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REPRESENTATIONS OF osp(1,4) IN TERMS OF THREE BOSON PAIRS AND MATRICES OF ARBITRARY EVEN ORDER.

Description of the Method

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I. INTRODUCTION AND SUMMARY OF RESULTS

I.1 The study of infinite-dimensional representations of the Lie superalgebra (LSA) osp(1,4) initiated in the preceding paper /1/ is continued. Throughout this work the symbol osp(1,2n), n=1,2,.., always denotes the (unique) real form /2/ of the complex LSA B(0,n). In Kac's classification of real Lie superalgebras /3/ the osp(1,2n) appears as osp(1,n;0;$\mathbb{R}$).

The representations we construct have two basic properties:

(i) even generators are represented by skew-symmetric operators (ESS ≡ even skew-symmetric);

(ii) both the independent Casimir operators are represented by multiples of unity (SCH ≡ Schur irreducibility).

The problem of studying representations with these properties has been motivated in ref. /1/. Here we only want to add that, due to isomorphism of osp(1,2n) with the algebra pB(n) of para-Bose operators for n degrees of freedom /4/, such representations can be used for describing the system of n non-interacting para-Bose oscillators /5/.

I.2 Constructing infinite-dimensional representations of osp(1,4) satisfying the conditions ESS and SCH is a very general problem. We shall delimit it by two requirements. The first one specifies which structure has the representation of the even subalgebra sp(4,$\mathbb{R}$). To this purpose we make use of one class $\mathcal{R}$ of infinite-dimensional skew-symmetric Schurean representations of sp(4,$\mathbb{R}$) obtained by the general method of canonical realizations of Lie algebras /6/. In each representation belonging to $\mathcal{R}$ the generators x_{jk} of sp(4,$\mathbb{R}$) are represented by

$$\hat{x}_{jk} = \sum_{r=1}^{\mu} \xi_r^{(j,k)}(q_\alpha, q_\alpha^{-1}, p_\alpha) \otimes \underline{t}_r \quad . \tag{1.1}$$

Here ξ are certain ordered polynomials in nine non-commuting variables fulfilling the canonical commutation relations (CCR)

$$[p_\alpha, q_\beta] = \delta_{\alpha-\beta} \quad , \quad [q_\alpha, q_\beta] = 0, \; [p_\alpha, p_\beta] = 0 \qquad \alpha, \beta = 1,2,3 \tag{1.2}$$

and relations that follow from the assumed existence of q_α^{-1}:

$$[p_\alpha, q_\beta^{-1}] = -\delta_{\alpha\beta} q_\alpha^{-2} , \quad [q_\alpha^r , q_\beta^s] = 0 , \; r,s = \pm 1 . \tag{1.2a}$$

Further $\tau_n := \{\underline{t}_1, \underline{t}_2, \ldots, \underline{t}_\mu\}$ is in End $\mathbb{C}^n$, the set of all linear operators on $\mathbb{C}^n$.

The class $\mathcal{R}$ is described in detail in sect.II. Here we only want to point out that the number μ of elements of the set τ_n is the same for all the representations in $\mathcal{R}$; individual representations differ only in the dimension n and in the form of operators $\underline{t}_r$.

The first limiting requirement reads: the restriction of any representation of osp(1,4) to the subalgebra sp(4,$\mathbb{R}$) is a finite direct sum of representations belonging to $\mathcal{R}$, i.e., the even generators of osp(1,4) are given by

$$\hat{X}_{jk} = \sum_{r=1}^{\mu} \xi_r^{(j,k)}(q_\alpha, q_\alpha^{-1}, p_\alpha) \otimes T_r , \tag{1.3a}$$

where the set $\mathcal{T}_N := \{T_1, \ldots, T_\mu\} \subset$ End $\mathbb{C}^N$ is a finite direct sum of sets τ_n .

The second limitation requires the odd generators to have the following form

$$\hat{Y}_l = \sum_{s=1}^{\nu} \eta_s^{(l)}(q_\alpha, q_\alpha^{-1}, p_\alpha) \otimes A_s , \tag{1.3b}$$

where $\eta_s^{(l)}$ are again ordered polynomials and $A_s \in$ End $\mathbb{C}^N$. However, unlike the set $\mathcal{T}_N$, there are no a priori limitations imposed on $\{A_s\}$.

Compatibility of these two requirements is examined in sect.III and is shown to be guaranteed at least for the class $\mathcal{R}$ we have chosen. Moreover, it appears that the polynomials $\xi_r^{(j,k)}$ uniquely determine (via the commutation relations connecting even and odd elements of osp(1,4)) the $\eta_s^{(l)}$, and that the number ν of operators A_s does not depend on N. On the basis of these results we further transform the relations of osp(1,4) and the conditions ESS and SCH to a system of conditions for the set $\mathcal{M}_N := \{T_1, \ldots, T_\mu, A_1, \ldots, A_\nu\}$. Each solution yields via Eqs.(1.3a-b) a representation $\Omega(\mathcal{M}_N)$ of osp(1,4) that has the properties ESS and SCH. The conditions determining $\mathcal{M}_N$ are analyzed in sect.IV, the main results being:

(i) a solution exists iff N is even, $N \geq 2$;

(ii) for a given N the solution depends on the eigenvalue $\varkappa$ of the second-order Casimir operator of osp(1,4), $\varkappa$ being confined to an interval $\mathcal{K}_N \subset \mathbb{R}$. For a given $\varkappa \in \mathcal{K}_N$ there is (up to a class of equivalence transformations) just one solution if $\frac{N}{2}$ is even and just two solutions if $\frac{N}{2}$ is odd;

(iii) there is a regular $R \in$ End $\mathbb{C}^N$ such that $\Omega(R\mathcal{M}_N R^{-1})$ if regarded as a representation of the complexification B(0,2) of osp(1,4) is a $*$-representation w.r.t. the involution ϑ defined on the Racah basis (see sect.II) by

$$\vartheta(X_{jk}) := -X_{jk}, \quad \vartheta(Y_l) := -iY_l ; \tag{1.4}$$

(iv) the restriction $\Omega(\mathcal{M}_N) \upharpoonright$ sp(4,$\mathbb{R}$) equals direct sum of two, three or four representations from the class $\mathcal{R}$ if N=2, N=4 and $N \geq 6$, respectively.

I.3 The above results represent the complete formal solution of the problem. The adjective "formal" reflects the fact that the functions $\xi_r^{(j,k)}$, $\eta_s^{(l)}$, as well as the canonical pairs p_α, q_α themselves, are not determined as linear operators on some specified Hilbert space $\mathcal{H}$. Due to the CCR these operators have to be unbounded and hence the missing specification, besides defining $\mathcal{H}$ and the action of operators p_α, q_α on the vectors of $\mathcal{H}$, must also determine a common dense domain $\mathcal{G} \subset \mathcal{H} \otimes \mathbb{C}^N$ of all the generators $\hat{X}_{jk}$, $\hat{Y}_l$ such that the conditions of invariance $\hat{X}_{jk}\mathcal{G} \subset \mathcal{G}$, $\hat{Y}_l \mathcal{G} \subset \mathcal{G}$ hold.

One can expect that there are several possibilities of choosing $\mathcal{G}$ with the required properties. Anyway, at least one such $\mathcal{G}$ always exists: the dense linear manifold $D^{(3)} \subset L^2(\mathbb{R}^3)$ considered in /1/ lies in the intersection of domains of self-adjoint operators Q_r, P_r of the r-th coordinate and momentum, r=1,2,3, the operators $q_r := Q_r \upharpoonright D^{(3)}$, $p_r := iP_r \upharpoonright D^{(3)}$ satisfy the CCR and $\mathcal{G} := D^{(3)} \otimes \mathbb{C}^N$ is invariant w.r.t. $\hat{X}_{jk}$, $\hat{Y}_l$ given by Eqs.(1.3a-b), irrespective of the form of functions ξ, η and operators T, A.

