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THE SOMMERFELD-WATSON SUMMATION OF PERTURBATION THEORY SERIES

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1. Introduction

In the scope of perturbation theory (PT) method the quantum field functions of the types of the Green functions, anomalous dimensions, β -function (or Gell-Mann-Low function), etc., are in correspondence with a formal series in the coupling constant g :

$$\beta(g) \sim \sum_{n=K_0}^{\infty} a_n (-g)^n \quad . \tag{1}$$

It has been shown in numerous models that the coefficients a_n rose factorially with increasing n, and the series (1) is divergent (see refs.[1,2] and the review[3]). There is a vast arbitrariness in the definition of the function $\beta(g)$ by its expansion (1). For example, the arbitrary function of the type of $f(g)\exp(-a/g)$ (f is regular in the vicinity of $g=0$), which has the expansion in g at $g=0$ with zero coefficients, may be added to $\beta(g)$. Therefore, we need an additional information about the properties of the function and its series for the unambiguous restoration of $\beta(g)$ from the expansion (1).

Such expansions are often summed by the Borel method, i.e., the sum is defined by the Laplace integral:

$$\beta(g) = \int_0^{\infty} dx\, e^{-x} B(xg), \tag{2}$$

$$B(x) = \sum_{n=K_0}^{\infty} \frac{a_n}{n!} (-x)^n .$$

By the Watson theorem[4] the function $\beta(g)$ may be represented in the form (2) provided it satisfies the strong asymptotic condition[5]:

a) the function $\beta(g)$ is analytic in the domain

$$G=\{g \mid 0<|g|<R, |\arg g|<\frac{\pi}{2}+\delta, \ \delta>0\};$$

b) such constants C and σ exist that for all N and all g in G

$$\left|\beta(g)-\sum_{n=K_0}^{N} a_n(-g)^n\right| \leq C\sigma^{N+1}(N+1)!\,|g|^{N+1}.$$

Hitherto, it has been proved only for some models that the Green functions satisfy the strong asymptotic condition. Besides, the effective methods of approximate restoration of $\beta(g)$ from a few of the first PT coefficients and asymptotics of a_n at $n\to\infty$ basing on formula (2) (for example, the conform - Borel method/6/) essentially use the knowledge of the asymptotic behaviour of $\beta(g)$ at $g\to\infty$.

We shall examine another method for the summation of the PT series. In sect. 2 the conditions sufficient for the summation of the expansion (1) in the Sommerfeld-Watson sense are formulated, and the relation between such a summation and the Borel summation is pointed out. The method of solution of the problem of approximate restoration of the sum of the series is presented in sect. 3. To see the efficiency of this procedure the latter has been used for the calculation of critical exponents of phase transitions basing on the ε-expansions and for the restoration of the β-function in the scalar massless $\varphi^4_{(4)}$ model (sect. 4).

2. The Sommerfeld-Watson Summation

Let us have the series (1), which is divergent in general, and the following conditions are true:

A) there is a function $a(z)$ (we shall call it the coefficient function) such, that $a(n)=a_n$ at $n=K_0, K_0+1, K_0+2, \ldots$;

B) $a(z)$ is analytic in the half-plane $\mathrm{Re}\, z>\sigma$, where σ is the definite value and $\sigma<K_0$;

C) $a(z)=\Gamma(z+\nu)\mu(z)$ with a parameter $\nu>0$ and at $|z|\to\infty$ in the analyticity domain

$$\mu(z)=Cz^{\alpha}\frac{1}{A^z}\left(1+\frac{A_1}{z}+\frac{A_2}{z^2}+\ldots\right) \quad (A>0). \quad (3)$$

This is a complex and for the present unsolved problem to prove the validity of the conditions (A)-(C) for the realistic quantum field models. Therefore, we shall consider here (A)-(C) to be suppositions leading to the reasonable results which are in good agreement with the other approaches, as we shall see below. Here, we can only present some arguments for the validity of the suppositions being made. Usually, the coefficients of the PT series (1) for the theory with action $S[\varphi]=S_0[\varphi]+gS_{int}[\varphi]$ (S_0 is the free part, S_{int} determines the interaction) are determined by the functional integrals of the type

$$a_n=\frac{1}{\Gamma(n+1)}\frac{1}{J_0}\int\prod_x \mathcal{D}\varphi(x)\prod_i\varphi(y_i)\,e^{-S_0[\varphi]}\left(S_{int}[\varphi]\right)^n. \quad (4)$$

