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V.K.Dobrev, A.Ch.Ganchev*

CONFORMAL OPERATORS FROM SPINOR FIELDS: ANTISYMMETRIC TENSOR CASE

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* Institute of Nuclear Research and Nuclear Energy, Sofia, Bulgaria.

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1. In this letter we continue work on the construction of the full system of composite conformal operators built bilinearly from spinor fields. In ref.[1] were constructed all conformal symmetric traceless tensor operators of Lorentz type $(\ell/2, \ell/2)$ with minimal twist that can be built from two spinor fields with noncanonical dimensions. The limit to canonical dimensions (i.e., free spinor fields) was also demonstrated. Here we shall construct the other possible class of operators that can be built from two spinor fields, namely tensor operators of Lorentz type $((\ell+2)/2, \ell/2) \oplus (\ell/2,(\ell+2)/2)$. In ref.[1] also a survey of the applications of composite operators of QFT was made and we shall not persure this here. In the meantime some of the results of ref.[1] were applied to the operator product expansion in asymptotically free QFT (cf.[2]).

2. An irreducible tensor $F^{IR}_{\mu\nu,a_1\ldots a_\ell}$ is of Lorentz type $((\ell+2)/2, \ell/2) \oplus (\ell/2, (\ell+2)/2)$ if it is:

(i) antisymmetric in μ, ν;

(ii) symmetric in $a_1 \ldots a_\ell$;

(iii) traceless in $a_1 \ldots a_\ell$;

(iv) traceless in one antisymmetric and one symmetric index;

(v) and $\epsilon_{\rho\mu\nu a_1} F^{IR}_{\mu\nu,a_1\ldots a_\ell} = 0$ is fulfilled.

It is easy to build a tensor fulfilling (i) and (ii) and we assume that we always start with such a tensor $F_{\mu\nu,a_1\ldots a_\ell}$. Now instead of fulfilling (iii) we replace (cf.[1,3]) $F_{\mu\nu,a_1\ldots a_\ell}$ by a homogeneous polynomial of degree ℓ

$$F_{\mu\nu}(\mathfrak{z}) = F_{\mu\nu,a_1\ldots a_\ell}\mathfrak{z}_{a_1}\cdots\mathfrak{z}_{a_\ell},$$
$$\mathfrak{z} \in K^n = \{\mathfrak{z} \in \mathbf{C}^n \mid \mathfrak{z}^2 = \mathfrak{z}_1^2 + \ldots + \mathfrak{z}_n^2 = 0\}, \tag{1}$$

or by the unique harmonic extension to the whole $\mathbf{C}^n$ (see [3])

$$F_{H\mu\nu}(\zeta) = F_{\mu\nu,a_1\ldots a_\ell}\zeta_{a_1}\cdots\zeta_{a_\ell}, \qquad \zeta \in \mathbf{C}^n, \tag{2}$$

$$\Delta_\zeta F_{H\mu\nu}(\zeta) \equiv \frac{\partial^2}{\partial\zeta_a \partial\zeta_a} F_{H\mu\nu}(\zeta) = 0, \quad F_{H\mu\nu}(\zeta)|_{\zeta^2=0} = F_{\mu\nu}(\zeta). \tag{3}$$

Now conditions (iv) and (v) are imposed respectively as

$$d_\mu F^{IR}_{\mu\nu}(z) = 0, \tag{4}$$

$$\epsilon_{\rho\lambda\mu\nu}\, d_\lambda F^{IR}_{\mu\nu}(z) = 0, \tag{5}$$

where

$$d_\mu \equiv \left(\frac{n}{2} - 1 + z_a \frac{\partial}{\partial z_a}\right)\frac{\partial}{\partial z_\mu} - \frac{1}{2} z_\mu \frac{\partial^2}{\partial z_a \partial z_a} \tag{6}$$

is the interior derivative on K^n (see [3]). Solving (4) and (5) for the irreducible part $F^{IR}_{\mu\nu}$ we obtain $(n = 2h)$

