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ON PHASE SPACE REPRESENTATIONS IN QUANTUM FIELD THEORY

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Standard formulations of the classical and quantum theories are far from each other, since they use different languages. Uniformity of quantum and classical mechanics is achieved by introducing phase space representations (PSRs) in quantum mechanics and by using Liouville equation (linear partial differential equation) of motion in both of them, instead of the Newton or Hamilton equations in classical mechanics and the Schrödinger equation in quantum mechanics.

Here we discuss an analogous common approach in field theory. We achieve a similar uniformity of classical and quantum field theories if we introduce in the latter analogs of the quantum-mechanical PSRs and assume analog of Liouville equation as equation of motion in both of them. Now the Liouville equation is a linear functional derivative equation. In classical field theory it contains only first functional derivatives. In quantum field theory it includes, in addition, third and higher (odd order) functional derivatives multiplied by the Planck constant $\hbar$. The PSR formulation of quantum field theory is one more functional method, which involves derivatives with respect to c-number canonical variables, e.g., in a scalar field theory case with respect to the coordinate $\varphi(\vec{x},t_0)$ and momentum $\dot{\varphi}(\vec{x},t_0)$ of quantum field $\hat{\varphi}(x)$ (two functions of 3-argument $\vec{x}$), or, equivalently, to one scalar function $\mathrm{J}(x)$ of 4-argument x_μ (the latter is more preferable due to covariance). Like in quantum mechanics, in quantum field theory one can define more than one PSR (unlike classical field theory).

<u>Completeness relations and nonoperator representatives.</u> We shall start with the completeness relations [x)] (e.g., for a scalar field theory)

$$
\begin{aligned}
|\mathbb{1}\|\otimes\|\mathbb{1}| &= \int \delta^2\varphi\; |\varphi\rangle\langle\varphi| \otimes \Lambda^{-2}\|\varphi\rangle\langle\varphi\| = \\
&= \int \delta^2\varphi\; \Lambda^{-1}|\varphi\rangle\langle\varphi| \otimes \Lambda^{-1}\|\varphi\rangle\langle\varphi\| = \\
&= \int \delta^2\varphi\; |:e^{i\hbar^{-1}(\hat{\varphi},\varphi)}:|\otimes\|\vdots e^{i\hbar^{-1}(\varphi,\hat{\varphi})}\vdots\| = \\
&= \int \delta^2\varphi\; |e^{i\hbar^{-1}(\hat{\varphi},\varphi)}|\otimes\|e^{i\hbar^{-1}(\varphi,\hat{\varphi})}\| \;, \qquad (1)
\end{aligned}
$$

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x) Of the same "matrix"-type like, for example, the completeness relation for γ-matrices $\sum_1^{16}|\gamma_A|\otimes\|\gamma_A\| = 4|1|\otimes\|1|$.

where $|\varphi\rangle$ (or more detailed would be $|\varphi\dot{\varphi}\rangle$ with an apparent dependence on coordinates and momenta) are coherent states of the free scalar field, $\hat{\varphi}(x)$ is a free scalar field operator, and $\varphi(x)=\langle\varphi|\hat{\varphi}(x)|\varphi\rangle$ is its classical counterpart, Λ is some Gauss transformation (for its definition see Appendix A and a footnote on p.6 below), $:\,:$ and $\vdots\;\vdots$ denote N- and anti-N-orderings, and $\delta^2\varphi = \prod_{\vec{x}}(2\pi\hbar)^{-1}\delta\varphi(\vec{x})\delta\dot{\varphi}(\vec{x})$.

The coherent states have the form

$$|\varphi\rangle = e^{i\hbar^{-1}(\varphi,\hat{\varphi})}|0\rangle = e^{i\hbar^{-1}\mathrm{J}\cdot\hat{\varphi}}|0\rangle, \quad \left(\varphi(x) = -\int d^4x\,\Delta(x-y)\mathrm{J}(y)\right) \qquad (2)$$

where $(\varphi,\hat{\varphi}) = i\int d^3x\,\varphi(x)\overleftrightarrow{\partial_4}\hat{\varphi}(x) = \mathrm{J}\cdot\hat{\varphi} = \int d^4x\,\mathrm{J}(x)\hat{\varphi}(x)$ and satisfy the condition

$$\hat{\varphi}^{(-)}(x)|\varphi\rangle = \varphi^{(-)}(x)|\varphi\rangle \qquad (3)$$

for uncertainties $\Delta\varphi(x) = \hat{\varphi}(x) - \varphi(x)$, $\Delta\dot{\varphi} = \dot{\hat{\varphi}}(x) - \dot{\varphi}(x)$ to be minimal. We have

$$
\begin{gathered}
\langle\varphi|\Delta\varphi(x)\Delta\varphi(y)|\varphi\rangle = \hbar\Delta^{(-)}(x-y) \\
\langle\varphi|\Delta\varphi(x)\Delta\varphi(y)|\varphi\rangle_{x_0=y_0} = \frac{\hbar}{2}\Delta^{(1)}(\vec{x}-\vec{y},0), \quad \langle\varphi|\Delta\dot{\varphi}(x)\Delta\dot{\varphi}(y)|\varphi\rangle_{x_0=y_0} = -\frac{\hbar}{2}\ddot{\Delta}^{(1)}(\vec{x}-\vec{y},0)
\end{gathered} \qquad (4)
$$

and their minimality is demonstrated by the equality

$$-\int d^3y\,\Delta^{(1)}(\vec{x}-\vec{y},0)\ddot{\Delta}^{(1)}(\vec{y}-\vec{z},0) = \delta(\vec{x}-\vec{z}). \qquad (5)$$

Note this role of the distribution $\Delta^{(1)}(x-y)$ as a momentum matrix. See also eqs. (A.6) - (A.8) and (A.12)-(A.18) of Appendix A.

Using (1), any operator $\hat{\mathbb{F}}$ can be represented

$$
\begin{aligned}
\hat{\mathbb{F}} &= \int\delta^2\varphi\,(\Lambda^{-2}|\varphi\rangle\langle\varphi|)\langle\varphi|\hat{\mathbb{F}}|\varphi\rangle \equiv \int\delta^2\varphi\,(\Lambda^{-2}|\varphi\rangle\langle\varphi|)\mathbb{F}_1(\varphi) = \\
&= \int\delta^2\varphi\,(\Lambda^{-1}|\varphi\rangle\langle\varphi|)\Lambda^{-1}\langle\varphi|\hat{\mathbb{F}}|\varphi\rangle \equiv \int\delta^2\varphi\,(\Lambda^{-1}|\varphi\rangle\langle\varphi|)\mathbb{F}_2(\varphi) = \\
&= \int\delta^2\varphi\,|\varphi\rangle\langle\varphi|\Lambda^{-2}\langle\varphi|\hat{\mathbb{F}}|\varphi\rangle \equiv \int\delta^2\varphi\,|\varphi\rangle\langle\varphi|\mathbb{F}_3(\varphi) = \\
&= \int\delta^2\varphi\,:e^{i\hbar^{-1}(\hat{\varphi},\varphi)}:\,\widetilde{\mathbb{F}}_1(\varphi) = \\
&= \int\delta^2\varphi\,e^{i\hbar^{-1}(\hat{\varphi},\varphi)}\,\widetilde{\mathbb{F}}_2(\varphi) = \\
&= \int\delta^2\varphi\,\vdots e^{i\hbar^{-1}(\hat{\varphi},\varphi)}\vdots\,\widetilde{\mathbb{F}}_3(\varphi).
\end{aligned} \qquad (6)
$$

Thus, we come to nonoperator representatives in PSR-1, PSR-2 and PSR-3

$$\mathbb{F}_1(\varphi) = \mathrm{Tr}(|\varphi\rangle\langle\varphi|\hat{\mathbb{F}}) = \langle\varphi|\hat{\mathbb{F}}|\varphi\rangle = \Lambda\mathbb{F}_2(\varphi) = \Lambda^2\mathbb{F}_3(\varphi), \qquad (7)$$

$$\mathbb{F}_2(\varphi) = \mathrm{Tr}(\Lambda^{-1}|\varphi\rangle\langle\varphi|\hat{\mathbb{F}}) = \Lambda^{-1}\langle\varphi|\hat{\mathbb{F}}|\varphi\rangle = \Lambda^{-1}\mathbb{F}_1(\varphi) = \Lambda\mathbb{F}_3(\varphi), \qquad (8)$$

$$\mathbb{F}_3(\varphi) = \mathrm{Tr}(\Lambda^{-2}|\varphi\rangle\langle\varphi|\hat{\mathbb{F}}) = \Lambda^{-2}\langle\varphi|\hat{\mathbb{F}}|\varphi\rangle = \Lambda^{-2}\mathbb{F}_1(\varphi) = \Lambda^{-1}\mathbb{F}_2(\varphi), \qquad (9)$$

and to other three PSRs

$$\widetilde{\mathbb{F}}_1(\varphi)=\mathrm{Tr}\left(:e^{i\hbar^{-1}(\varphi,\hat{\varphi})}:\hat{\mathbb{F}}\right)=\int\delta^2\varphi'\, e^{i\hbar^{-1}(\varphi,\varphi')}\mathbb{F}_1(\varphi')=\int\delta J'\, e^{i\hbar^{-1}\varphi\cdot J'}\mathbb{F}_1(\varphi), \tag{10}$$

$$\widetilde{\mathbb{F}}_2(\varphi)=\mathrm{Tr}\left(e^{i\hbar^{-1}(\varphi,\hat{\varphi})}\hat{\mathbb{F}}\right)=\int\delta^2\varphi'\, e^{i\hbar^{-1}(\varphi,\varphi')}\mathbb{F}_2(\varphi)=\int\delta J'\, e^{i\hbar^{-1}\varphi\cdot J'}\mathbb{F}_2(\varphi), \tag{11}$$

$$\widetilde{\mathbb{F}}_3(\varphi)=\mathrm{Tr}\left(:e^{i\hbar^{-1}(\varphi,\hat{\varphi})}:\hat{\mathbb{F}}\right)=\int\delta^2\varphi'\, e^{i\hbar^{-1}(\varphi,\varphi')}\mathbb{F}_3(\varphi)=\int\delta J'\, e^{i\hbar^{-1}\varphi\cdot J'}\mathbb{F}_3(\varphi), \tag{12}$$

which are related by the functional Fourier transformation with the above $\mathbb{F}_1$, $\mathbb{F}_2$ and $\mathbb{F}_3$, respectively (there are inverse transformations of the same form, see Appendix A). One can introduce other PSRs. The PSR-1 $(\mathbb{F}_1(\varphi))$ results from the coherent state representation (CSR) if one keeps only diagonal matrix elements. The off-diagonal matrix elements furnish a redundant information (due to overcompleteness of the coherent state set). They are deducible from the diagonal ones. The PSR-2 $(\mathbb{F}_2(\varphi))$ is in fact an analog of the Wigner representation in quantum mechanics /2/ x). The representatives $\widetilde{\mathbb{F}}_2(\varphi)$ form a representation, which is analogous to the Weyl representation in quantum mechanics/1/. According to eq.(6) we can reconstruct any operator via its representatives (<u>the reconstruction theorems</u>). There are more concise forms of these reconstruction theorems

$$\hat{\mathbb{F}}=:\mathbb{F}_1(\hat{\varphi}):=\mathrm{sym}\,\mathbb{F}_2(\hat{\varphi})=:\mathbb{F}_3(\hat{\varphi}):\,. \tag{13}$$

Let us give examples of the nonoperator representatives in PSR-1, PSR-2 and PSR-3. The simplest of them are

$$\langle\varphi_2|:\hat{\varphi}(x_1)\hat{\varphi}(x_2)\cdots\hat{\varphi}(x_n):|\varphi_1\rangle=\varphi_{21}(x_1)\cdots\varphi_{21}(x_n)\langle\varphi_2|\varphi_1\rangle,\quad \varphi_{21}(x)=\varphi_2^{(+)}(x)+\varphi_1^{(-)}(x) \quad \text{(CSR)}$$

$$\langle\varphi|:\hat{\varphi}(x_1)\hat{\varphi}(x_2)\cdots\hat{\varphi}(x_n):|\varphi\rangle=\Lambda^{-1}\langle\varphi|\mathrm{Sym}(\hat{\varphi}(x_1)\hat{\varphi}(x_2)\cdots\hat{\varphi}(x_n))|\varphi\rangle=$$
$$=\Lambda^{-2}\langle\varphi|:\hat{\varphi}(x_1)\hat{\varphi}(x_2)\cdots\hat{\varphi}(x_n):|\varphi\rangle=\varphi(x_1)\varphi(x_2)\cdots\varphi(x_n) \qquad \text{(PSR-1,2,3)}\ (14)$$