From the intuitive understanding of the functional integral (4), based on analogies with integrals of finite multiplicity, we may consider that the convergence in (4) will not be violated if n is complex and $\mathrm{Re}\, n>\sigma>0$. The evaluation of a_n by the steepest-descent method gives at high n /1-3/:

$$a_n=C\Gamma(n+\nu)n^{\alpha}\frac{1}{A^n}\left(1+\frac{A_1}{n}+\frac{A_2}{n^2}+\ldots\right).$$

During the derivation of this formula, however, the fact that n is a natural number is never used. It may be considered complex with $|\arg n|<\pi/2$, that is also confirmed by the analysis of the ordinary integral. The conditions (A)-(C) are satisfied in the theory with strong nonlinearity $\varphi^m_{(d)}$, $m\to\infty$ ($d=2m/(m-2)$)/7/ and in some other models (see, for example, ref./8/).

Let us now formulate the important statement (we shall restrict ourselves to the case $\nu=1$, $K_0=1$; the generalization is trivial):

let the formal series

$$\sum_{n=1}^{\infty}\alpha_n(-x)^n \quad (5)$$

be such that the coefficient function $\alpha(z)$ exists with the following properties:

1) $\alpha(n)=\alpha_n$, $n=1,2,\ldots$;

2) $\alpha(z)$ is analytical at $\mathrm{Re}\, z>\sigma$, $0<\sigma<1$;

3) $\alpha(z)/\Gamma(z+\nu)=\mu(z)$ is the function of exponential type at $\mathrm{Re}\, z>\sigma$;

4) $$\lim_{r\to\infty}\frac{\ln|\mu(\sigma+re^{\pm i\frac{\pi}{2}})|}{r}=\pi-\delta, \quad \delta>0.$$

Then, such a function $f(x)$ exists that be the rightest singularity of the function $a(z)$ in the complex z-plane. Then, using the formula of the same type as (6), one can show that

$$\beta(g) = \frac{\alpha_0 \sin\pi\gamma\, \Gamma(\gamma+1)}{\sin\pi\zeta_0} \frac{g^{\zeta_0}}{(\ln g)^{\gamma+1}} + o(|g|^{\zeta_0}) \qquad (12)$$

at $|g|\to\infty$ (if $\sin\pi\zeta_0\neq 0$).

3. The Approximation of the Sum of the Perturbation Theory Expansion

Usually a few of the first coefficients of PT only are known in quantum field theory. In some cases the leading asymptotics of a_n at $n\to\infty$ is known. Also the first correction A_1/n (see in eq. (3)) is known in the $\varphi^4_{(4)}$ -model/11/. All this information may be taken into account when approximately restoring the function $\beta(g)$ from the series (1) by the Sommerfeld-Watson method. Assume the poles and cuts to be the only singularities of the function $a(z)$. Then, the asymptotic behaviour of $\beta(g)$ at $g\to\infty$ will be of the type (12). Let us define the function $\mathcal{T}(z)=A^z\mu(z)=A^z a(z)/\Gamma(z+\nu)$. It follows from (3) that

$$\mathcal{T}(z)=Cz^{\alpha}(1+A_1/z+A_2/z^2+\dots) \qquad (13)$$

at $|z|\to\infty$ and $Re\, z>\sigma$. Let the parameter $\nu>0$ be so chosen that α be an integer. Then, it is reasonable to approximate the function $\mathcal{T}(z)$ by the rational function, the so-called multipoint Pade approximant /12/:

$$\left[\frac{M_1}{M_2}\right](z)=Q_{M_1}(z)/P_{M_2}(z),$$

where Q_{M_1} and P_{M_2} are polynomials in z of the degrees M_1 and M_2, respectively; $P_{M_2}(0)=1$. The polynomial coefficients are determined from the known K terms of the PT series (1), i.e., we deal with the problem of rational interpolation of $\mathcal{T}(z)$. We may also impose the condition of coincidence of the asymptotics of the approximant and the function $\mathcal{T}(z)$ (eq. (13)) at $|z|\to\infty$ up to the known terms.

For the case when the interpolation points tend to infinity the convergence of the sequence of Pade approximants has been established hitherto only for one class of the meromorphic functions, the so-called functions of the Stieltjes type/13/. But a good agreement

of the results obtained in sect. 4 and the results obtained in other theoretical approaches and in the experiment (for critical exponents) is the indication of the applicability of the procedure described above.