$$F^{IR}_{\mu\nu}(z) = F_{\mu\nu}(z) - \frac{\ell(\ell-1)}{4(h-1+\ell)} z_{[\mu} G_{\nu]}(z) - \frac{\ell}{2} L_{\mu\nu}(z) -$$
$$- \frac{\ell(3h+2\ell-2)}{2(h+\ell-1)(2h+\ell-2)} z_{[\mu} H_{\nu]}(z) + \frac{\ell(\ell-1)(h+\ell)}{2(h+\ell-1)(2h+\ell-2)} z_{[\mu} K_{\nu]}(z), \tag{7}$$

where

$$G_\nu(z) \equiv F_{\lambda\nu,\sigma\sigma a_3 \dots a_\ell} z_\lambda z_{a_3} \cdots z_{a_\ell},$$
$$H_\nu(z) \equiv F_{\sigma\nu,\sigma a_2 \dots a_\ell} z_{a_2} \cdots z_{a_\ell},$$
$$K_\nu(z) \equiv F_{\lambda\sigma,\nu\sigma a_3 \dots a_\ell} z_\lambda z_{a_3} \cdots z_{a_\ell},$$
$$L_{\mu\nu}(z) \equiv F_{\lambda[\mu,\nu] a_2 \dots a_\ell} z_\lambda z_{a_2} \cdots z_{a_\ell}. \tag{8}$$

We shall make a convention to denote the application of (7) by writing

$$F^{IR}_{\mu\nu}(z) = F_{\mu\nu}(z) - \text{traces}. \tag{7'}$$

3. Our method of derivation requires the knowledge of the conformal invariant 3-point functions of operators with the same type of Lorentz representation and two spinor fields. We recall the formulas (see [1]):

$$\langle \psi_{d_1}(x_1) F^d_{\mu\nu}(x_3, z) \bar\psi_{d_2}(x_2)\rangle_0 = \left(\frac{2}{x^2_{12}}\right)^{\frac{d_1+d_2-d+\ell+1}{2}} \left(\frac{2}{x^2_{13}}\right)^{\frac{d+d_1-d_2-\ell}{2}} \left(\frac{2}{x^2_{23}}\right)^{\frac{d+d_2-d_1-\ell}{2}} \tag{9}$$

$$\{ f_\ell (-\gamma_\lambda r_{\lambda[\mu}(13)\gamma_{\nu]} \not{x}_{32} + \not{x}_{13}\gamma_{[\nu} r_{\mu]\lambda}(23)\gamma_\lambda) + f'_\ell \not{x}_{13} \gamma_{[\mu} n_{\nu]} \not{x}_{32} \} (n \cdot z)^\ell - \text{traces},$$

$$x_{12} \equiv x_1 - x_2,\ \not{x}_{13} \equiv (x_{13})_\mu \gamma_\mu,\ r_{\mu\nu}(13) \equiv \frac{2(x_{13})_\mu (x_{13})_\nu}{x^2_{13}} - \delta_{\mu\nu},\ n_\mu \equiv 2\left(\frac{(x_{13})_\mu}{x^2_{13}} - \frac{(x_{23})_\mu}{x^2_{23}}\right)$$

for the γ_5-even function and

$$\langle \psi_{d_1}(x_1) F^{(5)d}_{\mu\nu}(x_3, z) \bar\psi_{d_2}(x_2)\rangle_0 = \left(\frac{2}{x^2_{12}}\right)^{\frac{d_1+d_2-d+\ell}{2}} \left(\frac{2}{x^2_{13}}\right)^{\frac{d+d_1-d_2-\ell+1}{2}} \left(\frac{2}{x^2_{23}}\right)^{\frac{d+d_2-d_1-\ell+1}{2}} \times$$