Starting with expansions of S-matrix in terms of N-products, symmetrized products and anti-N-products

$$\begin{matrix}\hat{U}(t,t_0)\\ \hat{U}^{-1}(t,t_0)\end{matrix}=\begin{matrix}T\\ \widetilde{T}\end{matrix}\; e^{\mp i\hbar^{-1}\int_{t_0}^{t}d^4x\,\mathcal{H}_I(\hat{\varphi})}=$$

$$=:e^{\int\hat{\varphi}\frac{\delta}{\delta\phi}}:\; e^{\mp\frac{i\hbar}{2}\iint\frac{\delta}{\delta\phi}\Delta_\pm\frac{\delta}{\delta\phi}}\; e^{\mp i\hbar^{-1}\int d^4x\,\mathcal{H}_I(\phi)}\Big|_{\phi=0}= \tag{15.a}$$

$$=e^{\int\hat{\varphi}\frac{\delta}{\delta\phi}}\; e^{\mp\frac{i\hbar}{2}\iint\frac{\delta}{\delta\phi}\Delta_{sym}\frac{\delta}{\delta\phi}}\; e^{\mp i\hbar^{-1}\int d^4x\,\mathcal{H}_I(\phi)}\Big|_{\phi=0}= \tag{15.b}$$

$$=:e^{\int\hat{\varphi}\frac{\delta}{\delta\phi}}:e^{\mp\frac{i\hbar}{2}\iint\frac{\delta}{\delta\phi}\Delta_\mp\frac{\delta}{\delta\phi}}\; e^{\mp i\hbar^{-1}\int d^4x\,\mathcal{H}_I(\phi)}\Big|_{\phi=0} \tag{15.c}$$

we obtain

x) The Wigner definition was not via coherent states. For the proof of equivalence of both definitions in quantum mechanics see Appendix to ref./15f/. In field theory the definition via the coherent states seems more convenient.

$$\langle\varphi|\begin{matrix}\hat{U}\\ \hat{U}^{-1}\end{matrix}|\varphi\rangle=e^{\mp\frac{i\hbar}{2}\iint\frac{\delta}{\delta\phi}\Delta_\pm\frac{\delta}{\delta\phi}}\; e^{\mp i\hbar^{-1}\int d^4x\,\mathcal{H}_I(\phi)}\Big|_{\phi=\varphi} \qquad \text{(PSR-1)}\ (16.a)$$

$$\Lambda^{-1}\langle\varphi|\begin{matrix}\hat{U}\\ \hat{U}^{-1}\end{matrix}|\varphi\rangle=e^{\mp\frac{i\hbar}{2}\iint\frac{\delta}{\delta\phi}\Delta_{sym}\frac{\delta}{\delta\phi}}\; e^{\mp i\hbar^{-1}\int d^4x\,\mathcal{H}_I(\phi)}\Big|_{\phi=\varphi} \qquad \text{(PSR-2)}\ (16.b)$$

$$\Lambda^{-2}\langle\varphi|\begin{matrix}\hat{U}\\ \hat{U}^{-1}\end{matrix}|\varphi\rangle=e^{\mp\frac{i\hbar}{2}\iint\frac{\delta}{\delta\phi}\Delta_\mp\frac{\delta}{\delta\phi}}\; e^{\mp i\hbar^{-1}\int d^4x\,\mathcal{H}_I(\phi)}\Big|_{\phi=\varphi} \qquad \text{(PSR-3)}.\ (16.c)$$

In some cases the S-matrix can be ordered explicitly: e.g., in the case of interaction with an external current $j^e(x)$, when

$$\hat{U}(t,t_0)=e^{i(2\hbar)^{-1}\iint d^4x\,d^4y\,j^e(x)\Delta_+(x-y)j^e(y)}:e^{i\hbar^{-1}\int d^4x\,\hat{\varphi}(x)j^e(x)}: \tag{17.a}$$

$$=e^{i(2\hbar)^{-1}\iint d^4x\,d^4y\,j^e(x)\Delta_{sym}(x-y)j^e(y)}\; e^{i\hbar^{-1}\int d^4x\,\hat{\varphi}(x)j^e(x)} \tag{17.b}$$

$$=e^{i(2\hbar)^{-1}\iint d^4x\,d^4y\,j^e(x)\Delta_-(x-y)j^e(y)}:e^{i\hbar^{-1}\int d^4x\,\hat{\varphi}(x)j^e(x)}: \tag{17.c}$$

and in PSR-1, PSR-2 and PSR-3

$$U(t,t_0)=e^{i(2\hbar)^{-1}\iint d^4x\,d^4y\,j^e(x)\Delta_{(\,)}(x-y)j^e(y)}\; e^{i\hbar^{-1}\int d^4x\,\varphi(x)j^e(x)} \tag{17.d}$$

with $\Delta_{(\,)}=\Delta_+,\Delta_{sym}$ and Δ_-, respectively; and in the case of interaction with an external (e.g., electromagnetic) field, when (cf./17/)

$$\hat{U}(t,t_0)=U_0(S_+):e^{-\hbar^{-1}\bar{e}\int d^4x\,\hat{\bar{\psi}}\gamma_\mu\hat{\psi}A_\mu+i\hbar^{-1}\bar{e}^2\iint d^4x\,d^4y\,\hat{\bar{\psi}}(x)\gamma_\mu A_\mu(x)S^A_+(x,y)\gamma_\nu A_\nu(y)\hat{\psi}(y)}:$$

$$=U_0(S_{sym})\,\widetilde{\mathrm{sym}}\,e^{-\hbar^{-1}\bar{e}\int d^4x\,\hat{\bar{\psi}}\gamma_\mu\hat{\psi}A_\mu+i\hbar^{-1}\bar{e}^2\iint d^4x\,d^4y\,\hat{\bar{\psi}}(x)\gamma_\mu A_\mu(x)S^A_{sym}(x,y)\gamma_\nu A_\nu(y)\hat{\psi}(y)}$$

$$=U_0(S_-):e^{-\hbar^{-1}\bar{e}\int d^4x\,\hat{\bar{\psi}}\gamma_\mu\hat{\psi}A_\mu+i\hbar^{-1}\bar{e}^2\iint d^4x\,d^4y\,\hat{\bar{\psi}}(x)\gamma_\mu A_\mu(x)S^A_-(x,y)\gamma_\nu A_\nu(y)\hat{\psi}(y)}: \tag{18.a}$$

with $\bar{e}=e/\hbar c$, and in PSR-1, PSR-2 and PSR-3

$$U(t,t_0)=U_0(S_{(\,)})\,e^{-\hbar^{-1}\bar{e}\int d^4x\,\bar{\psi}\gamma_\mu\psi A_\mu+i\hbar^{-1}\bar{e}^2\iint d^4x\,d^4y\,\bar{\psi}(x)\gamma_\mu A_\mu(x)S^A_{(\,)}(x,y)\gamma_\nu A_\nu(y)\psi(y)}, \tag{18.b}$$

where the case of the spinor electrodynamics is implied, $S^A_{(\,)}=S^A_+$, S^A_{sym} and S^A_- (the exact one-particle Green functions in an external electromagnetic field $A_\mu(x)$), respectively, and $S_{(\,)}$ are the corresponding free Green functions; $\psi(x)=\langle\psi|\hat{\psi}(x)|\psi\rangle$ is a (Grassmannian) counterpart of the free Dirac operator $\hat{\psi}(x)$. PSR representatives of the Heisenberg field operator $\hat{\varphi}(x)$ may serve as other interesting examples

$$\left.\begin{matrix}\langle\varphi|\hat{\varphi}(x)|\varphi\rangle\\ \Lambda^{-2}\langle\varphi|\hat{\varphi}(x)|\varphi\rangle\end{matrix}\right\}=e^{\pm\frac{\hbar}{4}\iint\frac{\delta}{\delta\phi}\Delta^{(1)}\frac{\delta}{\delta\phi}}\; e^{i\iint\frac{\delta}{\delta\phi}\Delta_{ret}\frac{\delta}{\delta\tilde{\Phi}}}\; e^{i\hbar^{-1}\int d^4x\left(\mathcal{H}_I(\phi+\frac{\hbar}{2}\tilde{\Phi})-\mathcal{H}_I(\phi-\frac{\hbar}{2}\tilde{\Phi})\right)}\phi(x)\Big|_{\tilde{\Phi}=0,\phi=\varphi} \qquad \begin{matrix}\text{(PSR-1)(19.a)}\\ \text{(PSR-3)(19.b)}\end{matrix}$$

$$\Lambda^{-1}\langle\varphi|\hat{\varphi}(x)|\varphi\rangle = e^{i\int\frac{\delta}{\delta\Phi}\Delta_{ret}\frac{\delta}{\delta\tilde{\Phi}}}\, e^{i\hbar^{-1}\int d^4x\left(\mathcal{H}_I(\phi+\frac{\hbar}{2}\tilde{\Phi})-\mathcal{H}_I(\phi-\frac{\hbar}{2}\tilde{\Phi})\right)}\phi(x)\Big|_{\substack{\tilde{\Phi}=0\\ \phi=\varphi}} \qquad \text{(PSR-2)}\ (20)$$

Thus, two functions $\Delta^{(1)}$ and Δ_{ret} take part in PSR-1 and PSR-3. PSR-1 (eq. (19a)) exactly corresponds to a Dyson double diagram (i.e. in terms of $\Delta^{(1)}$ and Δ_{ret}) N-ordering expansion for the Heisenberg operator $\hat{\varphi}(x)$ /13/. PSR-3 representative is also such a double diagram x). However in PSR-2 we have only "single diagram" expansion - only in terms of Δ_{ret}. Besides perturbation series eqs. (19)-(20) give power series in $\hbar$.

Operator representatives. Left and right operator representatives in PSR are defined as follows

$$\langle\varphi|\hat{\mathbb{F}}\,\hat{\mathbb{G}}|\varphi\rangle=\mathbb{F}_1^{\ell}\langle\varphi|\hat{\mathbb{G}}|\varphi\rangle=\mathbb{F}_1^{\ell}\mathbb{G}_1(\varphi)=\mathbb{G}_1^{r}\langle\varphi|\mathbb{F}|\varphi\rangle=\mathbb{G}_1^{r}\,\mathbb{F}_1(\varphi) \qquad \text{(PSR-1)}\ (21)$$

and similarly in any other representations (PSRs or not). The operators $\mathbb{F}^{\overset{\ell}{r}}$ $(\mathbb{G}^{r})$ are functional derivative operators. We have the relations

$$\mathbb{F}(\varphi)=\mathbb{F}^{\ell}\cdot 1=\mathbb{F}^{r}\cdot 1=\mathrm{Re}\,\mathbb{F}^{\overset{\ell}{r}}\cdot 1. \qquad (22)$$

The representatives $\mathbb{F}^{\ell}$ and $\mathbb{F}^{r}$ may be obtained in explicit forms by replacing in $\hat{\mathbb{F}}$ the operators $\hat{\varphi}(\vec{x})$, $\hat{\dot{\varphi}}(\vec{x})$ (or the interaction picture field operator $\hat{\varphi}(x)$) by their left and right representatives $\varphi^{\ell}(\vec{x})$, $\dot{\varphi}^{\ell}(\vec{x})$ (or $\varphi^{\ell}(x)$) and $\varphi^{r}(\vec{x})$, $\dot{\varphi}^{r}(\vec{x})$ (or $\varphi^{r}(x)$), respectively. The left representatives are multiplied in the same order as original operators, while the right representatives in the inverse order. The left representatives commute with the right ones (as a general rule for all associative theories xx)), and operators of one kind satisfy the usual commutation relations (except for the sign for r-representatives), namely, in the Schrödinger and interaction pictures

$$[\varphi^{\overset{\ell}{r}}(\vec{x}),\varphi^{\overset{\ell}{r}}(\vec{y})]=0,\ [\varphi^{\overset{\ell}{r}}(\vec{x}),\dot{\varphi}^{\overset{\ell}{r}}(\vec{y})]=\pm i\hbar\delta(\vec{x}-\vec{y}),\ [\dot{\varphi}^{\overset{\ell}{r}}(\vec{x}),\dot{\varphi}^{\overset{\ell}{r}}(\vec{y})]=0$$
$$[\varphi^{\overset{\ell}{r}}(x),\varphi^{\overset{\ell}{r}}(y)]=\pm i\hbar\Delta(x-y). \qquad (23)$$

The coordinate and momenta representatives are written explicitly as follows

x) In fact, the first exponential in eq. (19.a) ((19.b)) is one of representations of the operators Λ (Λ^{-1}) (of eqs. (7) and (9)). The same operators enter into eqs. (16.a) and (16.c) since $\Delta_{\pm}=\Delta_{sym}\pm\frac{i}{2}\Delta^{(1)}$.

xx) In nonassociative cases left and right representatives become noncommutative with each other (see ref. /18/).