The approximants for the Sommerfeld-Watson sum of the series (1) are calculated by the formula

$$\beta_{M_1,M_2}(g)=-\frac{1}{2i}\int_{\sigma_1-i\infty}^{\sigma_1+i\infty}dz\,\left(\frac{g}{A}\right)^z\frac{\Gamma(z+\nu)}{\sin\pi z}\left[\frac{M_1}{M_2}\right](z).$$

The calculations show one remarkable feature of this approach: if one constructs in a given physical problem the sequence of approximants including a different number of the PT coefficients a_n and terms in asymptotic formula (13) or corresponding to various integers α, then the rightest poles of the majority of approximants will lie relatively close to each other in the complex z-plane. This leads to the stable approximation of the asymptotic behaviour of $\beta(g)$ at $g\to\infty$.

4. Physical Results

The procedure of approximation described above has been used to calculate the critical exponents of phase transitions in the framework of quantum field approach to critical phenomena. The exponents η,ν,ω are represented by the expansion over the parameter 2ε up to the term $(2\varepsilon)^4$, involving the four-loop approximation in the O(N)-symmetrical $\varphi^4_{(4)}$ -model/14/. Taking into account the nature of the asymptotics of the PT coefficients a_n at $n\to\infty$, we have calculated approximate values of these exponents at the point $2\varepsilon=1$. Table 1 shows our results. It also presents the experimental data as well as the values calculated in other theoretical approaches; namely by the conform-Borel method in the framework of the $\varphi^4_{(3)}$ model/15/ and of the ε-expansion/14/. We have obtained also the values of f_0 and ζ_0 determining the asymptotics of the functions $\eta(2\varepsilon)$, $(1/\nu)(2\varepsilon)$, $\omega(2\varepsilon)$ at $2\varepsilon\to\infty$ ($f\sim f_0(2\varepsilon)^{\zeta_0}$) (see Table 2). The estimates of the parameter ζ_0 have been obtained in /14/ from the minimization of the relative error of approximation. The absolute errors indicated in Tables 1 and 2 are determined by the maximal deviation of the values given by various approximants, constructed with regard for the largest number of PT coefficients, from the averaged value.

The second problem that has been solved by the Sommerfeld-Watson summation is the problem of restoration of the β-function (or Gell-

Table 1.

The values of critical exponents calculated by the Sommerfeld-Watson summation (the first column), by the conform-Borel method in[14] (the second column) and [15] (the third column) and obtained experimentally[15] (the fourth column)

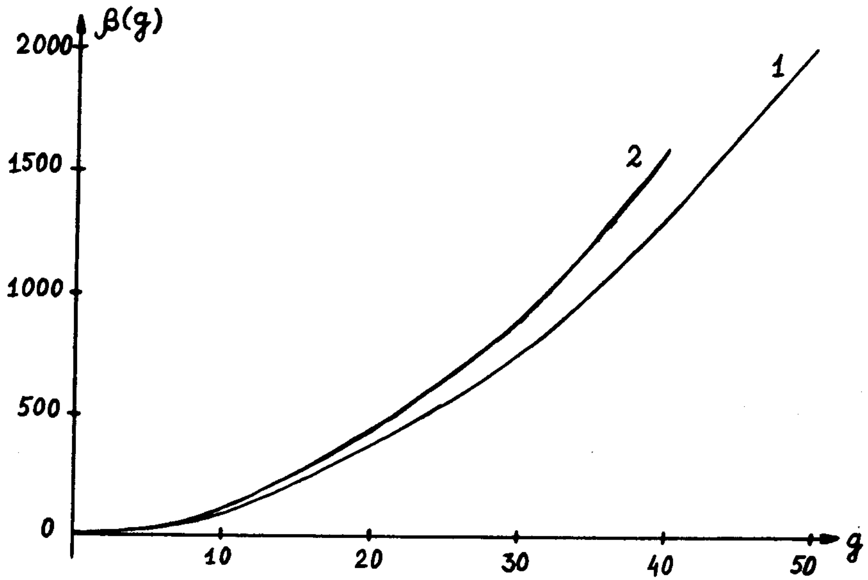
	N = 1			
η	0.0313±0.0005	0.0333±0.0001	0.0315±0.0025	0.016±0.014
ν	0.627 ± 0.006	0.628 ± 0.002	0.6300±0.0008	0.625±0.005
ω	0.786 ± 0.020	0.781 ± 0.015	0.782 ± 0.010	
	N = 2			
η	0.034 ± 0.001	0.0352±0.0001	0.0335±0.0025	
ν	0.673 ± 0.003	0.666 ± 0.004	0.6693±0.0010	0.675±0.001
ω	0.781 ± 0.026	0.777 ± 0.015	0.778 ± 0.008	
	N = 3			
η	0.036 ± 0.002	0.0354±0.0001	0.0340±0.0025	
ν	0.713 ± 0.007	0.700 ± 0.007	0.7054±0.0011	
ω	0.776 ± 0.041	0.779 ± 0.007	0.779± 0.006	