Another, physically more interesting choice is based on the fact that osp(1,4) is isomorphic to the para-Bose algebra pB(2). This is immediately obvious if one writes $\hat{a}_k$ and $\hat{b}_k$, k=1,2, instead of $\hat{Y}_k$ and $\hat{Y}_{-k}$, respectively. Now one readily identifies the commutation relations connecting even and odd generators of osp(1,4) with those of the pB(2) (cf. below Eqs.(2.1b-c)), the $\hat{a}_k$ having the meaning of para-Bose annihilation operators and the $\hat{b}_k$ of creation ones. One can then try to solve the equations

$$\hat{a}_k \Phi_0 = 0 \quad , k=1,2$$

for the "vacuum vector" Φ_0. If a unique solution exists, the corresponding formal representation become a fully specified representation having Φ_0 as cyclic vector. This representation has the usual physical interpretation only if $\hat{b}_k$ is the adjoint of $\hat{a}_k$, k=1,2. This is equivalent to requiring that the representation should be a $*$-representation of B(0,2) $^{+)}$. Our formal representations satisfy this requirement on the formal level (see (iii) of the list of results); one may thus expect that the above construction will yield a class of $*$-representations of B(0,2). Detailed investigation of these questions and of related topics such as irreducibility, equivalence, etc., which cannot be studied on the formal level, is in progress. Our preliminary results, as well as those of ref./7/ obtained in a different framework, confirm that the vacuum vector exists at least for several lowest values of N. On the domain $\mathcal{G}$ generated by applying polynomials in $\hat{a}_k, \hat{b}_k$ to Φ_0 the formal expressions (1.3a-b) become (up to inessential phase factors) symmetric operators and form an irreducible $*$-representation of the algebra pB(2) $\simeq$ B(0,2).

II. PRELIMINARIES

II.1 The LSA osp(1,4) has 14 generators X_{jk}, $k \le j$, Y_j, $j,k \in \mathcal{J} := \{-2,-1,1,2\}$. The X_{jk} span the 10-dimensional even subalgebra sp(4,$\mathbb{R}$). The basis of osp(1,4) we use is an extension of the Racah basis of sp(4,$\mathbb{R}$) /8/. With the help of six auxiliary quantities X_{kj}, $k<j$, defined by $X_{kj} := X_{jk}$ the law of multiplication in osp(1,4) assumes the following symmetric form:

$^{+)}$Let us recall that there is essentially only one involution on B(0,2) that corresponds to its unique real form osp(1,4)/2/. This involution is defined, e.g., by Eqs.(1.4) up to equivalence transformations $\vartheta \mapsto \varphi\vartheta\varphi^{-1}$ generated by any automorphism φ of B(0,2).

$$[X_{jk}, X_{lm}] = g_{jl}X_{km} + g_{jm}X_{kl} + g_{kl}X_{jm} + g_{km}X_{jl}; \quad (2.1a)$$

$$[X_{jk}, Y_l] = g_{jl}Y_k + g_{kl}Y_j, \quad (2.1b)$$

$$\{Y_j, Y_k\} = 2X_{jk}, \quad (2.1c)$$

where $g_{jk} := \varepsilon_j \delta_{j+k}$, $\varepsilon_j := \mathrm{sgn}(j)$.

II.2 There are two independent Casimir operators K_2 and K_4 in the enveloping algebra of osp(1,4), the former being quadratic and the latter quartic /9/. For the purposes of this study it is convenient to express them via Casimir operators C_2, C_4 of sp(4,$\mathbb{R}$) and quadratic quantities

$$V_{jk} := \tfrac{1}{2}[Y_j, Y_k] \quad , \quad W_{jk} := \sum_{l \in \mathcal{J}} \varepsilon_l X_{j,-l} X_{lk} + 3X_{jk}$$

as follows

$$K_2 = C_2 + 2(V_{1-1} + V_{2-2}) \quad (2.2a)$$

$$K_4 = C_4 - \tfrac{1}{2}(K_2 - C_2)^2 - 15\, C_2 + 4 \sum_{\substack{j,k \in \mathcal{J} \\ j>k}} \varepsilon_j \varepsilon_k V_{jk} W_{-j-k}. \quad (2.2b)$$

The C_2 and C_4 are themselves simple functions of W_{jk}:

$$C_2 = 2(W_{1-1} + W_{2-2}) \quad , \quad C_4 = 2 \sum_{\substack{j,k \in \mathcal{J} \\ j>k}} \varepsilon_j \varepsilon_k W_{jk} W_{-j-k}. \quad (2.3)$$

II.3 The starting point for constructing representations of osp(1,4) having the properties listed in sect.I.1 is a certain class $\mathcal{R}$ of representations of sp(4,$\mathbb{R}$). These representations are infinite-dimensional and their complete specification, which is in general a non-trivial problem of functional analysis, is not necessary for what we are considering here (cf.sect.I.3). The following algebraic specification is sufficient for our purposes.

Consider the complex associative algebra Ξ of ordered polynomials in $q_\alpha, q_\alpha^{-1}, p_\alpha$, $\alpha=1,2,3$, in which multiplication is defined with the help of ordering relations (1.2) $^{+)}$. An involution $\xi \mapsto \xi^*$ is defined on Ξ by the usual extension of $q_\alpha^* := q_\alpha$, $p_\alpha^* := -p_\alpha$, $\alpha=1,2,3$.

$^{+)}$Notice that Ξ is the so-called ring of fractions /10/ of the Weyl algebra W_6 defined by the set of monomials

$$q_1^{n_1} q_2^{n_2} q_3^{n_3} \quad , \quad n_1, n_2, n_3 = 0,1,\ldots .$$

Remark: Ξ can be realized as the algebra of formal linear differential operators [11] on some set C of sufficiently smooth functions on $\mathbb{R}^3 \setminus \{0\}$

$$\xi(q_\alpha, q_\alpha^{-1}, p_\alpha) \equiv \sum_{j,k,l} P_{jkl}(q_\alpha, q_\alpha^{-1})\, p_1^j p_2^k p_3^l \quad .$$

Here P_{jkl} are given polynomials and the action of ξ on any $\psi \in C$ is given by

$$\left(\xi[\psi]\right)(x_1,x_2,x_3) := \sum_{j,k,l} P_{jkl}(x_\alpha, x_\alpha^{-1}) \frac{\partial^{j+k+l}}{\partial x^j \partial x^k \partial x^l} \quad .$$

The involution $\xi \mapsto \xi^*$ is realized as formal adjoint operation, i.e., for any $\varphi, \psi \in C$ it holds $\varphi\, \xi[\psi] - \psi\, \xi^*[\varphi] = \operatorname{div} \vec{\eta}$ and $\vec{\eta} = (\eta_1, \eta_2, \eta_3)$ depends linearly on φ, ψ and their derivatives.

Let $\mathcal{A} \equiv \mathcal{A}_n := \Xi \otimes \operatorname{End} \mathbb{C}^n$; this is a complex associative algebra on which involution can be introduced with the help of the standard hermitian conjugation "+" on End $\mathbb{C}^n$

$$(\xi \otimes T)^* := \xi^* \otimes T^+ \quad .$$

The class $\mathcal{R}$ can be now algebraically specified as follows: $\mathcal{R}$ is the set of mappings $x_{jk} \mapsto \rho(x_{jk}) \equiv \hat{x}_{jk}$ of sp(4, $\mathbb{R}$) into $\mathcal{A}_n$, n=1,2,.. given by

$$\hat{x}_{-2-2} = iq_2^2 \;,\quad \hat{x}_{-1-2} = iq_1q_2 \;,\quad \hat{x}_{1-2} = p_1q_2$$
$$\hat{x}_{2-2} = q_2p_2 + \tfrac{1}{2} \;,\quad \hat{x}_{-1-1} = i(q_1^2 + q_3^2) \;,\quad \hat{x}_{1-1} = q_1p_1 + q_3p_3 + 1 \;,$$
$$\hat{x}_{2-1} = q_1(p_2 - \tfrac{1}{2}q_2^{-1}) - q_2^{-1}q_3(q_1p_3 - p_1q_3) - iq_2^{-1}q_3 \otimes \underline{h} \;,$$
$$\hat{x}_{11} = -i(p_1^2 + p_3^2) + iq_3^{-2} \otimes \underline{t} \;,\quad \hat{x}_{21} = -ip_1(p_2 + \tfrac{1}{2}q_2^{-1}) +$$
$$iq_2^{-1}(q_1p_3 - p_1q_3)p_3 - q_2^{-1}p_3 \otimes \underline{h} - iq_1q_2^{-1}q_3^{-2} \otimes \underline{t} +$$
$$\tfrac{1}{2}\, q_2^{-1}q_3^{-1} \otimes [\underline{t}, \underline{h}] \;,$$
$$\hat{x}_{22} = -ip_2^2 - iq_2^{-2}\Big((q_1p_3 - p_1q_3)^2 - \tfrac{15}{4} + 2i(q_1p_3 - p_1q_3) \otimes \underline{h} -$$
$$iq_1q_3^{-1} \otimes [\underline{t}, \underline{h}] - (q_1^2q_3^{-2} - 1) \otimes \underline{t} - \tfrac{1}{2} \otimes \underline{w}\Big) \qquad (2.4)$$