$$\left.\begin{aligned}
\varphi^{\overset{\ell}{r}}(\vec{x})&=\varphi(\vec{x})+\frac{\hbar}{2}\int d^3y\,\Delta^{(1)}(\vec{x}-\vec{y},0)\frac{\delta}{\delta\dot{\varphi}(\vec{y})}\pm\frac{i\hbar}{2}\frac{\delta}{\delta\dot{\varphi}(\vec{x})}=\Lambda\left(\varphi(\vec{x})\pm\frac{i\hbar}{2}\frac{\delta}{\delta\dot{\varphi}(\vec{x})}\right)\Lambda^{-1},\\
\dot{\varphi}^{\overset{\ell}{r}}(\vec{x})&=\dot{\varphi}(\vec{x})-\frac{\hbar}{2}\int d^3y\,\dot{\Delta}^{(1)}(\vec{x}-\vec{y},0)\frac{\delta}{\delta\dot{\varphi}(\vec{y})}\mp\frac{i\hbar}{2}\frac{\delta}{\delta\varphi(\vec{x})}=\Lambda\left(\dot{\varphi}(\vec{x})\mp\frac{i\hbar}{2}\frac{\delta}{\delta\varphi(\vec{x})}\right)\Lambda^{-1},\\
\varphi^{\overset{\ell}{r}}(x)&=\varphi(x)\mp\hbar\int d^3y\,\Delta^{(\mp)}(x-y)\overleftrightarrow{\partial}{}_4^{y}\frac{\delta}{\delta J(y)}=\Lambda\left(\varphi(x)\pm\frac{i\hbar}{2}\frac{\delta}{\delta J(x)}\right)\Lambda^{-1},
\end{aligned}\right\}\ \begin{matrix}\text{(PSR-1)}\\(24)\end{matrix}$$

$$\left.\begin{aligned}
&\varphi^{\overset{\ell}{r}}(\vec{x})=\varphi(\vec{x})\pm\frac{i\hbar}{2}\frac{\delta}{\delta\dot{\varphi}(\vec{x})},\quad \dot{\varphi}^{\overset{\ell}{r}}(\vec{x})=\dot{\varphi}(\vec{x})\mp\frac{i\hbar}{2}\frac{\delta}{\delta\varphi(\vec{x})}, && \text{(a)}\\
&\varphi^{\overset{\ell}{r}}(x)=\varphi(x)\pm\frac{i\hbar}{2}\frac{\delta}{\delta J(x)}\qquad\left(\frac{\delta}{\delta J(x)}\equiv i\int d^3y\,\Delta(x-y)\overleftrightarrow{\partial}{}_4^{y}\frac{\delta}{\delta J(y)}\right), && \text{(b)}
\end{aligned}\right\}\ \begin{matrix}\text{(PSR-2)}\\(25)\end{matrix}$$

$$\left.\begin{aligned}
\varphi^{\overset{\ell}{r}}(\vec{x})&=\varphi(\vec{x})-\frac{\hbar}{2}\int d^3y\,\Delta^{(1)}(\vec{x}-\vec{y},0)\frac{\delta}{\delta\dot{\varphi}(\vec{y})}\pm\frac{i\hbar}{2}\frac{\delta}{\delta\dot{\varphi}(\vec{x})}=\Lambda^{-1}\left(\varphi(\vec{x})\pm\frac{i\hbar}{2}\frac{\delta}{\delta\dot{\varphi}(\vec{x})}\right)\Lambda,\\
\dot{\varphi}^{\overset{\ell}{r}}(\vec{x})&=\dot{\varphi}(\vec{x})+\frac{\hbar}{2}\int d^3y\,\ddot{\Delta}^{(1)}(\vec{x}-\vec{y},0)\frac{\delta}{\delta\dot{\varphi}(\vec{y})}\mp\frac{i\hbar}{2}\frac{\delta}{\delta\varphi(\vec{x})}=\Lambda^{-1}\left(\dot{\varphi}(\vec{x})\mp\frac{i\hbar}{2}\frac{\delta}{\delta\varphi(\vec{x})}\right)\Lambda\\
\varphi^{\overset{\ell}{r}}(x)&=\varphi(x)\mp\hbar\int d^3y\,\Delta^{(\pm)}(x-y)\overleftrightarrow{\partial}{}_4^{y}\frac{\delta}{\delta J(y)}=\Lambda^{-1}\left(\varphi(x)\pm\frac{i\hbar}{2}\frac{\delta}{\delta J(x)}\right)\Lambda.
\end{aligned}\right\}\ \begin{matrix}\text{(PSR-3)}\\(26)\end{matrix}$$

Equations of motion in PSRs. Let us translate into PSR's the usual quantum evolution laws of a density operator ("matrix") $\hat{\rho}(t)$ in the Schrödinger picture and any operator $\hat{\mathbb{F}}(t)$, which does not depend explicitly on time, in the Heisenberg picture

$$\frac{d}{dt}\hat{\rho}(t)=-i\hbar^{-1}[\hat{H},\hat{\rho}(t)],\qquad \hat{\rho}(t)=e^{-i\hbar^{-1}\hat{H}(t-t_0)}\hat{\rho}(t_0)\,e^{i\hbar^{-1}\hat{H}(t-t_0)} \qquad (27)$$

$$\frac{d}{dt}\hat{\mathbb{F}}(t)=i\hbar^{-1}[\hat{H},\hat{\mathbb{F}}(t)],\qquad \hat{\mathbb{F}}(t)=e^{i\hbar^{-1}\hat{H}(t-t_0)}\hat{\mathbb{F}}(t_0)\,e^{-i\hbar^{-1}\hat{H}(t-t_0)}. \qquad (28)$$

Then we come to the Liouville type equations as equations of motion and their formal solutions of the form

$$\frac{d}{dt}\rho(\varphi\dot{\varphi}t)=-\mathcal{L}\,\rho(\varphi\dot{\varphi}t),\qquad \rho(\varphi\dot{\varphi}t)=e^{-\mathcal{L}(t-t_0)}\rho(\varphi\dot{\varphi}t_0), \qquad (29)$$

$$\frac{d}{dt}\mathbb{F}(\varphi\dot{\varphi}t)=\mathcal{L}\,\mathbb{F}(\varphi\dot{\varphi}t),\qquad \mathbb{F}(\varphi\dot{\varphi}t)=e^{\mathcal{L}(t-t_0)}\mathbb{F}(\varphi\dot{\varphi}t_0), \qquad (30)$$

where $\rho(\varphi\dot{\varphi}t)=\langle\varphi|\hat{\rho}(t)|\varphi\rangle$ in PSR-1 $=\Lambda^{-1}\langle\varphi|\hat{\rho}(t)|\varphi\rangle$ in PSR-2, etc., and the Liouvillian $\mathcal{L}$ is x)

$$\mathcal{L}=i\hbar^{-1}(H^{\ell}-H^{r}). \qquad (31)$$

As initial and final values for $\hat{\rho}$ and ρ it is natural to assume

$$\hat{\rho}_{1i}=|\varphi_0\rangle\langle\varphi_0|,\qquad \hat{\rho}_{1f}=|\varphi\rangle\langle\varphi|$$
$$\rho_1(\varphi\dot{\varphi}t_0,\varphi_0\dot{\varphi}_0t_0)=\mathrm{Tr}(\hat{\rho}_{1f}\hat{\rho}_{1i})=|\langle\varphi|\varphi_0\rangle|^2 \quad \text{(PSR-1)} \quad \text{(Gauss functional) xx)} \qquad (32)$$

x) For examples of $\mathcal{L}$ see Appendix B.

xx) See Appendix A, eq. (A.6).

$$\hat\rho_{2i}=\Lambda_0^{-1}|\varphi_0\rangle\langle\varphi_0|\,,\quad \hat\rho_{2f}=\Lambda^{-1}|\varphi\rangle\langle\varphi|$$

$$\rho_2(\varphi\dot\varphi t_0,\varphi_0\dot\varphi_0 t_0)=\mathrm{Tr}(\hat\rho_{2f}\hat\rho_{2i})=\Lambda^{-1}\Lambda_0^{-1}|\langle\dot\varphi|\varphi_0\rangle|^2= \qquad \text{(PSR-2)}$$

$$=\prod_{\vec x}2\pi\hbar\,\delta(\varphi(\vec x)-\varphi_0(\vec x))\,\delta(\dot\varphi(\vec x)-\dot\varphi_0(\vec x)) \qquad (\text{functional } \delta\text{-function})^{x)} \qquad (33)$$

Now the evolution law may be written also as follows

$$\rho(\varphi\dot\varphi t,\varphi_0\dot\varphi_0 t_0)=e^{-\mathcal{L}(t-t_0)}\rho(\varphi\dot\varphi t_0,\varphi_0\dot\varphi_0 t_0)=e^{\mathcal{L}^0(t-t_0)}\rho(\varphi\dot\varphi t_0,\varphi_0\dot\varphi_0 t_0)$$

$$\frac{d}{dt}\rho(\varphi\dot\varphi t,\varphi_0\dot\varphi_0 t_0)=\mathcal{L}^0\rho(\varphi\dot\varphi t,\varphi_0\dot\varphi_0 t_0), \qquad (34)$$

where the Liouvillian $\mathcal{L}^0$ acts on the variables φ_0 and $\dot\varphi_0$.

To solve the Liouville equation methods may be used similar to those for amplitudes: direct solution in the Schrödinger picture, T - exponential (interaction picture, see eq. (40) and (43)), Green function, path integral methods, etc. Exact solutions of the Liouville equation can be easily found in linear cases (equations of motion for field coordinates are linear and hence the Hamiltonian is at most bilinear) like the free case, interactions with an external current or an external field (see below). However we give now a symbolic solution of the PSR-2 Liouville equation in a general case

$$\rho_2(\varphi\dot\varphi t,\varphi_0\dot\varphi_0 t_0)=\Lambda^{-1}\Lambda_0^{-1}|\langle\varphi|e^{-i\hbar^{-1}\hat H(t-t_0)}|\varphi_0\rangle|^2=$$

$$=e^{i\iint d^4x\,d^3y\,\frac{\delta}{\delta\tilde\Phi}\left[\Delta(x-y)\frac{\delta}{\delta\varphi'(\vec y,t_0)}+\dot\Delta(x-y)\frac{\delta}{\delta\dot\varphi'(\vec y,t_0)}\right]}\;e^{i\iint\frac{\delta}{\delta\Phi}\Delta_{ret}\frac{\delta}{\delta\tilde\Phi}}$$

$$e^{i\hbar^{-1}\int d^4x\left(\mathcal{H}_I(\phi+\frac{\hbar}{2}\tilde\Phi)-\mathcal{H}_I(\phi-\frac{\hbar}{2}\tilde\Phi)\right)}\Lambda'^{-1}\Lambda'^{-1}|\langle\varphi'|\varphi\rangle|^2\Bigg|_{\substack{\varphi'(\vec x,t_0)\to\varphi_0(\vec x,t)\\ \dot\varphi'(\vec x,t_0)\to\dot\varphi_0(\vec x,t)\\ \tilde\Phi\to 0,\ \phi\to\varphi_0}} = \qquad (35.a)$$

$$=e^{i\iint\frac{\delta}{\delta\Phi}\Delta_{ret}\frac{\delta}{\delta\tilde\Phi}}\;e^{i\hbar^{-1}\int d^4x\left(\mathcal{H}_I(\phi+\frac{\hbar}{2}\tilde\Phi)-\mathcal{H}_I(\phi-\frac{\hbar}{2}\tilde\Phi)\right)}$$

$$\left\{\Lambda^{-1}\Lambda'^{-1}|\langle\varphi'|\varphi\rangle|^2\Bigg|_{\substack{\varphi'(\vec x,t_0)\to\phi(\vec x,t)\\ \dot\varphi'(\vec x,t_0)\to\dot\phi(\vec x,t)}}\right\}\Bigg|_{\substack{\tilde\Phi\to 0\\ \phi\to\varphi_0}}, \qquad (35.b)$$

where $y_0=t$, $\dot\Delta(x-y)=\frac{\partial}{\partial y_0}\Delta(x-y)$. It is obtained, using eqs.(15.b) (for a derivation see Appendix C) and permits to get a perturbation theory expansion and also power series in $\hbar$.

As an example of equations of motion in the Heisenberg picture we give the equation for the field operator

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x) See Appendix A, eq. (4).

$$\varphi^{\ell}_{r}(x)=\varphi^{\ell}_{r}(x)+\int_{t_0}d^4y\,\Delta_{ret}(x-y)\,j(\varphi^{\ell}_{r}(x)). \qquad (36)$$

In particular, in PSR-2

$$\varphi^{\ell}_{r}(x)=\varphi(x)\pm\frac{i\hbar}{2}\frac{\delta}{\delta\mathcal{J}(x)}+\int_{t_0}d^4y\,\Delta_{ret}(x-y)\,j(\varphi^{\ell}_{r}(x)). \qquad (37)$$

These Yang-Feldman equations can be easily converted into the usual Lagrange (Newton) or Hamilton equations (for operators $\varphi^{\ell}_{r}(x)$) with initial conditions of the form (25.a).