$$\times \{ f^{(5)}_\ell \not{x}_{13}\gamma_{[\mu}\gamma_{\nu]} \not{x}_{32} + f'^{(5)}_\ell \frac{x^2_{13} x^2_{23}}{2x^2_{12}} \left(\frac{2}{x^2_{13}} \not{x}_{13} (x_{13})_{[\mu} (x_{32})_{\nu]} \not{x}_{32} \frac{2}{x^2_{23}} + \right. \tag{10}$$

$$\left. + \not{x}_{13}\gamma_{[\mu} n_{\nu]} - \gamma_{[\mu} n_{\nu]} \not{x}_{32}\right)\} (n \cdot z)^\ell - \text{traces},$$

for the γ_5-odd one, with f_ℓ, f'_ℓ, $f^{(5)}_\ell$, $f'^{(5)}_\ell$ arbitrary constants, d is the dimension of $F^d_{\mu\nu}(x, z)$, d_1 and d_2 are the dimensions of the spinor fields. The two-point function of two spinor fields is well known

$$\langle \psi_{d_k}(x) \bar\psi_{d_k}(y)\rangle_0 = c_{d_k} (\not{x} - \not{y}) \left(\frac{2}{(x-y)^2}\right)^{d_k + 1/2}. \tag{11}$$

4. Let us recall the method for the construction of conformal operators used in ref. [1]. For a conformal covariant operator with the properties of a $((\ell+2)/2, \ell/2) \oplus (\ell/2, (\ell+2)/2)$ Lorentz tensor with dimension d $\mathcal{F}^d_{\mu\nu}$ we write

$$\mathcal{F}^{d}_{\mu\nu}(x, z) = :\bar\psi_{d_1}(x) H_{\mu\nu}(\overleftarrow{\partial}_x, \overrightarrow{\partial}_x, z)\psi_{d_2}(x): , \qquad (12)$$

where $H_{\mu\nu}$ is a polynomial in z and $\overleftarrow{\partial}_x, \overrightarrow{\partial}_x$ with unknown coefficients. We sandwich $\mathcal{F}^d_{\mu\nu}$ of (8) between two spinor states. On the one hand, from conformal invariance this three point function is given by (9) or (10). On the other hand, using the fact that the fields are generalized free fields we may apply Wick's theorem and get the operator in (12) between two conformal two point functions:

$$\langle\psi_{d_1}(x_1)\mathcal{F}^d_{\mu\nu}(x_3)\bar\psi_{d_2}(x_3)\rangle_0 = \langle\psi_{d_1}(x_1)\bar\psi_{d_1}(x_3)\rangle_0 H_{\mu\nu}(\overleftarrow{\partial}, \overrightarrow{\partial}, z)\langle\psi_{d_2}(x_3)\bar\psi_{d_2}(x_2)\rangle_0 . \qquad (13)$$

In this way we determine the coefficients in the polynomial $H_{\mu\nu}$.

The conformal operator of minimal twist $\tau_m \equiv d - \ell = d_1 + d_2 + 1$ reproducing the structures in the γ_5-even three point function is:

$$\mathcal{F}^d_{\mu\nu}(x, z) = \frac{\ell!\,\Gamma(d_1+1/2)\Gamma(d_2+1/2)}{c_{d_1}c_{d_2}} :\bar\psi_{d_1}(x)\{ f_\ell \tilde P_\ell^{(d_1-\frac12, d_2-\frac12)} \left[\frac{\overleftarrow{\not\partial}}{2d_1-3}\gamma_{[\mu}\gamma_{\nu]} - \gamma_{[\mu}\gamma_{\nu]}\frac{\overrightarrow{\not\partial}}{2d_2-3}\right] +$$

$$+ (f'_\ell - f_\ell)\tilde P_\ell^{(d_2+\frac12, d_1-\frac12)}\left[\overrightarrow{\partial}_{[\mu}\gamma_{\nu]} - \gamma_{[\mu}\gamma_{\nu]}\frac{\overrightarrow{\not\partial}}{2d_2-3}\right] -$$

