14) we can represent the matrices $\Omega^{\cdot b}_{a\cdot}$ in the following form

$$\Omega^{\cdot b}_{a\cdot|1}=\begin{vmatrix} 0 & \frac{a_{,2}}{a} & -\frac{b_{11}}{a} \\ \frac{a_{,2}}{a} & 0 & -\frac{b_{12}}{a} \\ -\frac{b_{11}}{a} & \frac{b_{12}}{a} & 0 \end{vmatrix}, \quad \Omega^{\cdot b}_{a\cdot|2}=\begin{vmatrix} 0 & \frac{a_{,1}}{a} & -\frac{b_{12}}{a} \\ \frac{a_{,1}}{a} & 0 & -\frac{b_{11}}{a} \\ -\frac{b_{12}}{a} & \frac{b_{11}}{a} & 0 \end{vmatrix}. \qquad (3.1)$$

The integrability condition (2.10) is written obviously as

$$\Omega_{1,2} - \Omega_{2,1} + [\Omega_1, \Omega_2] = 0. \qquad (3.2)$$

Putting $a^2 = \exp(-\phi)$ and substituting (3.1) into (3.2) we obtain

$$\phi_{,11} - \phi_{,22} = 2q_+(u^+)\, q_-(u^-)\, e^{\phi}, \qquad (3.3)$$

$$b_{11,1} - b_{12,2} = 0, \qquad (3.4)$$

$$b_{11,2} - b_{12,1} = 0, \qquad (3.5)$$

$$b_{11} + b_{12} = q_+(u^+), \quad b_{11} - b_{12} = q_-(u^-), \qquad u^{\pm} = u^1 \pm u^2. \qquad (3.6)$$

Without loss of generality one can choose the functions $q_{\pm}(u^{\pm})$ as constants[5]

$$q_+(u^+) = q_-(u^-) = q. \qquad (3.7)$$

Now we use the gauge freedom in the theory and introduce instead of the basis $\{e^{\mu}_{a}\}$ a new one $\{\bar{e}^{\mu}_{a}\}$

$$e_a = g^{\cdot b}_{a\cdot}\, \bar{e}_b \qquad (3.8)$$

with the matrix

$$g[\lambda(u^1,u^2)] = \begin{vmatrix} \operatorname{ch}\lambda & \operatorname{sh}\lambda & 0 \\ \operatorname{sh}\lambda & \operatorname{ch}\lambda & 0 \\ 0 & 0 & 1 \end{vmatrix}. \qquad (3.9)$$

The differential forms $\Omega^{\cdot b}_{a\cdot}$ are transformed as gauge fields

$$\bar{\Omega}_i = g^{-1}\Omega_i g - g^{-1}\partial_i g, \quad i = 1,2. \qquad (3.10)$$

Let us write explicitly the matrices $\bar{\Omega}_i$:

$$\bar{\Omega}^{\cdot b}_{a\cdot|1} = \begin{vmatrix} 0 & -\frac{\phi_{,2}}{2} - \lambda_{,1} & -q\exp(\frac{\phi}{2})\operatorname{ch}\lambda \\ -\frac{\phi_{,2}}{2} - \lambda_{,1} & 0 & q\exp(\frac{\phi}{2})\operatorname{sh}\lambda \\ -q\exp(\frac{\phi}{2})\operatorname{ch}\lambda & -q\exp(\frac{\phi}{2})\operatorname{sh}\lambda & 0 \end{vmatrix},$$

$$\bar{\Omega}^{\cdot b}_{a\cdot|2} = \begin{vmatrix} 0 & -\frac{\phi_{,1}}{2} - \lambda_{,2} & q\exp(\frac{\phi}{2})\operatorname{sh}\lambda \\ -\frac{\phi_{,1}}{2} - \lambda_{,2} & 0 & -q\exp(\frac{\phi}{2})\operatorname{ch}\lambda \\ q\exp(\frac{\phi}{2})\operatorname{sh}\lambda & q\exp(\frac{\phi}{2})\operatorname{ch}\lambda & 0 \end{vmatrix}, \qquad (3.11)$$