Let us give in parallel the evolution laws of the operator representatives in terms of the Hamiltonians and Liouvillians

$$\mathbb{F}^{\ell}(t)=e^{i\hbar^{-1}H^{\ell}\tau}\mathbb{F}^{\ell}(t_0)e^{-i\hbar^{-1}H^{\ell}\tau} \qquad \mathbb{F}^{r}(t)=e^{-i\hbar^{-1}H^{r}\tau}\mathbb{F}^{r}(t_0)e^{i\hbar^{-1}H^{r}\tau}=$$

$$=e^{\mathcal{L}\tau}\mathbb{F}^{\ell}(t_0)e^{-\mathcal{L}\tau} \qquad =e^{\mathcal{L}\tau}\mathbb{F}^{r}(t_0)e^{-\mathcal{L}\tau}=$$

$$=U^{\ell-1}(t,t_0)\mathbb{F}^{\ell}(t)U^{\ell}(t,t_0)= \qquad =U^{r}(t,t_0)\mathbb{F}^{r}(t)U^{r-1}(t,t_0)=$$

$$=W^{-1}(t,t_0)\mathbb{F}^{\ell}(t)W(t,t_0)\,, \qquad =W^{-1}(t,t_0)\mathbb{F}^{r}(t)W(t,t_0)\,, \qquad (38)$$

where $\tau=t-t_0$, $\mathbb{F}^{\ell}_{r}(t)$ are the interaction picture representatives

$$\mathbb{F}^{\ell}_{r}(t)=e^{\pm i\hbar^{-1}H^{\ell}_{0r}\tau}\mathbb{F}^{\ell}_{r}(t_0)e^{\mp i\hbar^{-1}H^{\ell}_{0r}\tau}=e^{\mathcal{L}_0\tau}\mathbb{F}^{\ell}_{r}(t_0)e^{-\mathcal{L}_0\tau}, \qquad (39)$$

$$W(t,t_0)=U^{\ell}(t,t_0)U^{r-1}(t,t_0)=T\exp\left\{-\int_{t_0}^{t}dt'\,\mathcal{L}_I(t')\right\}, \qquad (40)$$

$$\mathcal{L}_0=i\hbar^{-1}(H^{\ell}_0-H^{r}_0)\,,\quad \mathcal{L}_I(t)=i\hbar^{-1}\int d^3x\left(\mathcal{H}_I(\varphi^{\ell}(x))-\mathcal{H}_I(\varphi^{r}(x))\right), \qquad (41)$$

$$\left[\frac{d}{dt}+i\hbar^{-1}\int d^3x\left(\mathcal{H}_I(\varphi^{\ell}(x))-\mathcal{H}_I(\varphi^{r}(x))\right)\right]W(t,t_0)=0. \qquad (42)$$

The latter is in fact the Liouville equation in the interaction picture, and, e.g.,

$$\rho_2(\varphi\dot\varphi t,\varphi_0\dot\varphi_0 t_0)=\Lambda^{-1}\Lambda_0^{-1}|\langle\varphi|e^{-i\hbar^{-1}\hat H(t-t_0)}|\varphi_0\rangle|^2=$$

$$=W_{(0)}^{-1}(t,t_0)\,\Lambda^{-1}\Lambda_0^{-1}|\langle\varphi|e^{-i\hbar^{-1}\hat H_0(t-t_0)}|\varphi_0\rangle|^2\,. \qquad (43)$$

<u>Products and commutators of operators in terms of PSRs.</u> Due to eq.(13)

$$\hat{\mathbb{F}}\hat{\mathbb{G}}=e^{\int d^3x\left[\hat\varphi(\vec x)\frac{\delta}{\delta\varphi'(\vec x)}+\hat{\dot\varphi}(\vec x)\frac{\delta}{\delta\dot\varphi'(\vec x)}\right]}\,e^{\int d^3y\left[\hat\varphi(\vec y)\frac{\delta}{\delta\varphi''(\vec y)}+\hat{\dot\varphi}(\vec y)\frac{\delta}{\delta\dot\varphi''(\vec y)}\right]}$$

$$\mathbb{F}_2(\varphi',\dot\varphi')\,\mathbb{G}_2(\varphi'',\dot\varphi'')\Big|_{\substack{\varphi'=\dot\varphi'=0\\ \varphi''=\dot\varphi''=0}}=$$

$$= e^{\int d^3x\left[\hat{\varphi}(\vec{x})\left(\frac{\delta}{\delta\varphi'(\vec{x})}+\frac{\delta}{\delta\varphi''(\vec{x})}\right)+\hat{\dot{\varphi}}(\vec{x})\left(\frac{\delta}{\delta\dot{\varphi}'(\vec{x})}+\frac{\delta}{\delta\dot{\varphi}''(\vec{x})}\right)\right]}$$

$$e^{\frac{i\hbar}{2}\int d^3x\left[\frac{\delta}{\delta\varphi'(\vec{x})}\frac{\delta}{\delta\dot{\varphi}''(\vec{x})}-\frac{\delta}{\delta\dot{\varphi}'(\vec{x})}\frac{\delta}{\delta\varphi''(\vec{x})}\right]}\mathbb{F}_2(\varphi',\dot{\varphi}')\mathbb{G}_2(\varphi'',\dot{\varphi}'')\Big|_{\substack{\varphi'=\dot{\varphi}'=0\\ \varphi''=\dot{\varphi}''=0}} \qquad (44)$$

$$\Lambda^{-1}\langle\varphi|\hat{\mathbb{F}}\hat{\mathbb{G}}|\varphi\rangle = e^{\frac{i\hbar}{2}\int d^3x\left[\frac{\delta}{\delta\varphi'(\vec{x})}\frac{\delta}{\delta\dot{\varphi}''(\vec{x})}-\frac{\delta}{\delta\dot{\varphi}'(\vec{x})}\frac{\delta}{\delta\varphi''(\vec{x})}\right]} \qquad (45.a)$$

$$\mathbb{F}_2(\varphi+\varphi',\dot{\varphi}+\dot{\varphi}')\mathbb{G}_2(\varphi+\varphi'',\dot{\varphi}+\dot{\varphi}'')\Big|_{\substack{\varphi'=\dot{\varphi}'=0\\ \varphi''=\dot{\varphi}''=0}}$$

$$=\mathbb{F}_2^{ord}\left(\varphi+\frac{i\hbar}{2}\frac{\delta}{\delta\dot{\varphi}},\dot{\varphi}-\frac{i\hbar}{2}\frac{\delta}{\delta\varphi}\right)\mathbb{G}_2(\varphi,\dot{\varphi})= \qquad (45.b)$$

$$=\mathbb{G}_2^{ord}\left(\varphi-\frac{i\hbar}{2}\frac{\delta}{\delta\dot{\varphi}},\dot{\varphi}+\frac{i\hbar}{2}\frac{\delta}{\delta\varphi}\right)\mathbb{F}_2(\varphi,\dot{\varphi}), \qquad (45.c)$$

where ord (ordered) indicates that in $\mathbb{F}_2^{ord}$ and $\mathbb{G}_2^{ord}$ all the derivatives $\frac{\delta}{\delta\varphi}$ and $\frac{\delta}{\delta\dot{\varphi}}$ are placed on the right of all φ and $\dot{\varphi}$. In eq. (45.a) $\mathbb{F}_2$ and $\mathbb{G}_2$ are corresponding classical quantities, and quantum mechanics means some modified rule of (nonassociative and noncommutative) multiplication of classical quantities (Yu.M.Schirokov/14/). A commutator of operators takes the form (the Moyal bracket/3/)

$$\Lambda^{-1}\langle\varphi|[\hat{\mathbb{F}}\hat{\mathbb{G}}]|\varphi\rangle = 2i\sin\left\{\frac{\hbar}{2}\int d^3x\left[\frac{\delta}{\delta\varphi'(\vec{x})}\frac{\delta}{\delta\dot{\varphi}''(\vec{x})}-\frac{\delta}{\delta\dot{\varphi}'(\vec{x})}\frac{\delta}{\delta\varphi''(\vec{x})}\right]\right\}$$

$$\mathbb{F}_2(\varphi+\varphi',\dot{\varphi}+\dot{\varphi}')\mathbb{G}_2(\varphi+\varphi'',\dot{\varphi}+\dot{\varphi}'')\Big|_{\substack{\varphi'=\dot{\varphi}'=0\\ \varphi''=\dot{\varphi}''=0}}. \qquad (46)$$

The product and the commutator contains reiteration of the Poisson brackets. Thus, the famous Dirac statement on connection between commutators and Poisson brackets finds its final realization in PSRs. It is this fact, that leads directly to conclusion that a classically integrable system (Poisson brackets of a complete set of quantities equal to zero) is also integrable in the quantum sense (commutators of corresponding complete set of operators equal to zero), and conversely./19/

Now the Liouville equation can be written via the classical Hamiltonian (as Moyal does in quantum mechanics)

$$\frac{d}{dt}\rho(\varphi\dot{\varphi}t) = \frac{2}{\hbar}\sin\left\{\frac{\hbar}{2}\int d^3x\left[\frac{\delta}{\delta\varphi'(\vec{x})}\frac{\delta}{\delta\dot{\varphi}''(\vec{x})}-\frac{\delta}{\delta\dot{\varphi}'(\vec{x})}\frac{\delta}{\delta\varphi''(\vec{x})}\right]\right\}$$

$$H(\varphi+\varphi',\dot{\varphi}+\dot{\varphi}')\rho(\varphi+\varphi'',\dot{\varphi}+\dot{\varphi}'')\Big|_{\substack{\varphi'=\dot{\varphi}'=0\\ \varphi''=\dot{\varphi}''=0}}. \qquad (47)$$

Product and commutator of operators in PSR-1 and PSR-3 are derived similarly, but look more complicated (see ref./15g/, Appendix).

Transition to the classical limit $\hbar = 0$ easily proceeds (formally) in PSRs[x]. The Liouville equation turns into a Liouville equation of the classical field theory of the same form(29),(30) and (34) with the classical Liouvillian

$$\mathcal{L}_c=\mathcal{L}\Big|_{\hbar=0}=i\hbar^{-1}(H^l-H^r)\Big|_{\hbar=0}=\sum_{\varphi}\int d^3\vec{x}\left(\frac{\delta H_c}{\delta\dot{\varphi}(\vec{x})}\frac{\delta}{\delta\varphi(\vec{x})}-\frac{\delta H_c}{\delta\varphi(\vec{x})}\frac{\delta}{\delta\dot{\varphi}(\vec{x})}\right) \qquad (48)$$

(summation is over all the participating fields), which now is a first order functional derivative operator (usual Poisson bracket). A solution of the classical Liouville equation with the initial condition (33) occurs to be

$$\rho_c(\varphi\dot{\varphi}t,\varphi_0\dot{\varphi}_0t) = \left.\begin{matrix} e^{\mathcal{L}_c^0(t-t_0)} \\ e^{-\mathcal{L}_c(t-t_0)}\end{matrix}\right\}\prod_{\vec{x}}2\pi\hbar\,\delta(\varphi(\vec{x})-\varphi_0(\vec{x}))\delta(\dot{\varphi}(\vec{x})-\dot{\varphi}_0(\vec{x}))=$$

$$=\prod_{\vec{x}}2\pi\hbar\,\delta(\varphi(\vec{x})-\varphi_0(\vec{x},t))\,\delta(\dot{\varphi}(\vec{x})-\dot{\varphi}_0(\vec{x},t))=$$

$$=\prod_{\vec{x}}2\pi\hbar\,\delta(\varphi(\vec{x},t_0-\tau)-\varphi_0(\vec{x}))\,\delta(\dot{\varphi}(\vec{x},t_0-\tau)-\dot{\varphi}_0(\vec{x}))=$$

$$\equiv\prod_{\vec{x}}2\pi\hbar\,\delta(\Phi(\vec{x},t_0)-\varphi_0(\vec{x}))\,\delta(\dot{\Phi}(\vec{x},t_0)-\dot{\varphi}_0(\vec{x})), \qquad (49)$$

where $\tau=t-t_0$,

$$\varphi_0(\vec{x},t)=\Lambda_0^{-1}\langle\varphi_0|\hat{\varphi}(\vec{x},t)|\varphi_0\rangle\Big|_{\hbar=0}=e^{\tau\mathcal{L}_c^0}\varphi_0(\vec{x})\cdot 1=e^{\tau\mathcal{L}_c^0}\varphi_0(\vec{x})e^{-\tau\mathcal{L}_c^0},$$

$$\dot{\varphi}_0(\vec{x},t)=\Lambda_0^{-1}\langle\varphi_0|\hat{\dot{\varphi}}(\vec{x},t)|\varphi_0\rangle\Big|_{\hbar=0}=e^{\tau\mathcal{L}^0}\dot{\varphi}_0(\vec{x})\cdot 1=e^{\tau\mathcal{L}_c^0}\dot{\varphi}_0(\vec{x})e^{-\tau\mathcal{L}_c^0},$$

$$\Phi(\vec{x},t_0)=\Lambda^{-1}\langle\varphi|\hat{\Phi}(\vec{x},t_0)|\varphi\rangle\Big|_{\hbar=0}=$$

$$=\varphi(\vec{x},t_0-\tau)=\Lambda^{-1}\langle\varphi|\hat{\varphi}(\vec{x},t_0-\tau)|\varphi\rangle_{\hbar=0}=e^{-\tau\mathcal{L}_c}\varphi(\vec{x})\cdot 1=e^{-\tau\mathcal{L}_c}\varphi(\vec{x})e^{\tau\mathcal{L}_c},$$

$$\dot{\Phi}(\vec{x},t_0)=\Lambda^{-1}\langle\varphi|\hat{\dot{\Phi}}(\vec{x},t_0)|\varphi\rangle\Big|_{\hbar=0}=$$

$$=\dot{\varphi}(\vec{x},t_0-\tau)=\Lambda^{-1}\langle\varphi|\hat{\dot{\varphi}}(\vec{x},t_0-\tau)|\varphi\rangle\Big|_{\hbar=0}=e^{-\tau\mathcal{L}_c}\dot{\varphi}(\vec{x})\cdot 1=e^{-\tau\mathcal{L}_c}\dot{\varphi}(\vec{x})e^{\tau\mathcal{L}_c}, \qquad (50)$$

($\hat{\Phi}(\vec{x},x_0)=e^{i\hbar^{-1}\hat{H}(x_0-t)}\hat{\varphi}(\vec{x})e^{-i\hbar^{-1}\hat{H}(x_0-t)}$ is an analog of the Heisenberg field operator) are, in turn, particular solutions of the classical Liouville equation

$$\frac{d}{dt}\varphi_0(\vec{x},t)=\mathcal{L}_c^0\varphi_0(\vec{x},t),\ \varphi_0(\vec{x},t_0)=\varphi_0(\vec{x}),\ \frac{d}{dt}\dot{\varphi}_0(\vec{x},t)=\mathcal{L}_c^0\dot{\varphi}_0(\vec{x},t),\ \dot{\varphi}_0(\vec{x},t_0)=\dot{\varphi}_0(\vec{x}) \qquad (51)$$

and a general Cauchy problem solution of the Hamilton equations

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x) We keep fixed $\hbar$'s incoming into Compton wave lengths and coupling constants (cf./13/).