$$-(f_\ell + f'_\ell)\tilde P_\ell^{(d_2-\frac12, d_1+\frac12)}\left[\overleftarrow{\partial}_{[\mu}\gamma_{\nu]} + \frac{\overleftarrow{\not\partial}}{2d_1-3}\gamma_{[\mu}\gamma_{\nu]}\right] + \qquad (14)$$

$$+ \tilde P_{\ell-1}^{(d_2+\frac12, d_1+\frac12)}\left[(f_\ell + f'_\ell)\overleftarrow{\partial}_{[\mu}\gamma_{\nu]}(\gamma\cdot z)\frac{\overrightarrow{\not\partial}}{2d_2-3} + (f'_\ell - f_\ell)\frac{\overleftarrow{\not\partial}}{2d_1-3}(\gamma\cdot z)\overrightarrow{\partial}_{[\mu}\gamma_{\nu]}\right] -$$

$$(f_\ell + f'_\ell)\tilde P_{\ell-1}^{(d_2-\frac12, d_1+\frac32)}\frac{\overleftarrow{\not\partial}}{2d_1-3}(\gamma\cdot z)\overleftarrow{\partial}_{[\mu}\gamma_{\nu]}$$

$$+ (f_\ell - f'_\ell)\tilde P_{\ell-1}^{(d_2+\frac32, d_1-\frac12)}\overrightarrow{\partial}_{[\mu}\gamma_{\nu]}(\gamma\cdot z)\frac{\overrightarrow{\not\partial}}{2d_2-3} - \text{traces}\}\psi_{d_2}(x): ,$$

$$\left(\partial_\mu \equiv \frac{\partial}{\partial x_\mu}, \quad \not\partial \equiv \gamma_\mu\partial_\mu\right),$$

where

$$\tilde P_\ell^{(\alpha,\beta)} \equiv \frac{(z\cdot\overleftarrow{\partial} + z\cdot\overrightarrow{\partial})^\ell}{\Gamma(\alpha+1+\ell)\Gamma(\beta+1+\ell)} P_\ell^{(\alpha,\beta)}\left(\frac{z\cdot\overleftarrow{\partial} - z\cdot\overrightarrow{\partial}}{z\cdot\overleftarrow{\partial} + z\cdot\overrightarrow{\partial}}\right), \quad \tilde P_{-1}^{(\alpha,\beta)} \equiv 0, \qquad (15)$$

$P_\ell^{(\alpha,\beta)}(w)$ being the Jacobi polynomial /4/:

$$(x+y)^\ell P_\ell^{(\alpha,\beta)}\left(\frac{x-y}{x+y}\right) = \Gamma(\alpha+1+\ell)\Gamma(\beta+1+\ell)\sum_{k=0}^{\ell}\frac{(-1)^{\ell-k}x^k y^{\ell-k}}{k!\,(\ell-k)!\,\Gamma(\beta+1+k)\Gamma(\alpha+1+\ell-k)} .$$

The γ_5-odd three point function (10) with coefficient $f_\ell^{(5)}$ gives the following conformal operator with minimal twist $\tau_m = d_1 + d_2$:

$$\mathcal{F}^{(5)d}_{\mu\nu}(x, z) = \frac{f_\ell^{(5)}\ell!\,\Gamma(d_1+\frac12)\Gamma(d_2+\frac12)}{c_{d_1}c_{d_2}} :\bar\psi_{d_1}(x)\gamma_{[\mu}\gamma_{\nu]}\tilde P_\ell^{(d_2-\frac12, d_1-\frac12)}\psi_{d_2}(x): - \text{traces}. \qquad (16)$$