Now we take into account the Bäcklund transformation that connects the solutions of the Liouville and D'Alembert equations. It has the form[16,17]

$$\frac{\phi_{,1}}{2} + \lambda_{,2} = -|q|\exp(\frac{\phi}{2})\operatorname{ch}\lambda, \quad \frac{\phi_{,2}}{2} + \lambda_{,1} = |q|\exp(\frac{\phi}{2})\cdot\operatorname{sh}\lambda. \qquad (3.12)$$

Here the function $\phi(u^1,u^2)$ obeys the Liouville equation

$$\phi_{,11} - \phi_{,22} = 2q^2\cdot e^{\phi}; \qquad (3.13)$$

while the function $\lambda(u^1,u^2)$, the D'Alembert equation

$$\lambda_{,11} - \lambda_{,22} = 0. \qquad (3.14)$$

If

$$\exp\phi = \frac{4}{q^2}\cdot\frac{f'_+(u^+)\,f'_-(u^-)}{[f_+(u^+) + f_-(u^-)]^2} \qquad (3.15)$$

is the general solution of the Liouville equation (3.13), then the solution of the D'Alembert equation (3.14) entering into the

Bäcklund transformation (3.12) is expressed also in terms of the functions $f_+(u^+)$ and $f_-(u^-)$

$$\lambda(u^1,u^2) = -\frac{1}{2}\ln f'_+(u^+) + \frac{1}{2}\ln f'_-(u^-). \tag{3.16}$$

Now we take in matrices (3.11) as the transformation parameter $\lambda(u^1,u^2)$ the solution (3.16). Then by virtue of (3.12) it follows that the transformed matrices $\bar\Omega^{\cdot b}_{a\cdot}$ obey the conditions

$$\bar\Omega^{\cdot 1}_{0\cdot|i} = -\,\mathrm{sign}\,q\cdot\bar\Omega^{\cdot 2}_{1\cdot|i}\,, \quad i=1,2, \tag{3.17}$$

where sign q is

$$\mathrm{sign}\,q = \begin{cases} +1, & q>0,\\ -1, & q<0. \end{cases} \tag{3.18}$$

It is easy to verify by eqs. (3.1) and (3.2) that in terms of the transformed differential forms the theory of the relativistic string in three-dimensional space-time is defined by the D'Alembert equation[7]

$$\bar a_{,11} - \bar a_{,22} = 0. \tag{3.19}$$

Equations (3.13) with $q<0$ are the Kamimura gauge conditions in the relativistic string theory moving in the three-dimensional space-time[7].

Let us go to the four-dimensional Minkowski space-time. As is shown in paper[5] the string theory in this case is reduced to the nonlinear Liouville equation for the complex-valued function. It is convenient to direct unit normals at each point of the string world sheet along the vectors $\nabla_1 x^\mu_{,1}$ and $\nabla_1 x^\mu_{,2}$[8,9]. As a result, the matrices Ω describing the co-moving frame e^μ_a, $a=1,\dots,4$ on the string world sheet take the form

$$\Omega^{\cdot b}_{a\cdot|1} = \begin{vmatrix} 0 & -\frac{\phi_{,2}}{2} & -q\exp\frac{\phi}{2}\cos\frac{\theta}{2} & 0 \\ -\frac{\phi_{,2}}{2} & 0 & 0 & -q\exp\frac{\phi}{2}\cdot\sin\frac{\theta}{2} \\ -q\exp\frac{\phi}{2}\cos\frac{\theta}{2} & 0 & 0 & \frac{\theta_{,2}}{2} \\ 0 & q\exp\frac{\phi}{2}\sin\frac{\theta}{2} & -\frac{\theta_{,2}}{2} & 0 \end{vmatrix}$$

$$\Omega^{\cdot b}_{a\cdot|2} = \begin{vmatrix} 0 & -\frac{\phi_{,1}}{2} & 0 & -q\exp\frac{\phi}{2}\sin\frac{\theta}{2} \\ -\frac{\phi_{,1}}{2} & 0 & -q\exp\frac{\phi}{2}\cdot\cos\frac{\theta}{2} & 0 \\ 0 & q\exp\frac{\phi}{2}\cos\frac{\theta}{2} & 0 & \frac{\theta_{,1}}{2} \\ -q\exp\frac{\phi}{2}\cdot\sin\frac{\theta}{2} & 0 & -\frac{\theta_{,1}}{2} & 0 \end{vmatrix}. \tag{3.20}$$