Mann-Low function) in the O(N)-invariant massless theory $(16\pi^2/4!)g\varphi^4_{(4)}$. The questions about the existence of the nontrivial zero of the β-function and the behaviour of $\beta(g)$ at $g\to\infty$ are of the main interest. We have restored the behaviour of $\beta(g)$ in the $0<g\leq 50$ interval for N=1 (see the figure, curve 1) and N=10. It should be noted that we constructed the approximants using 4 known coefficients of PT[16], the leading term of the asymptotics of a_n at $n\to\infty$ [1,2] and the first correction A_1 (see eq. (13); A_1=-4.7 for N=1 and A_1=1.6 for N=10)[11]. The relative error of the approximation is about 20% for N=1 and 6% for N=10 at g=50. The Figure presents the averaged curves (the curve 1 is obtained by the Sommerfeld-Watson summation, and the curve 2 by the conform-Borel method[6]). Our result is in a good agreement with that obtained by the conform-Borel method in[6] (in this case

Table 2.

The values of the parameters f_0 and ζ_0 obtained by the Sommerfeld-Watson summation and the estimates of ζ_0 obtained in [14]

	N	η	$1/\nu$	ω
f_0	1	0.326 ± 0.008	1.32 ± 0.03	2.18 ± 0.11
	2	0.448 ± 0.003	1.82 ± 0.10	2.37 ± 0.09
	3	0.570 ± 0.010	2.33 ± 0.18	2.59 ± 0.13
ζ_0	1	2.40 ± 0.16	1.18 ± 0.05	0.88 ± 0.04
	2	2.44 ± 0.20	1.20 ± 0.07	0.89 ± 0.02
	3	2.46 ± 0.23	1.20 ± 0.12	0.89 ± 0.03
ζ_0 /14/		2 ÷ 3	1.0 ÷ 1.3	0.7 ÷ 0.9



the relative error is about 10% at $g=40$). We have obtained the following values of the parameters β_0 and ζ_0 ($\beta(g)\sim\beta_0 g^{\zeta_0}$ at $g\to\infty$):

$$\beta_0=\begin{cases}1.06 \pm 0.03\\ 2.13 \pm 0.06\end{cases} \qquad \zeta_0=\begin{cases}1.90 \pm 0.05 & \text{for } N=1\\ 1.90 \pm 0.01 & \text{for } N=10,\end{cases}$$

which are in a qualitative agreement with the asymptotics $\beta(g)\sim 0.9g^2$ obtained in [6] for N=1 from the minimization of relative error of approximation. This analysis of the behaviour of the function $\beta(g)$ shows that most likely the zero-charge situation takes place. This means that the $\varphi^4_{(4)}$ model is inherently inconsistent and can describe the interaction of the particles only together with other fields and interactions. It should be noted that all obtained results essentially used the suppositions (A)-(C) formulated in sect. 2.

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Appendix

Let us choose ε such that $0<\varepsilon<\delta-\delta_1$; then there is $R=R(\varepsilon)$ such that for $\tau>R$ the inequality (7) is valid, where the indicator $h(\theta)$ satisfies (8). Then at $\mathcal{J}_N=N+\gamma>R+\sigma$

$$|\tilde{\mu}(N+\gamma-\sigma+iy)| < \exp[\varkappa_1|y|+\varkappa_0\mathcal{J}_N+\varepsilon(\mathcal{J}_N+|y|)-(\varkappa_0+\varepsilon)\sigma].$$