$$\frac{d}{dt}\varphi_0(\vec{x},t)=\frac{\delta H_c[\varphi_0,\dot{\varphi}_0]}{\delta\dot{\varphi}_0(\vec{x},t)}\ ,\qquad \frac{d}{dt}\dot{\varphi}_0(\vec{x},t)=-\frac{\delta H_0[\varphi_0,\dot{\varphi}_0]}{\delta\varphi_0(\vec{x},t)} \qquad (52)$$

and, therefore, of Lagrange (Newton) equations

$$(\Box-m^2)\varphi_0(x)=-j(\varphi_0(x)) \qquad (53)$$

(or, equivalently, of the integral equation

$$\varphi_0(x)=\varphi_0(x)+\int_{t_0} d^4y\,\Delta_{ret}(x-y)\,j(\varphi_0(x))\,) \qquad (54)$$

of classical field theory. They ($\varphi_0(\vec{x},t)$ and $\dot{\varphi}_0(\vec{x},t)$) are characteristics of the classical Liouville equation, in terms of which any its solution can be expressed$^{x)}$: with an arbitrary initial function

$$\rho(\varphi\dot{\varphi}t,\varphi_0\dot{\varphi}_0t_0)=e^{\tau L_c^0}\rho(\varphi\dot{\varphi}t_0,\varphi_0\dot{\varphi}_0t_0)=\rho(\varphi\dot{\varphi}t_0,\varphi_0(\vec{x},t),\dot{\varphi}_0(\vec{x},t),t_0). \qquad (55)$$

Transition to $\hbar=0$ can be easily performed in many other PSR formulas, e.g., in eqs. (19), (20), (35), (42) and in others, thus generating corresponding classical expressions.

Equations (50), except for the last expressions, and (51) are valid for $h\neq0$ too, unlike in general eqs. (49) and (52)-(55) (cf. ref./15g/).

Causality. In PSR-2 linear cases (see the definition above) the Planck constant $\hbar$ falls out and all is like in the classics: the same Liouvillian, Liouville equation and its solutions of form (49), where, e.g., in the free case

$$\varphi_0(x)=\langle\varphi_0|\hat{\varphi}(x)|\varphi_0\rangle=\varphi_0(x)=i\int_{y_0=t_0} d^3y\,\Delta(x-y)\overleftrightarrow{\partial_4}\varphi_0(y)\,, \qquad (56)$$

in the case of interaction with an external current

$$\varphi_0(x)=\varphi_0(x)+\int_{t_0} d^4y\,\Delta_{ret}(x-y)\,j^e(y)\,, \qquad (57)$$

and in the theory of interaction with an external field

$$\rho_2(\psi\overline{\psi}t,\psi_0\overline{\psi}_0t_0)=\prod_{\vec{x}}(2\pi\hbar)^4\delta(\psi(\vec{x})-\psi_0(\vec{x},t))\,\delta(\overline{\psi}(\vec{x})-\overline{\psi}_0(\vec{x},t)), \qquad (58.a)$$

$$\psi_0(x)=\psi_0(x)-ie\int_{t_0} d^4y\,S_{ret}(x-y)\,\gamma_\mu A^e_\mu\psi_0(x)\,, \qquad (58.b)$$

$$\psi_0(x)=i\int_{y_0=t_0} d^3y\,S^A_{ret}(x-y)\gamma_4\psi_0(y)\qquad (\psi_0(x)=\Lambda_0^{-1}\langle\psi_0|\hat{\psi}(x)|\psi_0\rangle). \qquad (58.c)$$

In the linear cases the Planck constant falls out also from eqs. (20) and (35), and they correspond exactly to the classics and demonstrate (like eqs. (49), (56)-(58)) the causality inherent to the classics. Moreover these eqs. (20) and (35) demonstrate that the causality takes place in PSR-2 in general: they contain only the Δ_{ret}-"functions", although in a general (nonlinear) case the Planck constant does not fall out and the Liouville equation and its solutions have another nature, than in the classics.

x) The same is true for any $\mathbb{F}(\varphi,\dot{\varphi})$ in the Heisenberg picture.

Note that in PSR-2 even the S-matrix becomes explicitly causal (in any case more causal). According to eq. (16a) it is constructed out only of the Δ_{sym}-"functions", which equal zero outside the light cone. Similar construction encountered in the Wheeler-Feynman action-at-a-distance theory/20/.

These causal properties in PSR-2 seem to be in agreement with the famous causality condition of Bogolubov/21/.

Let us stress that it is just this operator Λ^{-1} that eliminates completely acausality (owing to $\Delta^{(1)}$'s) from PSR-1 and leads to PSR-2 (Λ does the same with PSR-3). See also Appendix C .

In PSR-1 in the free case and in the case of interaction with an external current the phase space density $\rho_1(\varphi\dot{\varphi}t,\varphi_0\dot{\varphi}_0t_0)$ is constructed out like in PSR-2, i.e., according to eq. (55). A distinction is only in the initial functional $\rho_1(\varphi\dot{\varphi}t_0,\varphi_0\dot{\varphi}_0t_0)$ (32), which is explicitly given by eq. (A.6). Thus, in these cases $\rho_1(\varphi\dot{\varphi}t,\varphi_0\dot{\varphi}_0t_0)$ is given by eq. (A.6) with the substitution $\varphi'(\vec{x})\equiv\varphi'(\vec{x},t_0)\longrightarrow\varphi_0(\vec{x},t)$.

Note that the derivatives $\frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_{2n})}\frac{|\varphi\rangle\langle\varphi|}{|\langle\varphi|0\rangle|^2}$ and

$$\frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_{2n})}\frac{\delta}{\delta J_0(y_1)}\cdots\frac{\delta}{\delta J_0(y_{2m})}\frac{|\langle\varphi|e^{-i\hbar^{-1}\hat{H}(t-t_0)}|\varphi_0\rangle|^2}{|\langle\varphi|0\rangle|^2\ |\langle\varphi_0|0\rangle|^2} \qquad (59)$$

after suitable projections give the n-quantum states $:\hat{\varphi}(x_1)\ldots\hat{\varphi}(x_n):|0\rangle\langle0|:\hat{\varphi}(x_{n+1})\cdots\hat{\varphi}(x_{2n}):$ and probabilities for transitions between such states. Here

$$\rho_1(\varphi\dot{\varphi}t,\varphi_0\dot{\varphi}_0t_0)=|\langle\varphi|e^{-i\hbar^{-1}\hat{H}(t-t_0)}|\varphi_0\rangle|^2=\Lambda\Lambda_0\rho_2(\varphi\dot{\varphi}t,\varphi_0\dot{\varphi}_0t_0) \qquad (60)$$

is the phase space density in PSR-1. Note that the real phase space densities in PSRs containe the same complete information as the complex amplitude. In PSRs it is no need to turn to complex amplitudes.

Appendix A. Using eqs. (6) for an operator $\hat{\mathbb{F}}$ we can express via the PSR representatives any matrix elements of $\hat{\mathbb{F}}$, e.g., the elements between n-particle states

$$\langle0|(:\hat{\varphi}(x_1)\cdots\hat{\varphi}(x_n):)\hat{\mathbb{F}}(:\hat{\varphi}(y_1)\cdots\hat{\varphi}(y_m):)|0\rangle$$

or between coherent states

$$\langle\varphi_2|\hat{\mathbb{F}}|\varphi_1\rangle=\mathbb{F}_1(\varphi_{21})\langle\varphi_2|\varphi_1\rangle \qquad \text{(CSR)}\ \ (A.1)$$

(for φ_{21} see eq. (14)). If we take matrix elements $\langle\varphi|\hat{\mathbb{F}}|\varphi\rangle$, $\Lambda^{-1}\langle\varphi|\hat{\mathbb{F}}|\varphi\rangle$ and $\Lambda^{-2}\langle\varphi|\hat{\mathbb{F}}|\varphi\rangle$, we obtain from eqs. (6) relations between PSR-1, PSR-2 and PSR-3

$$\mathbb{F}_1(\varphi)=\int\delta^2\varphi'\,\Lambda'^{-2}|\langle\varphi|\varphi'\rangle|^2\,\mathbb{F}_1(\varphi')=\int\delta J'\,\Lambda'^{-2}|\langle\varphi|\varphi'\rangle|^2\,\mathbb{F}_1(\varphi')=$$

$$=\Lambda\mathbb{F}_2(\varphi)=\int\delta^2\varphi'\,\Lambda'^{-1}|\langle\varphi|\varphi'\rangle|^2\,\mathbb{F}_2(\varphi')=\int\delta J'\,\Lambda'^{-1}|\langle\varphi|\varphi'\rangle|^2\,\mathbb{F}_2(\varphi')=$$

$$=\Lambda^2\mathbb{F}_3(\varphi)=\int\delta^2\varphi'\,|\langle\varphi|\varphi'\rangle|^2\,\mathbb{F}_3(\varphi')=\int\delta J'\,|\langle\varphi|\varphi'\rangle|^2\,\mathbb{F}_3(\varphi'), \qquad (\text{A.2})$$

$$\mathbb{F}_2(\varphi)=\Lambda\mathbb{F}_3(\varphi)=\int\delta^2\varphi'\,\Lambda^{-1}|\langle\varphi|\varphi'\rangle|^2\,\mathbb{F}_3(\varphi')=\int\delta J'\,\Lambda^{-1}|\langle\varphi|\varphi'\rangle|^2\,\mathbb{F}_3(\varphi'). \qquad (\text{A.3})$$

The first expression of eq. (A.2) demonstrates, that

$$\Lambda^{-2}|\langle\varphi|\varphi'\rangle|^2=\Lambda^{-1}\Lambda'^{-1}|\langle\varphi|\varphi'\rangle|^2=\Lambda'^{-2}|\langle\varphi|\varphi'\rangle|^2=\prod_{\vec{x}}2\pi\hbar\,\delta(\varphi(\vec{x})-\varphi'(\vec{x}))\,\delta(\dot\varphi(\vec{x})-\dot\varphi'(\vec{x})) \qquad (\text{A.4})$$

is a functional δ-function (see eq. (33)). Others show, that $\mathbb{F}_1(\varphi)$, $\mathbb{F}_2(\varphi)$ and $\mathbb{F}_3(\varphi)$ are related by Gauss type transformations (having been earlier symbolically denoted by Λ or Λ^2). This is clear from the formulas x)

$$\langle\varphi|\varphi'\rangle=\exp\{i(2\hbar)^{-1}(\varphi'^{(+)},\varphi'^{(-)})+i(2\hbar)^{-1}(\varphi^{(+)},\varphi^{(-)})-i\hbar^{-1}(\varphi^{(+)},\varphi'^{(-)})\}=$$

$$=\exp\{-i(4\hbar)^{-1}\int d^3x\,(\varphi^{(1)}(x)-\varphi'^{(1)}(x))\overleftrightarrow{\partial}_4(\varphi(x)-\varphi'(x))+(2\hbar)^{-1}\int d^3x\,\varphi(x)\overleftrightarrow{\partial}_4\varphi'(x)\}=$$

$$=\exp\{-(4\hbar)^{-1}\int d^4x\,d^4y\,(J(x)-J'(x))\Delta^{(1)}(x-y)(J(y)-J'(y))+i(2\hbar)^{-1}\int d^4x\,d^4y\,J(x)\Delta(x-y)J'(y)\} \qquad (\text{A.5})$$

$$|\langle\varphi|\varphi'\rangle|^2=\exp\{-(2\hbar)^{-1}i\int d^3x\,(\varphi^{(1)}(x)-\varphi'^{(1)}(x))\overleftrightarrow{\partial}_4(\varphi(x)-\varphi'(x))\}=$$

$$=\exp\{-(2\hbar)^{-1}\int d^4x\,d^4y\,(J(x)-J'(x))\Delta^{(1)}(x-y)(J(y)-J'(y))\}=$$

$$=\exp\{-(2\hbar)^{-1}\int d^3x\,d^3y\,[-\ddot\Delta^{(1)}(\vec{x}-\vec{y},0)(\varphi(\vec{x})-\varphi'(\vec{x}))(\varphi(\vec{y})-\varphi'(\vec{y}))+$$

$$+\Delta^{(1)}(\vec{x}-\vec{y},0)(\dot\varphi(\vec{x})-\dot\varphi'(\vec{y}))(\dot\varphi(\vec{y})-\dot\varphi'(\vec{y}))\,]\} \qquad (\text{A.6})$$

$$\Lambda^{-1}|\langle\varphi|\varphi'\rangle|^2=\Lambda'^{-1}|\langle\varphi|\varphi'\rangle|^2=2^{\infty}\exp\{-\hbar^{-1}\ldots\}=\ldots \qquad (\text{A.7})$$

where ... stand for the same expressions as in eq. (A.6), but with only two changes: $(2\hbar)^{-1}\to\hbar^{-1}$ (squaring the exponentials) and the normalizing factor 2^{∞} appears. The bilinear form in the exponents of eq. (A.6) and (A.7) is diagonalized by a Fourier transformation

$$\int d^4x\,d^4y\,J(x)\Delta^{(1)}(x-y)J(x)=\int d^4k\,J^*(k)\,\delta(k^2+m^2)\,J(k) \qquad (\text{A.8})$$

thus proving to be positive definite. Therefore eqs. (A.6) and (A.7) are negative real Gaussian functionals.