The compatibility conditions (3.2) with the matrices Ω given by (3.20) reduce to the Liouville equation for the complex-valued function $w=\phi+i\theta$[5]

$$w_{,11} - w_{,22} = 2q^2e^w. \tag{3.21}$$

The gauge freedom in the theory enables the transition from the co-moving frame e^μ_a, $a=1,\dots,4$ to a new basis $\bar e^\mu_a$, $a=1,\dots,4$ with the matrix from the SO(1,1)xSO(2)-group

$$g[\lambda(u^1,u^2),\psi(u^1,u^2)] = \begin{vmatrix} \mathrm{ch}\,\lambda & \mathrm{sh}\,\lambda & 0 & 0 \\ \mathrm{sh}\,\lambda & \mathrm{ch}\,\lambda & 0 & 0 \\ 0 & 0 & \cos\psi & -\sin\psi \\ 0 & 0 & \sin\psi & \cos\psi \end{vmatrix}. \tag{3.22}$$

We write now explicitly the matrix elements $\bar\Omega^{\cdot b}_{a\cdot}$, obtained by eq. (3.10)

$$\begin{aligned} &\bar\Omega^{\cdot 2}_{1\cdot|1} + i\bar\Omega^{\cdot 3}_{0\cdot|1} = q\exp[\tfrac{1}{2}(\phi+i\theta)]\cdot\mathrm{sh}(\lambda+i\psi),\\ &\bar\Omega^{\cdot 1}_{0\cdot|1} - i\bar\Omega^{\cdot 3}_{2\cdot|1} = -\tfrac{1}{2}(\phi_{,2}+i\theta_{,2}) - (\lambda_{,1}+i\psi_{,1}),\\ &\bar\Omega^{\cdot 2}_{1\cdot|2} + i\bar\Omega^{\cdot 3}_{0\cdot|2} = -q\exp[\tfrac{1}{2}(\phi+i\theta)]\cdot\mathrm{ch}(\lambda+i\psi),\\ &\bar\Omega^{\cdot 1}_{0\cdot|2} - i\bar\Omega^{\cdot 3}_{2\cdot|2} = -\tfrac{1}{2}(\phi_{,1}+i\theta_{,1}) - (\lambda_{,2}+i\psi_{,2}). \end{aligned} \tag{3.23}$$

Extending the Bäcklund transformation (3.12) on the complex-valued functions $\phi+i\theta$ and $\lambda+i\psi$ and taking into account (3.23) one can impose the following conditions on the matrices $\bar{\Omega}$

$$\bar{\Omega}^{\cdot 1}_{0\cdot} - i\bar{\Omega}^{\cdot 3}_{2\cdot} = -\operatorname{sign} q(\bar{\Omega}^{\cdot 2}_{1\cdot} + i\bar{\Omega}^{\cdot 3}_{0\cdot}). \tag{3.24}$$

From here the Kamimura gauge conditions in the theory of the relativistic string moving in four-dimensional space-time follow directly /7/

$$\bar{\Omega}^{\cdot 1}_{0\cdot} = \bar{\Omega}^{\cdot 2}_{1\cdot}, \quad \bar{\Omega}^{\cdot 3}_{2\cdot} = -\bar{\Omega}^{\cdot 3}_{0\cdot}, \quad q<0. \tag{3.25}$$

When these conditions are satisfied, the embedding equations (2.8)-(2.10) for the world sheet of the string are reduced to the D'Alembert equation for one complex-valued function /7/.