The γ_5-odd one with coefficient $f'^{(5)}_\ell$ has $\tau_m = d_1 + d_2 + 2$ and is:

$$\mathcal{F}'^{(5)d}_{\mu\nu}(x, z) = f'^{(5)}_\ell\frac{\ell!\,\Gamma(d_1+\frac12)\Gamma(d_2+\frac12)}{c_{d_1}c_{d_2}} :\bar\psi_{d_1}(x)\{\tilde P_\ell^{(d_2+\frac12, d_1+\frac12)}\left[-\overleftarrow{\partial}_{[\mu}\overrightarrow{\partial}_{\nu]} -\right.$$

$$\left. - \overleftarrow{\partial}_{[\mu}\gamma_{\nu]}\frac{\overrightarrow{\not\partial}}{2d_2-3} - \frac{\overleftarrow{\not\partial}}{2d_1-3}\gamma_{[\mu}\overrightarrow{\partial}_{\nu]} + (d_1 + d_2 + \ell - 1)\frac{\overleftarrow{\not\partial}}{2d_1-3}\gamma_{[\mu}\gamma_{\nu]}\frac{\overrightarrow{\not\partial}}{2d_2-3}\right] +$$

$$+ \tilde P_\ell^{(d_2-\frac12, d_1+\frac12)}\left[\overleftarrow{\partial}_{[\mu}\gamma_{\nu]}\frac{\overrightarrow{\not\partial}}{2d_2-3} + \frac{\overleftarrow{\not\partial}}{2d_1-3}\overleftarrow{\partial}_{[\mu}\gamma_{\nu]}\right] +$$

$$+ \tilde P_\ell^{(d_2+\frac12, d_1-\frac12)}\left[\frac{\overleftarrow{\not\partial}}{2d_1-3}\gamma_{[\mu}\overrightarrow{\partial}_{\nu]} + \gamma_{[\mu}\overrightarrow{\partial}_{\nu]}\frac{\overrightarrow{\not\partial}}{2d_2-3}\right] -$$

$$- \tilde P_{\ell-1}^{(d_2+\frac32, d_1+\frac12)}\left[(d_1 + d_2 + \ell - 1)\frac{\overleftarrow{\not\partial}}{2d_1-3}\gamma_{[\mu}\overrightarrow{\partial}_{\nu]}(\gamma\cdot z)\frac{\overrightarrow{\not\partial}}{2d_2-3} - \overleftarrow{\partial}_{[\mu}\overrightarrow{\partial}_{\nu]}(\gamma\cdot z)\frac{\overrightarrow{\not\partial}}{2d_2-3}\right] +$$

$$+ \tilde P_{\ell-1}^{(d_2+\frac12, d_1+\frac32)}\left[(d_1 + d_2 + \ell - 1)\frac{\overleftarrow{\not\partial}}{2d_1-3}(\gamma\cdot z)\overleftarrow{\partial}_{[\mu}\gamma_{\nu]}\frac{\overrightarrow{\not\partial}}{2d_2-3} -\right.$$

$$- \frac{\partial}{2d_1-3} (\gamma \cdot \mathfrak{z}) \partial_{[\mu} \vec{\partial}_{\nu]}] - \text{traces} \} \psi_{d_2}(x): . \tag{17}$$

Note that only the operator (16) has a limit to canonical dimensions $d_1 = d_2 = 3/2$.

5. We shall not write down the explicit application of (7) to (14), (16) and (17). However,we shall give a list of structures which are not included in (14), (16) and (17) because their irreducible counter-parts obtained by applying (7) are zero $(h=2)$

$$\begin{aligned} &\gamma_{[\mu} \gamma_{\nu]} (\gamma \cdot \mathfrak{z})(a \cdot \mathfrak{z})^p (b \cdot \mathfrak{z})^q , \\ &\gamma_{[\mu} \mathfrak{z}_{\nu]} (a \cdot \mathfrak{z})^p (b \cdot \mathfrak{z})^q , \\ &\gamma_{[\mu} \mathfrak{z}_{\nu]} (\gamma \cdot \mathfrak{z})(a \cdot \mathfrak{z})^p (b \cdot \mathfrak{z})^q , \\ &a_{[\mu} \mathfrak{z}_{\nu]} (a \cdot \mathfrak{z})^p (b \cdot \mathfrak{z})^q , \\ &a_{[\mu} \mathfrak{z}_{\nu]} (\gamma \cdot \mathfrak{z})(a \cdot \mathfrak{z})^p (b \cdot \mathfrak{z})^q , \end{aligned} \tag{18}$$

and a and b are arbitrary vectors (in our case they are the derivatives $\overleftarrow{\partial}$ and $\vec{\partial}$).