Let us give other useful representations of the operator Λ : in the scalar field theory

x) $(\varphi^{(+)},\varphi^{(-)})=i\int d^3x\,(\partial_4\varphi^{(+)}(x)\varphi^{(-)}(x)-\varphi^{(+)}(x)\partial_4\varphi^{(-)}(x))=i\int d^3x\,\varphi^{(+)}(x)\overleftrightarrow{\partial}_4\varphi^{(-)}(x).$

$$\Lambda=\exp\left(\frac{\hbar}{4}\int d^3x\,d^3y\,\frac{\delta}{\delta J(x)}\overleftrightarrow{\partial}{}^x_4\,\Delta^{(1)}(x-y)\overleftrightarrow{\partial}{}^y_4\,\frac{\delta}{\delta J(y)}\right)$$

$$=\exp\left[\frac{\hbar}{4}\int d^3x\,d^3y\left(\Delta^{(1)}(\vec{x}-\vec{y},0)\frac{\delta}{\delta\varphi(\vec{x})}\frac{\delta}{\delta\varphi(\vec{y})}-\ddot\Delta^{(1)}(\vec{x}-\vec{y},0)\frac{\delta}{\delta\dot\varphi(\vec{x})}\frac{\delta}{\delta\dot\varphi(\vec{y})}\right)\right] \qquad (\text{A.9})$$

(the times x_0 and y_0 are arbitrary, $\varphi(\vec{x})\equiv\varphi(\vec{x},t_0)$ and $\ddot\Delta^{(1)}(\vec{x}-\vec{y},0)=\left(\frac{\partial}{\partial x_0}\right)^2\Delta^{(1)}$); in the electrodynamics /15c,16/

$$\Lambda_A=\exp\left(\frac{\hbar}{4}\int d^3x\,d^3y\,\frac{\delta}{\delta J_\mu(x)}\overleftrightarrow{\partial}{}^x_4\Delta^{(1)}(x-y)\overleftrightarrow{\partial}{}^y_4\frac{\delta}{\delta J_\mu(y)}\right) \qquad (\text{A.10})$$

$$\Lambda_\psi=\exp\left(-\frac{\hbar}{2}\int d^3x\,d^3y\,\frac{\delta}{\delta\eta(x)}\gamma_4 S^{(1)}(x-y)\gamma_4\frac{\delta}{\delta\bar\eta(y)}\right). \qquad (\text{A.11})$$

Let us stress that Λ (and Λ^{-1}) acts in fact on c-number canonical variables, e.g., for a scalar field on functions $\varphi(\vec{x},t_0)$ and $\dot\varphi(\vec{x},t_0)$ or on J(x) (see eqs. (A.2) and (A.9)).

Once knowing eqs. (A.6) and (A.7) we conclude (according to eqs. (13)) immediately that

$$|\varphi\rangle\langle\varphi|=:\exp\{-(2\hbar)^{-1}i\int d^3x\,(\hat\varphi^{(1)}(x)-\varphi^{(1)}(x))\overleftrightarrow{\partial}_4(\hat\varphi(x)-\varphi(x))\}:=$$

$$=e^{-(2\hbar)^{-1}\int d^4x\,d^4y\,J(x)\Delta^{(1)}(x-y)J(y)}:e^{-(2\hbar)^{-1}i\int d^3x\,\hat\varphi^{(1)}(x)\overleftrightarrow{\partial}_4\hat\varphi(x)-\hbar^{-1}\int d^4x\,J(x)\hat\varphi^{(1)}(x)}:$$

$$=:\exp\{-(2\hbar)^{-1}\int d^3x\,d^3y\,[-\ddot\Delta^{(1)}(\vec{x}-\vec{y},0)(\hat\varphi(\vec{x})-\varphi(\vec{x}))(\hat\varphi(\vec{y})-\varphi(\vec{y}))+$$

$$+\Delta^{(1)}(\vec{x}-\vec{y},0)(\dot{\hat\varphi}(\vec{x})-\dot\varphi(\vec{x}))(\dot{\hat\varphi}(\vec{y})-\dot\varphi(\vec{y}))]\}:=$$

$$=2^{\infty}\,\mathrm{sym}\,\exp\{-\hbar^{-1}i\int d^3x\,(\hat\varphi^{(1)}(x)-\varphi^{(1)}(x))\overleftrightarrow{\partial}_4(\hat\varphi(x)-\varphi(x))\}=\ \ldots \qquad (\text{A.12})$$

(the ... stand for other expressions which are also constructed as above, but the exponentials are squared and the normalizing factor 2^{∞} appears),

$$\Lambda^{-1}|\varphi\rangle\langle\varphi|=2^{\infty}:\exp\{-i\hbar^{-1}\int d^3x\,(\hat\varphi^{(1)}(x)-\varphi^{(1)}(x))\overleftrightarrow{\partial}_4(\hat\varphi(x)-\varphi(x))\}:=\ldots \qquad (\text{A.13})$$

(the meaning of ... is as above). In particular

$$|0\rangle\langle 0|=:e^{-i(2\hbar)^{-1}\int d^3x\,\hat\varphi^{(1)}(x)\overleftrightarrow{\partial}_4\hat\varphi(x)}:=:e^{-\hat N}:=2^{\infty}\,\mathrm{sym}\,(e^{-2\hat N}) \qquad (\text{A.14})$$

$$\left(\Lambda^{-1}|\varphi\rangle\langle\varphi|\right)_{\varphi=0}=2^{\infty}:e^{-2\hat N}: \qquad (\text{A.15})$$

where

$$\hat N=\frac{i}{2\hbar}\int d^3x\,\hat\varphi^{(1)}(x)\overleftrightarrow{\partial}_4\hat\varphi(x)=\hbar^{-1}\int d^3x\,\hat\varphi^{(+)}(x)\overleftrightarrow{\partial}_4\hat\varphi^{(-)}(x) \qquad (\text{A.16})$$

are possible presentations of the operator of number of quanta (in passing note that they demonstrate a nonlocality of $\hat N$)

Using eqs. (A.10) and (A.11) the second and third lines of eq. (6) may be also written down as follows

$$\hat{\mathbb{F}} = \mathcal{Z}^\infty \int \delta^2\varphi : \exp\{-i\hbar^{-1}\int d^3x (\hat\varphi^{(1)}(x) - \varphi^{(1)}(x))\overleftrightarrow{\partial_4}(\hat\varphi(x)-\varphi(x))\} : \mathbb{F}_2(\varphi) =$$

$$= \mathcal{Z}^\infty : e^{-i\hbar^{-1}\int d^3x\, \hat\varphi^{(1)}(x)\overleftrightarrow{\partial_4}\hat\varphi(x)} \int \delta J\, e^{-2\hbar^{-1}\int d^4x\, \hat\varphi^{(1)}(x)J(x)} :$$

$$\cdot\, e^{-\hbar^{-1}\int d^4x\, d^4y\, J(x)\Delta^{(1)}(x-y)J(y)}\, \mathbb{F}_2(\varphi) = \qquad \text{(A.17.a)}$$

$$= \int\delta^2\varphi : \exp\{-i(2\hbar)^{-1}\int d^3x(\hat\varphi^{(1)}(x)-\varphi^{(1)}(x))\overleftrightarrow{\partial_4}(\hat\varphi(x)-\varphi(x))\} : \mathbb{F}_3(\varphi) =$$

$$= : e^{-i(2\hbar)^{-1}\int d^3x\,\hat\varphi^{(1)}(x)\overleftrightarrow{\partial_4}\hat\varphi(x)} \int\delta J\, e^{-\hbar^{-1}\int d^4x\,\hat\varphi^{(1)}(x)J(x)} :$$

$$\cdot\, e^{-(2\hbar)^{-1}\int d^4x\, d^4y\, J(x)\Delta^{(1)}(x-y)J(y)}\,\mathbb{F}_3(\varphi) = \qquad \text{(A.17.b)}$$

$$= \mathcal{Z}^\infty\int\delta^2\varphi\, \mathrm{sym}\left(\exp\{-i\hbar^{-1}\int d^3x(\hat\varphi^{(1)}(x)-\varphi^{(1)}(x))\overleftrightarrow{\partial_4}(\hat\varphi(x)-\varphi(x))\}\right)\mathbb{F}_3(\varphi)=$$

$$= \mathcal{Z}^\infty\, \mathrm{sym}\left(e^{-i\hbar^{-1}\int d^3x\,\hat\varphi^{(1)}(x)\overleftrightarrow{\partial_4}\hat\varphi(x)}\int\delta J\, e^{-2\hbar^{-1}\int d^4x\,\hat\varphi^{(1)}(x)J(x)}\right)$$

$$\cdot\, e^{-\hbar^{-1}\int d^4x\, d^4y\, J(x)\Delta^{(1)}(x-y)J(y)}\,\mathbb{F}_3(\varphi), \qquad \text{(A.18)}$$

where

$$\mathcal{Z}^\infty\delta^2\varphi = \prod_{\vec x}(\pi\hbar)^{-1}\delta\varphi(\vec x)\delta\dot\varphi(\vec x) \qquad \text{(A.19)}$$

Note useful forms of eqs. (13)

$$\hat{\mathbb F} = :\mathbb F_1(\hat\varphi): = : e^{\int d^3x\left(\hat\varphi(\vec x,t_0)\frac{\delta}{\delta\varphi(\vec x,t_0)} + \dot{\hat\varphi}(\vec x,t_0)\frac{\delta}{\delta\dot\varphi(\vec x,t_0)}\right)} : \mathbb F_1(\varphi)\Big|_{\substack{\varphi(\vec x,t_0)=0\\ \dot\varphi(\vec x,t_0)=0}} =$$

$$= \sum_{n=0}^\infty \frac{i^n}{n!}\int d^3y_1\dots d^3y_n : \hat\varphi(y_1)\dots\hat\varphi(y_n): \overleftrightarrow{\partial}_{14}\dots\overleftrightarrow{\partial}_{n4}\frac{\delta}{\delta J(y_1)}\cdots\frac{\delta}{\delta J(y_n)}\mathbb F_1(\varphi)\Big|_{J=0} =$$

$$= : \exp\left(i\int d^3y\,\hat\varphi(y)\overleftrightarrow{\partial_4}\frac{\delta}{\delta J(y)}\right) : \mathbb F_1(\varphi)\Big|_{J=0} = \qquad \text{(A.20)}$$

$$= \mathrm{sym}\,\mathbb F_2(\hat\varphi) = e^{\int d^3x\left(\hat\varphi(\vec x,t_0)\frac{\delta}{\delta\varphi(\vec x,t_0)} + \dot{\hat\varphi}(\vec x,t_0)\frac{\delta}{\delta\dot\varphi(\vec x,t_0)}\right)}\,\mathbb F_2(\varphi)\Big|_{\substack{\varphi(\vec x,t_0)=0\\ \dot\varphi(\vec x,t_0)=0}} =$$

$$= \sum_{n=0}^\infty \frac{i^n}{n!}\int d^3y_1\dots d^3y_n\, \mathrm{sym}(\hat\varphi(y_1)\dots\hat\varphi(y_n))\overleftrightarrow{\partial}_{14}\dots\overleftrightarrow{\partial}_{n4}\frac{\delta}{\delta J(y_1)}\cdots\frac{\delta}{\delta J(y_n)}\mathbb F_2(\varphi)\Big|_{J=0} =$$

$$= \mathrm{sym}\left(\exp\left(i\int d^3y\,\hat\varphi(y)\overleftrightarrow{\partial_4}\frac{\delta}{\delta J(y)}\right)\right)\mathbb F_2(\varphi)\Big|_{J=0} = \qquad \text{(A.21)}$$

$$= :\mathbb F_3(\hat\varphi): = : e^{\int d^3x\left(\hat\varphi(\vec x,t_0)\frac{\delta}{\delta\varphi(\vec x,t_0)} + \dot{\hat\varphi}(\vec x,t_0)\frac{\delta}{\delta\dot\varphi(\vec x,t_0)}\right)} : \mathbb F_3(\varphi)\Big|_{\substack{\varphi(\vec x,t_0)=0\\ \dot\varphi(\vec x,t_0)=0}} =$$

$$= \sum_{n=0}^\infty \frac{i^n}{n!}\int d^3y_1\dots d^3y_n : \hat\varphi(y_1)\dots\hat\varphi(y_n): \overleftrightarrow{\partial}_{14}\dots\overleftrightarrow{\partial}_{n4}\frac{\delta}{\delta J(y_1)}\cdots\frac{\delta}{\delta J(y_n)}\mathbb F_3(\varphi)\Big|_{J=0} =$$

$$= :\exp\left(i\int d^3y\,\hat\varphi(y)\overleftrightarrow{\partial_4}\frac{\delta}{\delta J(y)}\right): \mathbb F_3(\varphi)\Big|_{J=0}. \qquad \text{(A.22)}$$