Thus the Kamimura gauge in the theory of the relativistic string moving in 3- and 4-dimensional space-time is a direct consequence of the Bäcklund transformation for the Liouville equation.

4. CONFORMAL INVARIANCE IN STRING THEORY AND THE KAMIMURA GAUGE CONDITIONS

Equations (2.1) and (2.2) determining the string dynamics admit conformal transformations of the parameters u^1, u^2

$$u^{\pm} = u^1 \pm u^2 \to v_{\pm}(u^{\pm}) \tag{4.1}$$

with arbitrary functions $v_{\pm}$ (we do not consider now effects of the boundary conditions in string theory and assume for simplicity the string to be infinite). It appears that in the three-dimensional space-time the Kamimura gauge can be treated as a consequence of the conformal invariance in string theory. If the freedom connected with this invariance is not fixed, then the relativistic string dynamics is defined in terms of the differential forms by the Liouville equation (3.3) with arbitrary functions $q_{\pm}(u^{\pm})$ and by the second quadratic form (3.6). The general solution of eq. (3.3) has the form analogous to (3.15)

$$\exp[\phi(u^1,u^2)] = \frac{4}{q_+(u^+)q_-(u^-)}\,\frac{f'_+(u^+)f'_-(u^-)}{[f_+(u^+)+f_-(u^-)]^2}. \tag{4.2}$$

Let us use the transformation (4.1) and instead of choosing the functions $q_{\pm}(u^{\pm})$ be constants as it has been made in (3.7), we express them in terms of the functions $f_{\pm}(u^{\pm})$ entering into the general solution (4.2) in the following way

$$q_+(u^+) = -2f'_+(u^+), \quad q_-(u^-) = -2f'_-(u^-). \tag{4.3}$$

Substituting (4.3) into (4.2) one gets

$$a = \exp(-\frac{\phi}{2}) = f_+(u^+) + f_-(u^-). \tag{4.4}$$

Thus the metric on the world sheet of the string is determined now by the solution of the D'Alembert equation (3.19). Calculating by (4.3), (4.4) and (3.6) the matrix elements in (3.1) we obtain the Kamimura gauge conditions for the relativistic string in the three-dimensional space-time

$$\Omega^{\cdot 1}_{0\cdot|1} = \Omega^{\cdot 3}_{1\cdot|1} = \frac{f'_+ - f'_-}{f_+ + f_-}, \quad \Omega^{\cdot 1}_{0\cdot|2} = \Omega^{\cdot 3}_{1\cdot|2} = \frac{f'_+ + f'_-}{f_+ + f_-}. \tag{4.5}$$

These gauge conditions can be expressed in terms of the string coordinates

$$(x^{\mu}_{,11} \pm x^{\mu}_{,12})^2 = \pm 4[f'_{\pm}(u^{\pm})]^2, \tag{4.6}$$

where $f_{\pm}(u^{\pm})$ are functions in the general solution (4.2).

In the four-dimensional Minkowski space there is no such a simple relation between the Kamimura gauge and the conformal invariance in string theory.

5. GEOMETRIC DERIVATION OF THE BÄCKLUND TRANSFORMATION FOR THE LIOUVILLE EQUATION

The Bäcklund transformation for the sine-Gordon equation, as is well known /18/, is obtained naturally as a consequence of the so-called Bäcklund theorem for the pseudo-spherical surfaces. According to this theorem with each surface of a constant negative curvature in the three-dimensional Euclidean space one can relate a new pseudo-spherical surface. The co-moving frames of these two surfaces at the corresponding points appear to rotate relative to each other by the same angle.

It will be shown in this section in what way the Bäcklund transformations (3.12) relating the solutions of the Liouville and D'Alembert equations may be obtained by considering the geometry of the minimal surface in the three-dimensional pseudo-

Euclidean space. As far as we know the geometric derivation of the Bäcklund transformation for the Liouville equation has not been given in the literature.