and by the condition that $\{\underline{h}, \underline{t}, \underline{w}\} \subset \operatorname{End} \mathbb{C}^n$ is an irreducible set of hermitian operators satisfying $[\underline{h}, \underline{w}] = 0$,

$$[\underline{h}, [\underline{t}, \underline{h}]] = \underline{w} + 8 - 2\underline{h}^2 - 4\underline{t}, \quad [\underline{t}, [\underline{t}, \underline{h}]] = 2\{\underline{t}, \underline{h}\}. \quad (2.5)$$

Theorem: Let ρ be any element of $\mathcal{R}$. Then
(a) ρ is a homomorphism of sp(4, $\mathbb{R}$) into $\mathcal{A}$, i.e., the $\hat{x}_{jk}$ satisfy the relations (2.1a);
(b) $\hat{x}_{jk}^* = -\hat{x}_{jk}$ for all $j,k \in \mathcal{J}$;
(c) ρ maps both the independent Casimir operators of sp(4, $\mathbb{R}$) to multiples of unity (Schureanity).

Proof: The statements (a),(b) can be verified directly using the CCR. In the same way one gets by substituting (2.4) into (2.3)

$$\rho(C_2) = \underline{w} \;,\qquad \rho(C_4) = \tfrac{1}{4}\,\underline{w}^2 + \underline{x} \;, \qquad (2.6)$$

where $\underline{x} := \frac{1}{4}[\underline{h}, [\underline{t}, \underline{h}]]^2 - \underline{h}^4 - [\underline{t}, \underline{h}]^2 + (\underline{w}+4)\underline{h}^2$. Further Eqs.(2.5) imply $[\underline{t}, \underline{w}] = 0$, $[\underline{h}, \underline{x}] = [\underline{t}, \underline{x}] = [\underline{w}, \underline{x}] = 0$. Now the set $\{\underline{h}, \underline{t}, \underline{w}\}$ is irreducible and hence $\underline{w} = w\underline{I}$, $\underline{x} = x\underline{I}$. ■

II.4 The above specification of $\mathcal{R}$ is implicit. For getting its explicit form we have solved the problem of finding all the irreducible hermitian solutions to Eqs.(2.5). It is convenient to introduce

$$\underline{v}_\varepsilon := [\underline{t}, \underline{h}] + \tfrac{\varepsilon}{2}(\underline{w} + 8 - 4\underline{t} - 2\underline{h}^2) \quad , \varepsilon = \pm 1 \; .$$

Then (2.5) implies

$$[\underline{h}, \underline{v}_\varepsilon] = 2\varepsilon \underline{v}_\varepsilon \;,\; [\underline{v}_+, \underline{v}_-] = 4\underline{h}(2\underline{h}^2 - \underline{w} - 8),$$
$$[\underline{w}, \underline{v}_\varepsilon] = 0 \;,\; [\underline{w}, \underline{h}] = 0 \; . \qquad (2.7)$$

One can easily verify that $\{\underline{h}, \underline{t}, \underline{w}\}$ is an irreducible set of hermitian operators satisfying (2.5) iff the set $\{\underline{h}, \underline{v}_+, \underline{v}_-, \underline{w}\}$ is irreducible, fulfils (2.7), $\underline{h}, \underline{w}$ are hermitian and $\underline{v}_\varepsilon^* = -\underline{v}_{-\varepsilon}$.

The complete solution of the problem is described as follows. For n=1,2,.. consider the sets

$$\mathcal{C}_n^{(1)} := \begin{cases} \mathbb{R} \\ (2(n-1)^2 - 8, +\infty) \end{cases} \qquad \mathcal{C}_n^{(2)} := \begin{cases} \mathbb{R} & n=1 \\ (-1,0) \cup (0,1) & n=2,3,.. \end{cases}$$

Let $\underline{h}^{(r)}(\gamma)$, r=1,2, be the hermitian operator on $\mathbb{C}^n$ with non-degenerate eigenvalues

$$\lambda_k^{(r)}(\gamma) := 2k - 1 - n + (r-1)\gamma \;\; ; \;\; k = 1,2,..,n \;\; ; \;\; \gamma \in \mathcal{C}_n^{(2)} \qquad (2.8a)$$

and $\{\psi_k^{(r)} \mid k = 1,..,n\} \subset \mathbb{C}^n$ be a fixed set of orthonormal vectors satisfying $\underline{h}^{(r)}(\gamma)\psi_k^{(r)} = \lambda_k^{(r)}(\gamma)\psi_k^{(r)}$. Define for each $\gamma \in \mathcal{C}_n^{(r)}$ the operators $\underline{v}_+^{(r)}(\gamma), \underline{v}_-^{(r)}(\gamma) \in \operatorname{End} \mathbb{C}^n$ by

$$\underline{v}_+^{(r)}(\gamma)\,\psi_n^{(r)} := 0,$$

$$\underline{v}_+^{(r)}(\gamma)\,\psi_k^{(r)} := 2\left(k(n-k)u_k^{(r)}(\gamma)\right)^{\frac{1}{2}}\,\psi_{k+1}^{(r)},\quad k=1,..,n-1$$

$$\underline{v}_-^{(r)}(\gamma) := -\left(\underline{v}_+^{(r)}(\gamma)\right)^{*} \qquad (2.8b)$$

with

$$u_k^{(r)}(\gamma) := \begin{cases} \gamma+10-n^2+(2k-n)^2 & \ldots\ r=1 \\ n^2-(2k-n-2\gamma)^2 & \ldots\ r=2\ . \end{cases}$$

Further let

$$\mathcal{S}_r(n,\gamma) := \{\underline{h}^{(r)}(\gamma), \underline{v}_{\pm}^{(r)}(\gamma),\ w_r\underline{I}_n\},\quad w_1 := \gamma\ ,\quad w_2 := 2\gamma^2+2n^2-10,$$

$$\mathcal{M}_r(n,\gamma) := \{[\underline{h}^{(r)}(\gamma)]_r, [\underline{v}_{\pm}^{(r)}(\gamma)]_r\},\quad r=1,2\ , \qquad (2.9)$$

where $[\underline{t}]_r$ denotes the matrix of any given $\underline{t} \in \mathrm{End}\,\mathbb{C}^n$ w.r.t the basis $\{\psi_1^{(r)},\ldots,\psi_n^{(r)}\}$.

Proposition:

(a) Each element of the set $\mathcal{S} := \bigcup_{n=1}^{\infty}\bigcup_{r=1}^{2}\{\mathcal{S}_r(n,\gamma)\,|\,\gamma\in\mathcal{C}_n^{(r)}\}$ is an irreducible solution of Eqs.(2.7) fulfilling

$$\underline{h}^{*} = \underline{h},\ \underline{w}^{*} = w,\ \underline{v}_{\varepsilon}^{*} = -\underline{v}_{\varepsilon}\ . \qquad (+)$$

(b) Any two elements of $\mathcal{S}$ are non-equivalent.

(c) Any irreducible solution of Eqs.(2.7) which satisfies (+) is equivalent to some element of $\mathcal{S}$.