By the Stirling formula the constant B exists such that at large N

$$|\Gamma(\mathcal{J}_N+iy)|\le B(\mathcal{J}_N^2+y^2)^{\frac{1}{2}(\mathcal{J}_N-1/2)}\exp[-y\,\mathrm{arctg}\frac{y}{\mathcal{J}_N}-\mathcal{J}_N].$$

The remainder $R_N(x)$ (see eq. (10)) satisfies the inequality

$$|R_N(x)|<\frac{4|x|^{\mathcal{J}_N}e^{(\varepsilon+\varkappa_0)(\mathcal{J}_N-\sigma)}}{|\sin\pi\gamma|}B\int_0^\infty dy\,((\mathcal{J}_N+1)^2+y^2)^{\frac{1}{2}(\mathcal{J}_N+\frac{1}{2})}\cdot \tag{A.1}$$

$$\cdot\exp(\mathcal{V}y-\mathcal{J}_N-1),$$

$$\mathcal{V}=\frac{\pi}{2}-(\delta-\delta_1-\varepsilon)-\mathrm{arctg}(y/(\mathcal{J}_N+1)).$$

Let us choose λ such that $0<\lambda<\delta-(\delta_1+\varepsilon)$ and then find ω such that for $y/(\mathcal{J}_N+1)\ge\omega$ the inequality

$$\frac{\pi}{2}>\mathrm{arctg}\frac{y}{\mathcal{J}_N+1}\ge\frac{\pi}{2}-\lambda\equiv\mathrm{arctg}\,\omega$$

is valid. We shall divide the interval of integration in (A.1) into two parts: $(0,\omega(\mathcal{J}_N+1))$ and $(\omega(\mathcal{J}_N+1),\infty)$ and evaluate each of the integrals separately:

a) $$I_1=\int_0^{\omega(\mathcal{J}_N+1)}dy\,e^{\mathcal{V}y}[(\mathcal{J}_N+1)^2+y^2]^{\frac{1}{2}(\mathcal{J}_N+\frac{1}{2})}e^{-\mathcal{J}_N-1}<$$

$$<(\mathcal{J}_N+1)^{\mathcal{J}_N+1/2}e^{-\mathcal{J}_N-1}\int_0^{\omega(\mathcal{J}_N+1)}dy\,e^{\mathcal{V}y}[1+\frac{y^2}{(\mathcal{J}_N+1)^2}]^{\frac{1}{2}(\mathcal{J}_N+\frac{1}{2})}<$$

$$<C_1(N+1)!\,\sigma_1^{N+1}N^{\gamma};$$

$$\sigma_1=e^{\mathcal{V}_0\omega}\sqrt{1+\omega^2};\quad \mathcal{V}_0=\frac{\pi}{2}-(\delta-\delta_1-\varepsilon);$$

b) for $y>\omega(\mathcal{J}_N+1)$ $\mathcal{V}<-\Delta(\varepsilon,\lambda)<0$, where $\Delta(\varepsilon,\lambda)\equiv\delta-\delta_1-\varepsilon-\lambda$

$$I_2<e^{-\mathcal{J}_N-1}(1+\omega^2)^{\frac{1}{2}(\mathcal{J}_N+\frac{1}{2})}\int_{\omega(\mathcal{J}_N+1)}^\infty e^{-\Delta(\varepsilon,\lambda)y}y^{\mathcal{J}_N+1/2}dy<$$

$$<C_2(N+1)!\,N^{\gamma-1/2}\sigma_2^{N+1};$$

$$\sigma_2=\sqrt{1+\omega^2}/(e\Delta(\varepsilon,\lambda)).$$

Finally, we obtain the estimate (11) for the remainder $R_N(x)$, where $\sigma=\exp(\varepsilon+\varkappa_0)\max(\sigma_1,\sigma_2)$, which is valid for all $N\ge N^*>R-\sigma$ and for all $x\in\mathcal{D}$.

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Суммирование рядов теории возмущений по Зоммерфельду-Ватсону

Сформулированы условия, при которых ряд теории возмущений суммируется по Зоммерфельду-Ватсону. Разработана процедура приближенного восстановления суммы ряда. Приведены результаты для некоторых физических задач.

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The Sommerfeld-Watson Summation of Perturbation Theory Series

The conditions for the Sommerfeld-Watson summation of perturbation theory series are formulated. The procedure for an approximate restoration of the series sum has been developed. The results for some physical problems are presented.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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