From the three last expressions (6) it is easy to find relations between $\tilde{\mathbb F}_1(\varphi)$, $\tilde{\mathbb F}_2(\varphi)$ and $\tilde{\mathbb F}_3(\varphi)$. Namely, the exponential factors can be transformed into each other, using the Baker-Campbell-Hausdorff formula $e^A e^B = e^{A+B} e^{\frac12[A,B]}$ what leads to Gaussian factors, and we obtain

$$\tilde{\mathbb F}_1(\varphi) = e^{i(2\hbar)^{-1}(\varphi^{(+)},\varphi^{(-)})}\tilde{\mathbb F}_2(\varphi) = e^{-i(4\hbar)^{-1}(\varphi^{(1)},\varphi)}\tilde{\mathbb F}_2(\varphi)$$
$$\tilde{\mathbb F}_3(\varphi) = e^{-i(2\hbar)^{-1}(\varphi^{(+)},\varphi^{(-)})}\tilde{\mathbb F}_2(\varphi) = e^{i(4\hbar)^{-1}(\varphi^{(1)},\varphi)}\tilde{\mathbb F}_2(\varphi) \qquad \text{(A.23)}$$

$$(-i\hbar^{-1}(\varphi^{(+)},\varphi^{(-)}) = i(2\hbar)^{-1}(\varphi^{(1)},\varphi) = (2\hbar)^{-1}\int d^4x\, d^4y\, J(x)\Delta^{(1)}(x-y)J(y)) \qquad \text{(A.24)}$$

Relations inverse to eqs. (10)-(12) are

$$\mathbb F_j(\varphi) = \int\delta^2\varphi'\, e^{i(\varphi,\varphi')}\tilde{\mathbb F}_j(\varphi') = \int\delta J'\, e^{-i\hbar^{-1}\varphi\cdot J'}\tilde{\mathbb F}_j(\varphi') \quad (j=1,2,3). \qquad \text{(A.25)}$$

Let us give some notations adopted. The positive- and negative-frequency parts are defined with the Schrödinger equations

$$\partial_4\varphi^{(\mp)}(x) = \mp\sqrt{-\Delta+m^2}\,\varphi^{(\mp)}(x) \qquad (x_4 = it), \qquad \text{(A.26)}$$

and we can split into those as follows

$$\varphi^{(\mp)}(x) = \pm\frac{1}{2\pi i}\int\frac{1}{t-s\mp i\varepsilon}\varphi(\vec x,s) = \qquad (\varepsilon\to 0_+)$$

$$= i\int d^3x'\,\Delta^{(\mp)}(x-x')\overleftrightarrow{\partial'_4}\varphi(x') = \left(=-\int d^4y\,\Delta^{(\mp)}(x-y)J(y)\right)$$

$$= \frac12\left(1\mp\frac{\partial_4}{H}\right)\varphi(x) = \frac12\varphi(x)\mp\frac{i}{2}\varphi^{(1)}(x) \qquad (H=\sqrt{-\Delta+m^2}), \qquad \text{(A.27)}$$

$$\varphi^{(1)}(x) = \frac1\pi P\int\frac{ds}{t-s}\varphi(\vec x,s) =$$

$$= i\int d^3x'\,\Delta^{(1)}(x-x')\overleftrightarrow{\partial'_4}\varphi(x') = \left(=-\int d^4y\,\Delta^{(1)}(x-y)J(y)\right)$$

$$= \frac{\partial_4}{H}\varphi(x) = i(\varphi^{(-)}(x)-\varphi^{(+)}(x)). \qquad \text{(A.28)}$$

(The Hilbert transformation is, in fact, the operator of sign of energy). In particular,

$$\Delta^{(\mp)}(x) = \pm\frac{1}{2\pi i}\int\frac{ds}{t-s\mp i\varepsilon}\Delta(\vec x,s), \quad \Delta^{(1)}(x) = \frac1\pi P\int\frac{ds}{t-s}\Delta(\vec x,s). \qquad \text{(A.29)}$$

The Δ-functions are defined as follows

$$[\hat\varphi^{(\mp)}(x),\hat\varphi^{(\pm)}(y)] = \pm\frac{\hbar}{(2\pi)^3}\int\frac{d^3k}{2\omega}\,e^{\pm ik(x-y)} = i\hbar\Delta^{(\mp)}(x-y)\quad(\omega=\sqrt{\vec k^2+m^2})\,,$$

$$[\hat\varphi(x),\hat\varphi(y)]=i\hbar\Delta(x-y),\ [\hat\varphi^{(1)}(x),\hat\varphi(y)]=i\hbar\Delta^{(1)}(x-y),\ [\hat\varphi^{(1)}(x),\hat\varphi^{(1)}(y)]=i\hbar\Delta(x-y)\,,$$

$$\Delta_{ret}(x)=-\theta(t)\Delta(x)=\Delta_{sym}(x)-\tfrac{1}{2}\Delta(x)=\frac{1}{(2\pi)^4}\int d^4k\,\frac{\exp(ikx)}{k^2+m^2-i\varepsilon k_0}\,,$$

$$\Delta_{adv}(x)=\theta(-t)\Delta(x)=\Delta_{sym}(x)+\tfrac{1}{2}\Delta(x)=\frac{1}{(2\pi)^4}\int d^4k\,\frac{\exp(ikx)}{k^2+m^2+i\varepsilon k_0}\,,$$

$$\Delta_{sym}(x)=-\tfrac{1}{2}\varepsilon(t)\Delta(x)=\tfrac{1}{2}(\Delta_{ret}(x)+\Delta_{adv}(x))=\frac{1}{(2\pi)^4}P\int d^4k\,\frac{\exp(ikx)}{k^2+m^2}\,,$$

$$\Delta_{+}(x)=-\theta(t)\Delta^{(-)}(x)+\theta(-t)\Delta^{(+)}(x)=\Delta_{sym}(x)+\tfrac{i}{2}\Delta^{(1)}(x)=\frac{1}{(2\pi)^4}\int d^4k\,\frac{\exp(ikx)}{k^2+m^2-i\varepsilon}\,,$$

$$\Delta_{-}(x)=-\theta(t)\Delta^{(+)}(x)+\theta(-t)\Delta^{(-)}(x)=\Delta_{sym}(x)-\tfrac{i}{2}\Delta^{(1)}(x)=\frac{1}{(2\pi)^4}\int d^4k\,\frac{\exp(ikx)}{k^2+m^2+i\varepsilon}\,,$$

$$S_{(\ )}=(\gamma\partial-m)\Delta_{(\ )}(x)\,. \qquad \text{(A.30)}$$

Appendix B. Let us give examples of Hamiltonians and Liouvillians in PSRs. One can split up the total Liouvillian $\mathbb{L}$ into the free and interaction Liouvillians

$$\mathbb{L}=\mathbb{L}_0+\mathbb{L}_I\,,\quad \mathbb{L}_0=i\hbar^{-1}(H_0^{\ell}-H_0^{r})\,,\quad \mathbb{L}_I=i\hbar^{-1}(H_I^{\ell}-H_I^{r})\,.$$

The free and interaction Hamiltonians

$$\hat H_0=\tfrac{1}{2}\int d^3x\,[\dot{\hat\varphi}(x)\dot{\hat\varphi}(x)+\partial_k\hat\varphi(x)\partial_k\hat\varphi(x)+m^2\hat\varphi(x)\hat\varphi(x)]\,,$$

$$\hat H_I=\int d^3x\, j^e(x)\hat\varphi(x)\,,\qquad g\int d^3x\,\hat\varphi^3(x)$$

lead in the Schrodinger picture to

$$H_0^{\ell,r}=\tfrac{1}{2}\int d^3x\Big[\Big(\dot\varphi(\vec x)\mp\tfrac{i\hbar}{2}\tfrac{\delta}{\delta\varphi(\vec x)}\Big)^2+\Big(\partial_k\big(\varphi(\vec x)\pm\tfrac{i\hbar}{2}\tfrac{\delta}{\delta\dot\varphi(\vec x)}\big)\Big)^2+m^2\Big(\varphi(\vec x)\pm\tfrac{i\hbar}{2}\tfrac{\delta}{\delta\dot\varphi(\vec x)}\Big)^2\Big] \quad \text{(PSR-2)}$$

$$\mathbb{L}_0=\int d^3x\Big[\dot\varphi(\vec x)\frac{\delta}{\delta\varphi(\vec x)}-\partial_k\varphi(\vec x)\partial_k\frac{\delta}{\delta\dot\varphi(\vec x)}-m^2\varphi(\vec x)\frac{\delta}{\delta\dot\varphi(\vec x)}\Big]=$$

$$=\int d^3x\Big[\dot\varphi(\vec x)\frac{\delta}{\delta\varphi(\vec x)}-(-\Delta+m^2)\varphi(\vec x)\frac{\delta}{\delta\dot\varphi(\vec x)}\Big] \quad \text{(PSR-1,2,3)}$$

$$H_I^{\ell,r}=\int d^3x\, j^e(x)\Big(\varphi(\vec x)\pm\tfrac{i\hbar}{2}\tfrac{\delta}{\delta\dot\varphi(\vec x)}\Big)\,,\qquad g\int d^3x\Big(\varphi(\vec x)\pm\tfrac{i\hbar}{2}\tfrac{\delta}{\delta\dot\varphi(\vec x)}\Big)^3 \quad \text{(PSR-2)}$$

$$\mathbb{L}_I=-\int d^3x\, j^e(x)\frac{\delta}{\delta\dot\varphi(\vec x)}\ \text{(PSR-1,2,3)},\quad g\int d^3x\Big[-3\varphi^2(\vec x)\frac{\delta}{\delta\dot\varphi(\vec x)}+\frac{\hbar^2}{4}\Big(\frac{\delta}{\delta\dot\varphi(\vec x)}\Big)^3\Big]$$

and in the interaction picture to (PSR-2)

$$H_0^{\ell,r}=\tfrac{1}{2}\int d^3x\Big[-\Big(\partial_4\big(\varphi(x)\pm\tfrac{i\hbar}{2}\tfrac{\delta}{\delta J(x)}\big)\Big)^2+\Big(\partial_k\big(\varphi(x)\pm\tfrac{i\hbar}{2}\tfrac{\delta}{\delta J(x)}\big)\Big)^2+m^2\Big(\varphi(x)\pm\tfrac{i\hbar}{2}\tfrac{\delta}{\delta J(x)}\Big)^2\Big] \quad \text{(PSR-2)}$$

$$\mathbb{L}_0=\int d^3x\Big[\partial_4\varphi(x)\partial_4\frac{\delta}{\delta J(x)}-(-\Delta+m^2)\varphi(x)\frac{\delta}{\delta J(x)}\Big] \quad \text{(PSR-1,2,3)}$$

$$H_I^{\ell,r}=\int d^3x\, j^e(x)\Big(\varphi(x)\pm\tfrac{i\hbar}{2}\tfrac{\delta}{\delta J(x)}\Big)\,,\qquad g\int d^3x\Big(\varphi(x)\pm\tfrac{i\hbar}{2}\tfrac{\delta}{\delta J(x)}\Big)^3 \quad \text{(PSR-2)}$$

$$\mathbb{L}_I=-\int d^3x\, j^e(x)\frac{\delta}{\delta J(x)}\ \text{(PSR-1,2,3)},\quad g\int d^3x\Big[-3\varphi^2(x)\frac{\delta}{\delta J(x)}+\frac{\hbar^2}{4}\Big(\frac{\delta}{\delta J(x)}\Big)^3\Big]. \quad \text{(PSR-2)}$$

Appendix C. Let us consider the phase space density [x)]

$$\rho(\varphi\dot\varphi t,\varphi_0\dot\varphi_0 t_0)=\Lambda_0^{-1}|\langle\varphi|e^{-i\hbar^{-1}\hat H(t-t_0)}|\varphi_0\rangle|^2=\Lambda_0^{-1}|\langle\varphi|e^{-i\hbar^{-1}\hat H_0(t-t_0)}\hat U(t,t_0)|\varphi_0\rangle|^2 \quad \text{(C.1)}$$

and represent the expression between $\langle\varphi_0|$ and $|\varphi_0\rangle$ in a symmetrized product form. To this end we use the symmetrized product expansions of incoming factors: eqs. (15.b) for $\hat U$ and $\hat U^{-1}$ and

$$|\varphi\rangle\langle\varphi|=e^{\int d^3z\,[\hat\varphi(\vec z,t_0)\frac{\delta}{\delta\varphi'(\vec z,t_0)}+\dot{\hat\varphi}(\vec z,t_0)\frac{\delta}{\delta\dot\varphi'(\vec z,t_0)}]}\Lambda'^{-1}|\langle\varphi'|\varphi\rangle|^2\Big|_{\varphi'=\dot\varphi'=0}$$

$$e^{i\hbar^{-1}\hat H_0(t-t_0)}|\varphi\rangle\langle\varphi|e^{-i\hbar^{-1}\hat H_0(t-t_0)}= \quad \text{(C.2)}$$

$$=e^{\int d^3z\,[\hat\varphi(\vec z,t)\frac{\delta}{\delta\varphi'(\vec z,t_0)}+\dot{\hat\varphi}(\vec z,t)\frac{\delta}{\delta\dot\varphi'(\vec z,t_0)}]}\Lambda'^{-1}|\langle\varphi'|\varphi\rangle|^2\Big|_{\varphi'=\dot\varphi'=0}. \quad \text{(C.3)}$$