Proof: Assertion (a) can be verified directly and the validity of (b) is based on the following obvious statement: two elements $\{\underline{h},\underline{v}_+,\underline{v}_-,w\underline{I}_n\} \subset \mathrm{End}\,\mathbb{C}^n$, $\{\underline{h}',\underline{v}_+',\underline{v}_-',w'\underline{I}_m\} \subset \mathrm{End}\,\mathbb{C}^m$ of $\mathcal{S}$ are equivalent iff n=m, Tr $\underline{h}$ = Tr $\underline{h}'$ and $w = w'$. The proof of (c) will be given elsewhere. ∎

Remark: Let ρ_r be the element of $\mathcal{R}$ that is obtained by substituting $\mathcal{S}_r$ into Eqs.(2.4). Then the Casimir operators $\rho_r(C_2)$, $\rho_r(C_4)$ (see (2.6)) can be expressed via n and w_r ($w_1 := \gamma_1$, $w_2 := 2\gamma_2^2+2n^2-10$) as follows: $\rho_r(C_2) = w_r$, $r = 1,2$

$$\rho_1(C_4) = \tfrac{1}{4}\,w_1^2+(n^2-1)(w_1+9-n^2)\ ,$$

$$\rho_2(C_4) = \tfrac{1}{2}\,w_2^2 + 2w_2(2-n^2)+4(n^2-1)(n^2-4)\ .$$

III. REDUCTION OF THE PROBLEM TO "MATRIX LEVEL"

III.1 The restrictions formulated in sect.I.2 and the specification of the class $\mathcal{R}$ given in sect.II.3-4 determine almost completely the formal operators $\hat{X}_{jk}\in\mathcal{A}$ that represent the even generators of osp(1,4). They are given by Eqs.(2.4) in which the set $\{\underline{h},\underline{t},\underline{w}\}$ is replaced by some (finite) direct sum of elements $\{\underline{h}_s,\underline{t}_s,\underline{w}_s\}$ belonging to the set $\mathcal{S}$ (see proposition II.4) and there only remains to specify which elements of $\mathcal{S}$ enter the direct sum. Accordingly, the $\hat{X}_{jk}$ will be briefly written as

$$\hat{X}_{jk} = \hat{x}_{jk}(H,T,W)\ ,\quad \{H,T,W\} = \bigoplus_s\{\underline{h}_s,\underline{t}_s,\underline{w}_s\}. \qquad (3.1)$$

As to the odd generators, Eq.(1.3b) states that they also have to be represented by elements of $\mathcal{A}$. This requirement together with (3.1) determines uniquely the dependence of $\hat{Y}_l$ on p,q.

Proposition: Let $\hat{X}_{jk}$ be given by (3.1). Then the relations

$$[\hat{X}_{jk},\hat{Y}_{-2}] = g_{j,-2}\,\hat{Y}_k + g_{k,-2}\,\hat{Y}_j\ ,\quad j,k\in\mathcal{J}$$

are satisfied by $\hat{Y}_l\in\mathcal{A}$ iff

$$\hat{Y}_{-2} = \epsilon\, q_2\otimes A\ ,\quad \hat{Y}_{-1} = \epsilon(q_1\otimes A - iq_3\otimes[H,A])\ ,\quad \epsilon := \exp(i\pi/4)$$

$$\hat{Y}_1 = -i\epsilon(p_1\otimes A - ip_3\otimes[H,A] + \tfrac{i}{2}q_3^{-1}\otimes[[T,H],A]),$$

$$\hat{Y}_2 = -i\epsilon\Big(p_2\otimes A + iq_2^{-1}(q_1p_3 - p_1q_3)\otimes[H,A] - \tfrac{i}{2}q_1q_2^{-1}q_3^{-1}\otimes[[T,H],A] - \tfrac{1}{4}q_2^{-1}\otimes[W,A] \qquad (3.2)$$

and $[T,A] = 0$. (3.3a)

The proof can be performed in the same way as in /1/ (proposition 4.2); however, now a stronger and more general assertion is obtained.

III.2 By Eqs. (3.1-2) a mapping Ω of osp(1,4) into $\mathcal{A}$ is defined. According to what has been said in sect.I, the following conditions are imposed on Ω :

(Ω1) Ω is a homomorphism of osp(1,4) into $\mathcal{A}$, i.e., $\hat{X}_{jk} := \Omega(X_{jk})$ $\hat{Y}_l := \Omega(Y_l)$ satisfy the relations (2.1);

(Ω2) $\hat{X}_{jk}^{*} = -\hat{X}_{jk}$ for all $j,k\in\mathcal{J}$ (ESS property) ;

(Ω3) $\Omega(K_2) = \varkappa\hat{I}$, $\Omega(K_4) = \varkappa'\hat{I}$, $\varkappa,\varkappa'\in\mathbb{C}$ (SCH property).

As the dependence of Ω on q_α, p_α is fixed and there is a one-to-one correspondence of Ω's and systems $\mathcal{H} := \{A,H,T,W\} \subset \mathrm{End}\,\mathbb{C}^N$, the ($\Omega$1-3) can be reformulated to conditions for A,H,T,W. Some of them have just been established: requiring that H,T,W should be hermitian is clearly equivalent to (Ω2) and Eqs.(2.5)(3.3a) are equivalent to a subset of relations (2.1)(see proposition III.1). Two further equivalences can easily be verified:

$$\{\hat{Y}_{-2}, \hat{Y}_{-2}\} = 2\hat{X}_{-2-2} \Longleftrightarrow A^2 = I \tag{3.3b}$$

$$[\hat{X}_{2-1}, \hat{Y}_{-1}] = 0 \quad \Longleftrightarrow \quad [H,[H,A]] = A . \tag{3.3c}$$

This is all we need for the reformulation of conditions (Ω1)-(Ω3) in terms of operators A,H,T,W which is given in the next theorem.

III.3 <u>Theorem</u>: The mapping $\Omega = \Omega_{\mathcal{H}}(.)$ of osp(1,4) into $\mathfrak{A}$ related to $\mathcal{H} := \{A,H,T,W\} \subset \mathrm{End}\,\mathbb{C}^N$ according to Eqs.(3.1-2),i.e.,

$$\hat{X}_{-2-2} = iq_2^2, \quad \hat{X}_{-1-2} = iq_1q_2, \quad \hat{X}_{1-2} = p_1q_2, \quad \hat{X}_{2-2} = q_2p_2 + \tfrac{1}{2},$$

$$\hat{X}_{-1-1} = i(q_1^2 + q_3^2), \quad \hat{X}_{1-1} = q_1p_1 + q_3p_3 + 1,$$

$$\hat{X}_{2-1} = q_1(p_2 - \tfrac{1}{2}q_2^{-1}) - q_2^{-1}q_3j_2 - iq_2^{-1}q_3 \otimes H ,$$

$$\hat{X}_{11} = -i(p_1^2 + p_3^2) + iq_3^{-2} \otimes T,$$

$$\hat{X}_{21} = -ip_1(p_2 + \tfrac{1}{2}q_2^{-1}) + iq_2^{-1}(j_2p_3 + ip_3 \otimes H - q_1q_3^{-2} \otimes T - \tfrac{i}{2}q_3^{-1} \otimes V),$$

$$\hat{X}_{22} = -ip_2^2 - iq_2^{-2}(j_2^2 - \tfrac{15}{4} + 2ij_2 \otimes H - iq_1q_3^{-1} \otimes V + (1 - q_1^2q_3^{-2}) \otimes T - \tfrac{1}{2} \otimes W),$$

$$\hat{Y}_{-2} = \epsilon q_2 \otimes A, \quad \hat{Y}_{-1} = \epsilon(q_1 \otimes A - iq_3 \otimes B),$$

$$\hat{Y}_1 = -i\epsilon(p_1 \otimes A - ip_3 \otimes B + \tfrac{i}{2}q_3^{-1} \otimes Z),$$

$$\hat{Y}_2 = -i\epsilon(p_2 \otimes A + iq_2^{-1}j_2 \otimes B - \tfrac{i}{2}q_1q_2^{-1}q_3^{-1} \otimes Z - \tfrac{1}{2}q_2^{-1} \otimes \tilde{Z}),$$

with $j_2 := q_1p_3 - p_1q_3$, $\epsilon := \exp(i\pi/4)$

$$V := [T,H], \; B := [H,A], \; Z := [[T,H],A], \; \tilde{Z} := \tfrac{1}{2}[W,A], \tag{3.3d}$$

has the following properties:

(a) Ω is a homomorphism of osp(1,4) into $\mathfrak{A}$ iff

($\mathcal{H}$1) $[V,H] = 4T - 8 - W + 2H^2$, $[V,T] = -4HT - 2V$, $[W,H] = 0$,

$[T,A] = 0$, $[H,B] = A$, $A^2 = I$.