If we want the most explicit expression for our operators we must apply not only (7) but also write their harmonic extensions (since $\mathfrak{z}^2 = 0$ kills all terms that have $\delta_{\alpha_i \alpha_j}$ in them).

The general structure which we have to extend harmonically is of the form

$$f(\mathfrak{z}) = \prod_{i=1}^{p} (a_i \cdot \mathfrak{z})^{s_i} , \tag{19}$$

where s_i are integers, a_i and vectors, $\mathfrak{z}^2 = 0$. Its harmonic extension is

$$f_H(\zeta) = \Big(\prod_{\ell=1}^{p} (a_\ell \cdot \zeta)^{s_\ell} \Big) \Big(\prod_{1 \le i \le j \le p} \sum_{k_{ij}=0}^{2k_{ii} + \tilde{k}_i \le s_i} \Big) \frac{1}{(2-h-|s|)_{|k|} 2^{|k|'}} \cdot$$

$$\cdot \prod_{n=1}^{p} (-s_n)_{2k_{nn} + \tilde{k}_n} \prod_{m \ge n}^{p} \Big(\frac{-\zeta^2 (a_n \cdot a_m)}{2(a_n \cdot \zeta)(a_m \cdot \zeta)} \Big)^{k_{nm}} \frac{1}{(k_{nm})!} ; \quad |k| \equiv \sum_{1 \le i \le j \le p} k_{ij} , \quad |k|' \equiv \sum_{i=1}^{p} k_{ii} ,$$

$$|s| \equiv \sum_{i=1}^{p} s_i ; \quad \tilde{k}_n \equiv \sum_{\substack{j \ne n \\ j=1}}^{p} k_{nj} , \quad k_{ij} \equiv k_{ji} \ (i > j), \tag{20}$$

$$(\alpha)_k \equiv \frac{\Gamma(\alpha + k)}{\Gamma(\alpha)} .$$

6. Further on the operators with higher twist $\tau = \tau_m + 2k \ (k=1,2,\ldots)$ will be given (τ_m is the minimal twist displayed above for each series). Work on the operators in QCD is also in progress. As is noted in ref./2/ conformal operators can be used in 1-loop QCD calculations and the relevant operators can be obtained by replacing in the above formulas ∂ with the covariant derivative D (colour indices being suppressed).

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Добрев В.К., Ганчев А.Х.

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Конформные операторы из спинорных полей: случай антисимметрических тензоров

Построена полная система конформных антисимметрических тензорных операторов лоренцевского типа $((\ell+2)/2,\ell/2)\oplus(\ell/2,(\ell+2)/2)$ и минимального твиста, составленных из двух спинорных полей. Вместе с ранее сконструированными симметрическими тензорными операторами лоренцевского типа $(\ell/2,\ell/2)$ это дает все конформные локальные операторы минимального твиста, которые можно построить из двух спинорных полей.

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Dobrev V.K., Ganchev A.Ch.

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Conformal Operators from Spinor Fields: Antisymmetric Tensor Case

The full system of conformal antisymmetric tensor operators of Lorentz type $((\ell+2)/2,\ell/2)\oplus(\ell/2,(\ell+2)/2)$ with minimal twist constructed from two conformal spinor fields is given. Together with the earlier constructed symmetric tensor operator of Lorentz type $(\ell/2,\ell/2)$ this gives all conformal local operators with minimal twist that can be constructed from two spinor fields

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