Now using the simplest case of the Baker-Campbell-Hausdorf formula $e^Ae^B=e^{A+B}e^{\frac{1}{2}[A,B]}$ we find

$$e^{i\hbar^{-1}\hat H(t-t_0)}|\varphi\rangle\langle\varphi|e^{-i\hbar^{-1}\hat H(t-t_0)}=\hat U^{-1}(t,t_0)e^{i\hbar^{-1}\hat H_0(t-t_0)}|\varphi\rangle\langle\varphi|e^{-i\hbar^{-1}\hat H_0(t-t_0)}\hat U(t,t_0)=$$

$$=e^{\int\hat\varphi\frac{\delta}{\delta\Phi}+\int d^3z\,[\hat\varphi(\vec z,t)\frac{\delta}{\delta\varphi'(\vec z,t_0)}+\dot{\hat\varphi}(\vec z,t)\frac{\delta}{\delta\dot\varphi'(\vec z,t_0)}]}\;e^{i\iint d^4x\,d^3y\,\frac{\delta}{\delta\tilde\Phi}[\Delta(x-y)\frac{\delta}{\delta\varphi'(\vec y,t_0)}+\dot\Delta(x-y)\frac{\delta}{\delta\dot\varphi'(\vec y,t_0)}]}$$

$$e^{i\iint\frac{\delta}{\delta\Phi}\Delta_{ret}\frac{\delta}{\delta\tilde\Phi}}\;e^{i\hbar^{-1}\int d^4x\,(\mathcal{H}_I(\Phi+\frac{\hbar}{2}\tilde\Phi)-\mathcal{H}_I(\Phi-\frac{\hbar}{2}\tilde\Phi))}\Lambda'^{-1}|\langle\varphi'|\varphi\rangle|^2\Big|_{\substack{\varphi'(\vec x,t_0)=0\\ \dot\varphi'(\vec x,t_0)=0\\ \tilde\Phi=\Phi=0}}, \quad \text{(C.4)}$$

where $y_0=t$, $\dot\Delta(x-y)=\frac{\partial}{\partial y_0}\Delta(x-y)$. We obtain finally that

$$\rho(\varphi\dot\varphi t,\varphi_0\dot\varphi_0 t_0)=\Lambda_0^{-1}|\langle\varphi|e^{-i\hbar^{-1}\hat H(t-t_0)}|\varphi_0\rangle|^2=$$

$$=e^{i\iint d^4x\,d^3y\,\frac{\delta}{\delta\tilde\Phi}[\Delta(x-y)\frac{\delta}{\delta\varphi'(\vec y,t_0)}+\dot\Delta(x-y)\frac{\delta}{\delta\dot\varphi'(\vec y,t_0)}]}\;e^{i\iint\frac{\delta}{\delta\Phi}\Delta_{ret}\frac{\delta}{\delta\tilde\Phi}}$$

$$e^{i\hbar^{-1}\int d^4x\,(\mathcal{H}_I(\Phi+\frac{\hbar}{2}\tilde\Phi)-\mathcal{H}_I(\Phi-\frac{\hbar}{2}\tilde\Phi))}\Lambda'^{-1}|\langle\varphi'|\varphi\rangle|^2\Big|_{\substack{\varphi'(\vec x,t_0)\to\varphi_0(\vec x,t)\\ \dot\varphi'(\vec x,t_0)\to\dot\varphi_0(\vec x,t)\\ \tilde\Phi\to 0,\ \Phi\to\varphi_0}}. \quad \text{(C.5)}$$

---

x) We prefer to deal with the Gauss functional rather than with the functional δ-function.

Note that Δ -functions can be replaced here by $\Delta_{ret}(y-x)=\theta(y_0-x_0)\Delta(x-y)$ and that the last expression may be written down in a short form as

$$\rho(\varphi\dot{\varphi}t,\varphi_0\dot{\varphi}_0t_0)=\Lambda_0^{-1}|\langle\varphi|\,e^{-i\hbar^{-1}\hat{H}(t-t_0)}|\varphi_0\rangle|^2=e^{i\iint\frac{\delta}{\delta\phi}\Delta_{ret}\frac{\delta}{\delta\widetilde{\Phi}}}$$

$$e^{i\hbar^{-1}\int d^4x\left(\mathcal{H}_I(\phi+\frac{\hbar}{2}\widetilde{\Phi})-\mathcal{H}_I(\phi-\frac{\hbar}{2}\widetilde{\Phi})\right)}\left[\Lambda'^{-1}|\langle\varphi'|\varphi\rangle|^2\Big|_{\substack{\varphi'(\vec{x},t_0)\to\phi(\vec{x},t)\\ \dot{\varphi}'(\vec{x},t_0)\to\dot{\phi}(\vec{x},t)}}\right]\Big|_{\substack{\widetilde{\Phi}=0\\ \phi=\varphi_0}} \qquad (C.6)$$

We can derive eq. (C.6) also as follows. Starting with the identity

$$sym(\hat{\varphi}(x_1)\hat{\varphi}(x_2)\dots\hat{\varphi}(x_n))=\frac{1}{n!}\{\hat{\varphi}(x_1)\hat{\varphi}(x_2)\dots\hat{\varphi}(x_n)\}=$$

$$=\frac{1}{2^n n!}\sum_{\substack{\text{over all } n!\\ \text{permutations}\\ \text{of } 1,2,\dots n}}\sum_{m=0}^{n}C_n^m\,\widetilde{T}(\hat{\varphi}(x_1)\dots\hat{\varphi}(x_m))\,T(\hat{\varphi}(x_{m+1})\dots\hat{\varphi}(x_n)) \qquad (C.7)$$

where $C_n^m=\frac{n!}{m!(n-m)!}$. We have for $x_{1o},x_{2o},\dots x_{no}\geqslant t$

$$\hat{U}^{-1}(t,t_0)\{\hat{\varphi}(x_1)\dots\hat{\varphi}(x_n)\}U(t,t_0)=$$

$$=\frac{1}{2^n}\sum_{\substack{\text{over all } n!\\ \text{permutations}\\ \text{of } 1,2,\dots n}}\sum_{m=0}^{n}C_n^m\,\widetilde{T}(\hat{U}^{-1}(t,t_0)\hat{\varphi}(x_1)\dots\hat{\varphi}(x_m))\,T(\hat{\varphi}(x_{m+1})\dots\hat{\varphi}(x_n)\hat{U}(t,t_0))=$$

$$=e^{\int\hat{\varphi}\frac{\delta}{\delta\phi_1}}e^{\int\hat{\varphi}\frac{\delta}{\delta\phi_2}}e^{\frac{i\hbar}{2}\iint\frac{\delta}{\delta\phi_1}\Delta_{sym}\frac{\delta}{\delta\phi_1}}e^{-\frac{i\hbar}{2}\iint\frac{\delta}{\delta\phi_2}\Delta_{sym}\frac{\delta}{\delta\phi_2}}e^{i\hbar^{-1}\int d^4x(\mathcal{H}_I(\phi_1)-\mathcal{H}(\phi_2))}$$

$$\frac{1}{2^n}\sum_{\substack{\text{over all } n!\\ \text{permutations}\\ \text{of } 1,2,\dots n}}\sum_{m=0}^{n}C_n^m\,\phi_1(x_1)\dots\phi_1(x_m)\phi_2(x_{m+1})\dots\phi_2(x_n)\Big|_{\phi_1=\phi_2=0}=$$

$$=e^{\int\hat{\varphi}\left(\frac{\delta}{\delta\phi_1}+\frac{\delta}{\delta\phi_2}\right)}e^{\frac{i\hbar}{2}\iint\left(\frac{\delta}{\delta\phi_1}+\frac{\delta}{\delta\phi_2}\right)\Delta_{ret}\left(\frac{\delta}{\delta\phi_1}-\frac{\delta}{\delta\phi_2}\right)}$$

$$e^{i\hbar^{-1}\int d^4x(\mathcal{H}_I(\phi_1)-\mathcal{H}_I(\phi_2))}\,\frac{n!}{2^n}(\phi_1(x_1)+\phi_2(x_1))\dots(\phi_1(x_n)+\phi_2(x_n))\Big|_{\phi_1=\phi_2=0}= \qquad (C.8)$$

$$=n!\,e^{\int\hat{\varphi}\frac{\delta}{\delta\phi}}e^{i\iint\frac{\delta}{\delta\phi}\Delta_{ret}\frac{\delta}{\delta\widetilde{\Phi}}}e^{i\hbar^{-1}\int d^4x\left(\mathcal{H}_I(\phi+\frac{\hbar}{2}\widetilde{\Phi})-\mathcal{H}_I(\phi-\frac{\hbar}{2}\widetilde{\Phi})\right)}\phi(x_1)\dots\phi(x_n)\Big|_{\substack{\phi=0\\ \widetilde{\Phi}=0}}$$

where the latter expression is obtained by the change of variables

$$\phi=\frac{1}{2}(\phi_1+\phi_2)\ ,\quad \hbar\widetilde{\Phi}=\phi_1-\phi_2$$

Note that above derivation gives the initial identity (C.7) too, if we set $\mathcal{H}_I = 0$ (hence $\hat{U}=\hat{U}^{-1}=1$) and take into account the relation

$$e^{\int\hat{\varphi}\frac{\delta}{\delta\phi}}\phi(x_1)\phi(x_2)\dots\phi(x_n)\Big|_{\phi=0}=\frac{1}{n!}\{\hat{\varphi}(x_1)\hat{\varphi}(x_2)\dots\hat{\varphi}(x_n)\} \qquad (C.9)$$

Since eq. (C.3) is constructed out of the symmetrized products, eq. (C.8) leads to eq. (C.6). In PSR-1 and PSR-3 eq. (C.5) or eq. (C.6) is modified by including additional Λ, and Λ_0^{-1}, respectively, in the same fashion as in eqs. (19.a) and (19.b).

It is worthwhile to note that eq. (C.4) may be written down in terms of symmetrized products of the Heisenberg operators.

$$e^{i\hbar^{-1}\hat{H}(t-t_0)}|\varphi\rangle\langle\varphi|e^{-i\hbar^{-1}\hat{H}(t-t_0)}=$$

$$=e^{\int d^3z\left[\hat{\varphi}(\vec{z},t)\frac{\delta}{\delta\varphi'(\vec{z},t_0)}+\hat{\dot{\varphi}}(\vec{z},t)\frac{\delta}{\delta\dot{\varphi}'(\vec{z},t_0)}\right]}\Lambda'^{-1}|\langle\varphi'|\varphi\rangle|^2\Big|_{\varphi'=\dot{\varphi}'=0}\ . \qquad (C.10)$$

Then ρ is the following matrix element of the latter expression

$$\rho(\varphi\dot{\varphi}t,\varphi_0\dot{\varphi}_0t_0)=\Lambda_0^{-1}\langle\varphi_0|\,e^{\int d^3z\left[\hat{\varphi}(\vec{z},t)\frac{\delta}{\delta\varphi'(\vec{z},t_0)}+\hat{\dot{\varphi}}(\vec{z},t)\frac{\delta}{\delta\dot{\varphi}'(\vec{z},t_0)}\right]}|\varphi_0\rangle$$

$$\Lambda'^{-1}|\langle\varphi'|\varphi\rangle|^2\Big|_{\varphi'=\dot{\varphi}'=0}\ . \qquad (C.11)$$

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Полубаринов И.В. E2-82-800
О представлениях фазового пространства в квантовой теории поля

В квантовой теории поля вводятся представления фазового пространства /ПФП/ в духе теории представлений Дирака. Определены неоператорные и левые и правые операторные представители для операторов /наблюдаемых/. В качестве уравнения движения принято уравнение Лиувилля /линейное уравнение в функциональных производных/ для плотности в фазовом пространстве в шредингеровской картине и для неоператорных представителей операторов в гейзенберговской картине. Квантовые законы эволюции операторных представителей также записаны через лиувиллиан. Показано, что в ПФП вигнеровского типа эволюция полей и плотности в фазовом пространстве является явно причинной, т.е. выражается исключительно в терминах Δ_{ret} -функций. В этом же ПФП линейные теории /лагранжиан максимум квадратичен/ совпадают с их классическими аналогами. В общем случае некоторый формальный переход к $\hbar = 0$ приводит к уравнениям и величинам классической теории поля, соответствующим квантовым в шредингеровской и гейзенберговской картинах и картине взаимодействия. Уравнения Гамильтона и Лагранжа классической теории поля естественно возникают как уравнения для характеристик уравнения Лиувилля.

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Polubarinov I.V. E2-82-800
On Phase Space Representations in Quantum Field Theory

Phase space representations (PSRs) are introduced in quantum field theory along the line of the Dirac representation theory. Nonoperator and left and right operator representatives for operators (observables) are defined. A Liouville equation (functional derivative one) for a phase space density in the Schrödinger picture and for the nonoperator representatives of operators in the Heisenberg picture is adopted as an equation of motion. Laws of quantum evolution for the operator representatives are also given in terms of a Liouvillian. It is shown that in PSR of the Wigner type evolution of fields and the phase space density is manifestly causal, i.e., is expressed only in terms of Δ_{ret} -"functions". In this PSR linear theories (Lagrangian at most bilinear) coincide with their classical analogs. In general case a some formal transition to $\hbar = 0$ leads to classical field theory equations and quantities, corresponding to quantum ones in the Schrödinger, Heisenberg and interaction pictures. The Hamilton and Lagrange equations of classical field theory appear naturally as ones for characteristics of a classical Liouville equation.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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