(b) Ω has the ESS property iff

($\mathcal{H}$2) $H^+ = H$, $T^+ = T$, $W^+ = W$.

(c) Ω is Schurean if

($\mathcal{H}$3) $\mathcal{H}$ is irreducible.

(d) The Casimir operators given by Eqs.(2.2) become

$$\hat{K}_2 \equiv \Omega_{\mathcal{H}}(K_2) = W - BZ + A\tilde{Z} + 3, \tag{3.4}$$

$$\hat{K}_4 \equiv \Omega_{\mathcal{H}}(K_4) = \hat{K}_4^{(1)} + \hat{K}_4^{(2)}$$

$$\hat{K}_4^{(1)} := \Omega_{\mathcal{H}}(C_4 - \tfrac{1}{2}(K_2 - C_2)^2 - 15C_2) = 4H^2T + 4T^2 - V^2 + 4HV + \tfrac{1}{2}\hat{K}_2^2 + (A\tilde{Z} - BZ)(2T - \hat{K}_2 + 4) - (2\hat{K}_2 + 2)T - 7\hat{K}_2 + 12 \tag{3.5a}$$

$$\hat{K}_4^{(2)} := 4\Omega_{\mathcal{H}}\Big(\sum_{\substack{j,k \in \mathcal{J} \\ j > k}} \varepsilon_j \varepsilon_k V_{jk} W_{-j-k}\Big) = AB(4HT + Z\tilde{Z}) - 2(BZ + A\tilde{Z})T + (AZ + Z\tilde{Z} - 2AB - 2B\tilde{Z})H + (3AB - AZ - B\tilde{Z})V + A\tilde{Z}(\hat{K}_2 - 2) + 14T - BZ - 3\hat{K}_2 - 22 . \tag{3.5b}$$

<u>Proof</u>: (a,b,d) The equivalences ($\mathcal{H}_j$) $\Leftrightarrow$ (Ω_j), j=1,2, can be verified directly. In principle one could obtain in this way also the formulae for $\hat{K}_2$ and $\hat{K}_4$, however, this would be an enormous calculation and we have used another approach that will be described elsewhere (see also the Appendix).

(c) Let $\hat{K} \in \mathfrak{A}$ commute with all the $\hat{X}_{jk}, \hat{Y}_l$ (in particular the operators $\hat{K}_2$ and $\hat{K}_4$ have this property). One easily finds that the six conditions

$$[\hat{X}_{jk}, \hat{K}] = 0 , \quad k = -2,-1, \quad j \in \mathcal{J} , \quad k \le j \le |k|$$

are equivalent to

$$[q_\alpha, \hat{K}] = [p_\alpha, \hat{K}] = 0 , \quad \alpha = 1,2,3,$$

which is further equivalent to $\hat{K} = I \otimes K$. Then

$$[\hat{X}_{2-1}, \hat{K}] = 0 \Rightarrow [H,K] = 0, \quad [\hat{X}_{11}, \hat{K}] = 0 \Rightarrow [T,K] = 0$$

$$[\hat{X}_{22}, \hat{K}] = 0 \wedge [H,K] = 0 \wedge [T,K] = 0 \Rightarrow [W,K] = 0$$

and finally $[\hat{Y}_{-2}, \hat{K}] = 0 \Rightarrow [A,K] = 0$. If $\mathcal{H}$ is irreducible, then by Schur lemma $\hat{K} = \varkappa\hat{I}$, $\varkappa \in \mathbb{C}$. ■

We shall denote

$$\mathcal{E} := \{\Omega \equiv \Omega_{\mathcal{H}}(.) \mid \mathcal{H} \text{ satisfies } (\mathcal{H}1)-(\mathcal{H}3)\}$$

$$\mathcal{E}_0 := \{\Omega \in \mathcal{E} \mid A^{*} = A\}. \qquad (3.6)$$

From the formulae for $\hat{Y}_1$ one finds that $\Omega \in \mathcal{E}_0$ iff Ω is a $*$-representation of B(0,2) w.r.t the involution (1.4).

Remark: Notice that $(\mathcal{H}3)$ is not equivalent to $(\Omega 3)$: if, e.g., $\mathcal{H}' = \mathcal{H} \oplus \mathcal{H}$, where $\mathcal{H}$ fulfils $(\mathcal{H}1)-(\mathcal{H}3)$, then $\mathcal{H}'$ is reducible and yet $\Omega_{\mathcal{H}'}$ is Schurean. Thus the set $\mathcal{E}$ does not contain all the representations specified by $(\Omega 1)-(\Omega 3)$. On the other hand, the representations out of $\mathcal{E}$ are probably of little interest because one can expect them reducible; however this question cannot be solved on the present formal level (see the discussion in sect.I.3).

IV. ANALYSIS OF THE CONDITIONS $(\mathcal{H}1)$ - $(\mathcal{H}3)$

IV.1 In this section the following problem is solved: for a given positive integer N find all the mutually nonequivalent systems $\{A,H,T,W\} \subset \mathrm{End}\, \mathbb{C}^N$ fulfilling the conditions $(\mathcal{H}1)-(\mathcal{H}3)$.

It is convenient to reformulate the problem by passing from A,T (and auxiliary operators $B \equiv [H,A]$, $V \equiv [T,H]$) to

$$A_\varepsilon := \tfrac{1}{2}(A + \varepsilon B), \quad V_\varepsilon := V + \tfrac{\varepsilon}{2}(W + 8 - 4T - 2H^2), \ \varepsilon = \pm 1 \qquad (4.1)$$

(cf. sect.II.4). The inverse transformation reads

$$A = \sum_\varepsilon A_\varepsilon, \quad B = \sum_\varepsilon \varepsilon A_\varepsilon, \quad T = \tfrac{1}{4}(W + 8 - 2H^2 - \sum_\varepsilon \varepsilon V_\varepsilon), \quad V = \tfrac{1}{2}\sum_\varepsilon V_\varepsilon. \qquad (4.2)$$

Lemma: 1. If the set $\{A,H,T,W\}$ fulfills $(\mathcal{H}1)-(\mathcal{H}3)$, then

(a) there is a real $\varkappa$ such that the operators $A_\pm, H, V_\pm$ satisfy

$$\{A_\varepsilon, A_\eta\} = \delta_{\varepsilon+\eta}, \ [H,A_\varepsilon] = \varepsilon A_\varepsilon, \ [V_\varepsilon, A_\varepsilon] = 0, \ [H,V_\varepsilon] = 2\varepsilon V_\varepsilon,$$

$$[V_+,V_-] = 4H(2H^2 - \tilde{W} - \varkappa - 4), \ [V_\varepsilon,[V_\varepsilon,A_{-\varepsilon}]] = 8\varepsilon A_\varepsilon V_\varepsilon \qquad (4.3)$$

with $\tilde{W} := \sum_\varepsilon \varepsilon A_\varepsilon V_{-\varepsilon} A_\varepsilon + 2\sum_\varepsilon \varepsilon A_\varepsilon A_{-\varepsilon} H$; (4.3a)

(b) the set $\{A_\pm, H, V_\pm\}$ is irreducible and

(c) has the following **"star"** property:

$$V_\varepsilon^{*} = -V_{-\varepsilon}, \quad H^{*} = H, \quad \tilde{W}^{*} = \tilde{W}. \qquad (4.4)$$

2. If the set $\{A_\pm, H, V_\pm\}$ is irreducible, satisfies (4.3) for some $\varkappa \in \mathbb{R}$ and has the star properties (4.4), then $\{A,H,T,W\}$, $W := \tilde{W} + \varkappa - 4$, where $A, T, \tilde{W}$ are given by (4.2) and (4.3a), fulfils $(\mathcal{H}1)-(\mathcal{H}3)$.