At the outset we prove the following algebraic lemma. Let Ω be a linear differential form in the basis $\{du^1, du^2\}$ taking values in the Lie algebra of the SO(1,2)-group and obeying the equation

$$dG = G \wedge G \quad . \tag{5.1}$$

Here $\wedge$ is the wedge product[14]. Then by the gauge transformation

$$\bar{\Omega} = g^{-1}\Omega g - g^{-1}dg, \tag{5.2}$$

where g is a matrix from the SO(1,1)-group of the type (3.9), one can satisfy always the condition

$$\bar{\Omega}^{\cdot 1}_{0\cdot} = \pm \bar{\Omega}^{\cdot 2}_{1\cdot} \quad . \tag{5.3}$$

The proof of this lemma is reduced to verifying that the condition (5.2) enables one to determine the corresponding transformation parameter $\lambda(u^1,u^2)$ in the matrix g in eq. (3.9). From (5.3) one obtains the following Pfaff equation for the function $\lambda(u^1,u^2)$

$$d\lambda = \Omega^{\cdot 1}_{0\cdot} \pm (\Omega^{\cdot 2}_{0\cdot}\,\mathrm{sh}\,\lambda - \Omega^{\cdot 2}_{1\cdot}\,\mathrm{ch}\,\lambda). \tag{5.4}$$

It can be verified by the direct calculation that the integrability condition of this equation

$$d^2\lambda = d \wedge d\lambda = 0$$

is fulfilled by virtue of (5.2). Thus conditions (5.3) can be satisfied always by means of the gauge transformation (5.2).

Now we consider the minimal surface in the three-dimensional Minkowski space. In conformally-flat coordinate system (2.2) the matrices Ω describing the co-moving frame on this surface have the form (3.1). The arbitrary functions $q_{\pm}(u^{\pm})$ are assumed again to be constants. Applying to the transformed matrices $\bar{\Omega}$ (3.11) the lemma proved above (the equality (5.3)) we obtain immediately the Bäcklund transformation (3.12). In this approach we do not know beforehand the equation satisfied by the function $\lambda(u^1,u^2)$. However, using (3.12) one can be easily convinced by the differentiation that the function $\lambda(u^1,u^2)$ is a solution of the D'Alembert equation (3.14).

6. CONCLUSION

From the results obtained it follows that the world sheet of the relativistic string may be described equally well from the geometric viewpoint by the nonlinear Liouville equation and by the free D'Alembert equation. Which way will be more suitable for constructing the noncontradictory quantum theory of the string will be shown by further investigations in this field.

One can give another interpretation of the result obtained: the nonlinear Liouville equation with two independent variables is gauge-equivalent to the free D'Alembert equation. Recall that in the theory of nonlinear evolution equations integrable by the inverse scattering method two equations are called gauge-equivalent[19] if their linear spectral problems are related by the gauge transformation (3.10).

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Преобразование Бэклунда для уравнения Лиувилля и калибровочные условия в теории релятивистской струны

Показано, что калибровочные условия в теории релятивистской струны, которые позволяют использовать здесь вместо нелинейного уравнения Лиувилля уравнение Даламбера, являются прямым следствием преобразования Бэклунда, связывающего решения этих уравнений. Предложена еще одна интерпретация данной калибровки, а именно, показано, что в 3-мерном пространстве-времени ее можно трактовать как следствие конформной инвариантности в теории релятивистской струны. Дан чисто геометрический вывод преобразований Бэклунда для уравнения Лиувилля.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

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Barbashov B.M., Nesterenko V.V. E2-82-922
Bäcklund Transformation for the Liouville Equation and Gauge Conditions in the Relativistic String Theory

It is shown that the gauge conditions in the geometric theory of the relativistic string which make it possible to use here the D'Alembert equation instead of the nonlinear Liouville equation are a direct consequence of the Bäcklund transformation relating the solutions of these equations. Just one more interpretation of this gauge is proposed. It is shown that in the three-dimensional space-time it can be treated as a consequence of the conformal invariance in the relativistic string theory. A purely geometric derivation of the Bäcklund transformation for the Liouville equation is given also.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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