Proof: 1.(a) The first four of relations (4.3) and the equality

$$[V_+,V_-] = 4H(2H^2 - W - 8) \qquad (*)$$

follow directly from $(\mathcal{H}1)$ and the definitions (4.1). According to $(\mathcal{H}3)$ there is a complex $\varkappa$ such that $\hat{K}_2 = \varkappa \hat{I}$; from (3.4) one them has

$$W = \varkappa - 3 + BZ - A\tilde{Z}. \qquad (4.5a)$$

By using the relations $(\mathcal{H}1)$ one gets

$$\tilde{Z} \equiv \tfrac{1}{2}[W,A] = [B,V] + 2BH + A$$

and the first and third of (4.3) now yield

$$W = \varkappa - 4 + \sum_\varepsilon \varepsilon A_\varepsilon V_{-\varepsilon} A_\varepsilon + 2\sum_\varepsilon \varepsilon A_\varepsilon A_{-\varepsilon} H. \qquad (4.5b)$$

By substituting into $(*)$ the fifth of (4.3) is obtained. For getting the last relation one first derives from (4.5b) and just proved part of (4.3)

$$[W,A_\varepsilon] = 2A_\varepsilon + \varepsilon[V_\varepsilon, A_{-\varepsilon}] + 4\varepsilon A_\varepsilon H, \quad [W,V_\varepsilon] = 0.$$

Then

$$[V_\varepsilon,[V_\varepsilon,A_{-\varepsilon}]] = [V_\varepsilon, \varepsilon[W,A_\varepsilon] - 2\varepsilon A_\varepsilon - 4A_\varepsilon H] = 8\varepsilon A_\varepsilon V_\varepsilon.$$

For proving $\varkappa \in \mathbb{R}$ consider the minimal eigenvalue of H. In view of $(\mathcal{H}2)$ this is a real number, say λ_0. Thus there is some non-zero $\psi_0 \in \mathbb{C}^N$ fulfilling $H\psi_0 = \lambda_0 \psi_0$. Now $[H,A_\varepsilon] = \varepsilon A_\varepsilon$ implies that $\psi_1 := A_+\psi_0$ is an eigenvector of H, the corresponding eigenvalue being $\lambda_0 + 1$, or $\psi_1 = 0$. For $A_-\psi_0$ we always get $A_-\psi_0 = 0$ because otherwise λ_0 is not minimal. If $\psi_1 \neq 0$, then $[H,V_-] = -2V_-$ yields $V_-\psi_1 = 0$, i.e., always $V_-A_+\psi_0 = 0$. Then

$$A_\varepsilon V_{-\varepsilon} A_\varepsilon \psi_0 = 0, \quad A_\varepsilon A_{-\varepsilon}\psi_0 = \delta_{\varepsilon+1}\psi_0, \ \varepsilon = \pm 1$$

and substituting into (4.5b) gives

$$W\psi_0 = (\varkappa - 4)\psi_0 - 2\lambda_0 \psi_0.$$

Thus ψ_0 is an eigenvector of the hermitian operator W and as λ_0 is real so is $\varkappa$.

(b) Eqs.(4.1,2,5b) show that A,H,T,W are polynomial functions of $A_\pm, H, V_\pm$ and vice versa. Hence $\{A,H,T,W\}$ is irreducible iff $\{A_\pm, H, V_\pm\}$ is so.

(c) ($\mathcal{H}$2) implies $V^{\star} = -V$ which gives $V_{\varepsilon}^{\star} = -V_{-\varepsilon}$. Further $W^{\star}=W$ and $\varkappa \in \mathbb{R}$ imply $\widetilde{W}^{\star} = \widetilde{W}$.

2. The relations $A^2=1$, $[H,B]=A$, $[H,V] = \widetilde{W} + \varkappa + 4 - 4T - 2H^2$ immediately follow from (4.2,3). The remaining of ($\mathcal{H}$1), i.e., $[H,W]=0$, $[T,A]=0$ can be verified by using the following auxiliary relations that can be derived from the "basic" ones (4.3):

$$\{Z_{\varepsilon}, A_{\eta}\}=0, \quad [\widetilde{W}, A_{\varepsilon}] = 2A_{\varepsilon} + \varepsilon Z_{\varepsilon} + 4\varepsilon A_{\varepsilon} H, \quad [H, Z_{\varepsilon}] = \varepsilon Z_{\varepsilon}, \quad [V_{\varepsilon}, \widetilde{W}] = 0 \;^{+)};$$

where $Z_{\varepsilon} := [V_{\varepsilon}, A_{-\varepsilon}]$.

Verifying ($\mathcal{H}$2) and ($\mathcal{H}$3) is elementary (see also (b) of the first part of the proof). ∎

IV.2 The above lemma makes possible the following reformulation of the problem presented in the beginnig of this section:

(i') For a given positive integer N find $\mathcal{K}_N \subset \mathbb{R}$ such that $\varkappa \in \mathcal{K}_N$ iff an irreducible set $\{A_{\pm}, H, V_{\pm}\} \subset \mathrm{End}\, \mathbb{C}^N$ exists that satisfies the relations (4.3) and has the star properties (4.4).

Each such set will be denoted $\mathcal{H}_{\varkappa} \equiv \{A_{\pm}, H, V_{\pm}\}_{\varkappa}$ and called "solution" (for the given $\varkappa \in \mathcal{K}_N$).

(ii) For each $\varkappa \in \mathcal{K}_N$ find all the mutually non-equivalent solutions.

To each solution $\mathcal{H}_{\varkappa}$ corresponds just one ESS Schurean representation $\Omega_{\mathcal{H}}(\cdot) \equiv \Omega_{\mathcal{H}_{\varkappa}}(\cdot)$ of osp(1,4) (see theorem III.3), the sets $\mathcal{H}_{\varkappa}$ and $\mathcal{H}$ being uniquely assigned to each other by Eqs.(4.1), (4.2) and by the condition that $\varkappa$ is the value of the second-order Casimir operator K_2 (cf.Eq.(4.5a)): $\Omega_{\mathcal{H}}(K_2) = \varkappa$.

The following theorem gives the complete solution to this problem. We present it here without proof which is rather bulky and can be found in the second part of this paper.

Theorem:

(a) If N is odd, then $\mathcal{K}_N = \emptyset$, i.e., no solution exists.

(b) If $N=4M$, $M=1,2,\ldots$, then $\mathcal{K}_N = (2M(M-1)-4, +\infty)$ and for each $\varkappa \in \mathcal{K}_N$ there is just one solution.

(c) If $N=4M-2$, $M=1,2,\ldots$, then

$$\mathcal{K}_2 = [-\tfrac{9}{2}, +\infty), \quad \mathcal{K}_N = [2M(M-1)-\tfrac{9}{2},\; 2M(M-1)-4) \quad M=2,3\ldots .$$

The solution for a given $\varkappa \in \mathcal{K}_N$ depends on a real parameter ϑ

+) For getting $[V_{\varepsilon}, \widetilde{W}] = 0$ two additional "intermediate" relations have been derived: $Z_{\varepsilon}^2 = -4\varepsilon(V_{\varepsilon} + A_{\varepsilon} Z_{\varepsilon})$, $A_{\varepsilon}[V_{\varepsilon}, Z_{-\varepsilon}] = -4\varepsilon A_{\varepsilon} Z_{\varepsilon}(1+\varepsilon H)$.

that is related to $\varkappa$ by $\varkappa = (\vartheta^2 - 9)/2 + 2M(M-1)$. For $\varkappa = 2M(M-1) - 9/2$ there is just one solution ($\vartheta = 0$), whereas for all the other values $\varkappa \in \mathcal{K}_N$ there are just two non-equivalent solutions ($\vartheta = \pm\sqrt{2\varkappa + 9 - 4M(M-1)}$).

(d) Let $n=1,2,\ldots$, $\varkappa \in \mathcal{K}_{2n}$ and $\{A_{\pm}, H, V_{\pm}\}_{\varkappa} \subset \mathrm{End}\, \mathbb{C}^{2n}$ be a solution. Then a regular $R_D \in \mathrm{End}\, \mathbb{C}^{2n}$ exists such that the operators $A_{\pm}^{(D)}, H^{(D)}, V_{\pm}^{(D)}, \widetilde{W}^{(D)}$ defined by $Q^{(D)} := R_D Q R_D^{-1}$, $Q = A_{\pm}, H, V_{\pm}, \widetilde{W}$ +) have the following properties:

(i) The set $\mathcal{P}^{(D)} := \{H^{(D)}, V_{\pm}^{(D)}, W^{(D)}\}$, $W^{(D)} := \widetilde{W}^{(D)} + \varkappa - 4$, is reduced by four mutually orthogonal projections $F^{(\alpha)}$ onto subspaces $\mathcal{V}^{(\alpha)} \subset \mathbb{C}^{2n}$, $\dim \mathcal{V}^{(\alpha)} = n_{\alpha}$, $n_1 := E(n/2+1)$, $n_2 := E(\frac{n-1}{2})$, $n_3 := E(\frac{n+1}{2})$, $n_4 := E(n/2)$ $^{\star)}$, such that

$$\sum_{\alpha=1}^{4} F^{(\alpha)} = I_{2n}.$$

(ii) (complete description of $H^{(D)}, V_{\pm}^{(D)}, W^{(D)}$)

If $n_{\alpha} > 0$, then the restriction $\mathcal{P}_{\alpha}^{(D)} := \mathcal{P}^{(D)} \upharpoonright \mathcal{V}^{(\alpha)}$ is irreducible and

$$W^{(D)} \upharpoonright \mathcal{V}^{(\alpha)} = (\widetilde{w}_{\alpha} + \varkappa - 4)\, I_{n_{\alpha}}, \tag{4.6}$$

where

$$\widetilde{w}_1 = -\widetilde{w}_2 = \mu_0 := \begin{cases} n \\ n-\vartheta \end{cases}, \quad \widetilde{w}_3 = -\widetilde{w}_4 = \mu_1 := \begin{cases} (4\varkappa + 20 - n^2)^{1/2} & \ldots\; n=2,4,\ldots \\ n+\vartheta & \ldots\; n=1,3,\ldots \end{cases}$$

Further an orthonormal basis $\mathcal{E}_{\alpha} \subset \mathcal{V}^{(\alpha)}$ exists such that the matrices of operators $H_{\alpha}^{(D)} := H^{(D)} \upharpoonright \mathcal{V}^{(\alpha)}$, $(V_{\pm}^{(D)})_{\alpha} := V_{\pm}^{(D)} \upharpoonright \mathcal{V}^{(\alpha)}$ w.r.t. $\mathcal{E}_{\alpha}$ satisfy

$$\{[H_{\alpha}^{(D)}], [(V_{\pm}^{(D)})_{\alpha}]\} = \begin{cases} \mathcal{M}_1(n_{\alpha}, \widetilde{w}_{\alpha} + \varkappa - 4) & \ldots\; n=2,4,\ldots \\ \mathcal{M}_2(n_{\alpha}, \tfrac{1}{2}(\vartheta - (-1)^{E(\alpha/2)}) & \ldots\; n=1,3,\ldots \end{cases} \tag{4.7}$$

the sets $\mathcal{M}_r$ being given by Eq.(2.9).

(iii) $A_{-}^{(D)} = (A_{+}^{(D)})^{\star}$.

(iv) (complete description of $A_{+}^{(D)}$)

Let $A^{(\alpha\beta)}$ be the operator from $\mathcal{V}^{(\beta)}$ to $\mathcal{V}^{(\alpha)}$ that is obtained by restricting $i\sqrt{\mu_0 \mu_1}\, F^{(\alpha)} A_{+}^{(D)} F^{(\beta)}$ to $\mathcal{V}^{(\beta)}$.

+) $\widetilde{W}$ is the polynomial (4.3a).

$\star$) The function $E: \mathbb{R} \to \mathbb{Z} \equiv \{0, \pm 1, \pm 2, \ldots\}$ is defined by $E(x) := \sup\{n \mid n \in \mathbb{Z},\, n \leq x\}$.

Then $A^{(\alpha,\beta)} = A^{(\alpha+2,\beta+2)} = 0$, $\alpha,\beta=1,2$ and for the remaining eight pairs (α,β) the matrices of $A^{(\alpha,\beta)}$ w.r.t. the bases introduced in (ii) have the following elements $a_{kl}^{(\alpha,\beta)}$, $1 \leq k \leq n_\alpha$, $1 \leq l \leq n_\beta$:

$$a_{kl}^{(13)} = \delta_{l+1-k}\sqrt{2l(\mu_1-\tau_l)}\ , \quad a_{kl}^{(14)} = -\delta_{l+1-k}\sqrt{2l\tau_l}\ ,$$
$$a_{kl}^{(23)} = \delta_{l-k}\sqrt{(\mu_0-2l)(\mu_1-\tau_l)}\ , \quad a_{kl}^{(24)} = -\delta_{l-k}\sqrt{(\mu_0-2l)\tau_l}\ ,$$
$$a_{kl}^{(31)} = \delta_{l-k}\sqrt{(\mu_0+2-2l)\tau_l}\ , \quad a_{kl}^{(32)} = -\delta_{l+1-k}\sqrt{2l\tau_{l+1}}\ ,$$
$$a_{kl}^{(41)} = \delta_{l-k}\sqrt{(\mu_0+2-2l)(\mu_1-\tau_l)}, \quad a_{kl}^{(42)} = -\delta_{l+1-k}\sqrt{2l(\mu_1-\tau_{l+1})},$$

where $\tau_l := 2l-1+(\mu_1-\mu_0)/2$.

<u>Corollary</u>: The representation $\Omega_{\mathcal{H}_\varkappa^{(D)}}$ is a $*$-representation of $B(0,2)$.

<u>Remark</u>: The statement (d) can be expressed in terms of classes $\mathcal{C}$, $\mathcal{C}_0$ (see Eq.(3.6)): to any $\Omega_{\mathcal{H}_\varkappa} \in \mathcal{C}$ there is a regular R such that $\Omega_{R\mathcal{H}_\varkappa R^{-1}} \in \mathcal{C}_0$. Thus the classes $\mathcal{C}$, $\mathcal{C}_0$ are equal up to equivalence transformations $\mathcal{H}_\varkappa \mapsto R\mathcal{H}_\varkappa R^{-1}$. In this context it should be emphasized that, in view of the star properties (4.4), <u>not each</u> regular R leaves $\mathcal{C}$ invariant, i.e., transforms any solution $\mathcal{H}_\varkappa$ again to a solution. If R_D is the regular operator such that $\Omega_{R_D\mathcal{H}_\varkappa R_D^{-1}} \in \mathcal{C}_0$, then, using the conditions (4.4), one sees that $R\mathcal{H}_\varkappa R^{-1}$ is a solution iff $S := R_D R^* R R_D^{-1}$ commutes with each element of the set $\mathcal{P}^{(D)}$. Now this set equals direct sum of four irreducible sets $\{H_\alpha^{(D)}, (V_\pm^{(D)})_\alpha, \widetilde{W}_\alpha^{(D)}+\varkappa-4\}$, where $\widetilde{W}_\alpha^{(D)} = \widetilde{w}_\alpha I_{n_\alpha}$. Thus the "blocks" $S^{(\alpha\beta)} := F^{(\alpha)}SF^{(\beta)}$ satisfy $(\widetilde{w}_\alpha-\widetilde{w}_\beta)S^{(\alpha\beta)}=0$. It may happen that $\widetilde{w}_\alpha = \widetilde{w}_\beta$ for $\alpha \neq \beta$: Eq.(4.6) shows that this occurs for $N=4M$ iff $\varkappa = 2M^2-5$, $|\alpha-\beta|=2$ and for $n=4M-2$ iff $\vartheta=0$, $|\alpha-\beta|=2$. In all such cases the condition $[H^{(D)},S]=0$ implies $S^{(\alpha\beta)}=0$ – this can be verified by using Eq.(4.7) and the explicit form (2.8a) of $H^{(D)}$. Thus $S^{(\alpha\beta)} = \delta_{\alpha-\beta}S^{(\alpha\alpha)}$ and, by taking into account irreducibility of $\{H_\alpha^{(D)}, (V_\pm^{(D)})_\alpha, \widetilde{W}_\alpha^{(D)}\}$ and the fact that S is equivalent to the regular positive operator R^*R, one finally gets

$$S = \bigoplus_{\alpha=1}^{4} s_\alpha I_n\ , \quad s_\alpha > 0\ .$$

Hence for a given solution $\mathcal{H}_\varkappa \subset \mathrm{End}\,\mathbb{C}^N$ the set $R\mathcal{H}_\varkappa R^{-1}$ is solution iff there are positive $s_1,\ldots,s_4$ such that

$$R^*R = R_D^{-1} \bigoplus_{\alpha=1}^{4} s_\alpha I_{n_\alpha} R_D\ .$$

Similarly, irreducibility of $\mathcal{H}_\varkappa$ implies: if $\mathcal{H}_\varkappa \subset \mathrm{End}\,\mathbb{C}^N$ is a solution such that $\Omega_{\mathcal{H}_\varkappa} \in \mathcal{C}_0$, then $\Omega_{R\mathcal{H}_\varkappa R^{-1}} \in \mathcal{C}_0$ iff $R^* = cR^{-1}$, $c>0$.

APPENDIX

The following lemma is very useful for verifying assertion (a) of theorem III.3 and also if one wants to check that Eqs.(3.4,5) are correct: commutativity of $\hat{K}_2$, $\hat{K}_4$ with A,H,T,W is obviously a necessary condition.

<u>Lemma</u>: If the set $\mathcal{M} \equiv \{A,H,T,W\}$ satisfies the conditions $(\mathcal{M}1)$, then the extended set $\widetilde{\mathcal{M}} := \mathcal{M} \cup \{V,B,Z,\widetilde{Z}\}$ fulfils the following relations [+)]:

$$[T,H]=V,\quad [V,H]=4T-W-8+2H^2,\quad [W,H]=0,$$
$$[V,T]=-4HT-2V,\quad [W,T]=0,\quad [W,V]=0, \tag{A.1}$$

$$[H,A]=B,\quad [H,B]=A,\quad [H,Z]=\widetilde{Z}-A-2BH,\quad [H,\widetilde{Z}]=Z+B+2AH,$$
$$[T,A]=0,\quad [T,B]=Z,\quad [T,Z]=2Z+4BT,\quad [T,\widetilde{Z}]=0,$$
$$[V,A]=Z,\quad [V,B]=A-\widetilde{Z}+2BH,\quad [V,Z]=2(A-\widetilde{Z})+2(BV-ZH)+4(BH-AT),$$
$$[V,\widetilde{Z}]=3Z+4BT+2AV,$$
$$[W,A]=2\widetilde{Z},\quad [W,B]=2(Z+B)+4AH,\quad [W,Z]=6Z+8BT+4AV,$$
$$[W,\widetilde{Z}]=8\widetilde{Z}+26A+4(AW+ZH-BV)-8(AT+BH), \tag{A.2}$$

$$\{A,A\}=2,\quad \{B,A\}=0,\quad \{Z,A\}=0,\quad \{\widetilde{Z},A\}=0,$$
$$\{B,B\}=-2,\quad \{Z,B\}=0,\quad \{\widetilde{Z},B\}=-4H-2AB,$$
$$\{Z,Z\}=8T-4BZ,\quad \{\widetilde{Z},Z\}=-4V-2AZ,$$
$$\{\widetilde{Z},\widetilde{Z}\}=-30+8T-4W+4(BZ-A\widetilde{Z}). \tag{A.3}$$

<u>Proof</u>: Most of these relations are obtained directly from the basic ones $(\mathcal{M}1)$ and definitions (3.3d) with the help of the Jacobi identity and/or of its generalized form

$$\{K,[L,M]\} = [L,\{M,K\}] + \{M,[K,L]\}.$$

E.g. $[T,B] = [T,[H,A]] = -[H,[A,T]] - [A,[T,H]] = [V,A] = Z$,
$\{A,B\} = \{A,[H,A]\} = [H,\{A,A\}] + \{A,[A,H]\} = -\{A,B\}$.

+) For the sake of completeness the definitions (3.3d) as well as the relations $(\mathcal{M}1)$ are repeated.

In some cases one has to calculate two commutators simultaneously. As an illustration consider $[Z,H]$. The substitution $Z=[V,A]$ gives

$$[Z,H] = -[[A,H],V] - [[H,V],A] = [B,V] - [W,A] + 2[H^2,A]$$
$$= [B,V] - 2\tilde{Z} + 2A + 4BH.$$

On the other hand, by using $[T,B] = Z$, we **get**

$$[Z,H] = -[[B,H],T] - [[H,T],B] = [A,T] + [V,B] = [V,B] .$$

Hence $[Z,H] = [V,B] = A + 2BH - \tilde{Z}$. $\blacksquare$

The lemma can help in calculations in which polynomial functions of operators A,H,T,W occur. In view of relations $(\mathcal{M}1)$ not all elements of the tensor algebra $\mathcal{A}(\mathcal{M})$ generated by A,H,T,W are independent and it is important to pass to a factor algebra with independent elements. Performing such a factorization is difficult in the framework of $\mathcal{A}(\mathcal{M})$ since the relations $(\mathcal{M}1)$, if written in terms of A,H,T,W only, are complicated (double commutators). On the other hand, the structure of (A.1-3) is much simpler: after introducing ordering in $\widetilde{\mathcal{M}}$ by $A \prec B \prec Z \prec \tilde{Z} \prec H \prec T \prec V \prec W$, one sees that (A.1-3) automatically guarantee that any polynomial function of A,B,..,W of the second order can be "brought to the ordered form", i.e., expressed as a linear combination of ordered polynomials of at most second order.

One can thus try to factorize the tensor algebra $\mathcal{A}(\widetilde{\mathcal{M}})$ along its ideal $\mathcal{J}$ generated by the relations (A.1-3) and check whether in $\mathcal{A}(\widetilde{\mathcal{M}})/\mathcal{J}$ a basis formed by ordered polynomials exists (this would be, e.g., the case if the Poincaré-Birkhoff-Witt theorem applies). Up to now we have not solved this question; we only can add that using (A.1-3) we were able to bring to the ordered form all the polynomials up to the fourth order that occurred in calculating $\hat{K}_2$ and $\hat{K}_4$.

Finally, we want to draw attention to the almost superalgebraic structure of (A.1-3) with $A,B,Z,\tilde{Z}$ odd and H,T,V,W even elements, the only difference from the usual LSA being that the r.h.s are quadratic (ordered) polynomials in generators.

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Беднарж М. и др. E2-82-771

Представления $osp(1,4)$ при помощи трех бозонных пар и матриц произвольного четного порядка. Описание метода

Построено двухпараметрическое семейство бесконечномерных шуровски-неприводимых представлений супералгебры Ли $osp(1,4)$. Эти представления выведены чисто алгебраическим путем. Генераторы выражаются при помощи тензорных произведений матриц $2n \times 2n$ и полиномов в девяти некоммутирующих переменных p_j, q_j, q_j^{-1}, $j = 1,2,3$, где p_j, q_j удовлетворяют каноническим коммутационным соотношениям. Помимо дискретного параметра n, представления характеризуются вещественным непрерывным параметром, который является собственным значением оператора Казимира второго порядка. Четные генераторы представлены кососимметрическим образом. Показано, что это свойство влечет за собой то, что каждое представление из описанного семейства эквивалентно представлению парабозевской алгебры $pB(2)$ со следующим физическим "свойством сопряжения": операторы рождения и уничтожения взаимно сопряжены.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Сообщение Объединенного института ядерных исследований. Дубна 1982

Bednář M. et al. E2-82-771

Representations of $osp(1,4)$ in Terms of Three Boson Pairs and Matrices of Arbitrary Even Order. Description of the Method

Two-parameter family of infinite-dimensional Schur-irreducible representations of the Lie superalgebra $osp(1,4)$ is presented. The representations are derived in a purely algebraic way. The generators are represented in terms of tensor products of matrices $2n$ by $2n$ and polynomial functions of nine non-commuting variables p_j, q_j, q_j^{-1}, $j = 1,2,3$, the p_j, q_j satisfying the canonical commutation relations. Besides the discrete parameter n, the representations are labelled by a continuous real parameter - the eigenvalue of the second-order Casimir operator. The even generators are represented skew-symmetrically and this property is shown to imply that each of the representations in the family is equivalent to a representation of the para-Bose algebra $pB(2)$ having the physical "star properties": the creation and annihilation operators are adjoint to each other.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna 1982