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дубна

E2-82-761

V.V.Nesterenko

NONLINEAR TWO-DIMENSIONAL SIGMA MODEL WITH THE PSEUDOORTHOGONAL SYMMETRY GROUP

Submitted to "ЯФ"

1982

1. Introduction

Nonlinear sigma models considered in elementary particle physics are constructed on the basis of the compact symmetry groups /1-4/. However, recently it has been noted in some papers that the sigma models with noncompact, in particular, with pseudo-orthogonal groups of symmetry, are seemingly interesting. For example, the two-dimensional Gross-Neveu model appears to be connected with the nonlinear sigma model on SO(1,2)/SO(2) /5,6/. In the classical theory of the relativistic string there arise naturally the nonlinear two-dimensional sigma models on the symmetric spaces SO(1,n-1)/[SO(1,1)xSO(n-2)], where n is a dimension of the space-time in which the string is moving /7,8/.

The nonlinear two-dimensional sigma model with the fields taking values on spheres S^n can be reduced to a set of interacting fields, which are scalars under the SO(n)-group /9-11/. Such a treatment of the sigma model appears to be convenient for the investigation of its equations of motion by the inverse scattering method, for obtaining the infinite sets of the local and nonlocal conservation laws, etc.

In this note the procedure of reduction to the interacting scalar fields will be carried out for nonlinear two-dimensional sigma models on symmetric spaces SO(1,2)/SO(1,1) and SO(1,3)/[SO(1,1)xSO(2)]. First, these models will be formulated in terms of the gauge fields interacting with the vector massless fields /12/. Further a special gauge will be imposed on these fields in which the gauge fields are related with the vector fields. It is important that this gauge can be chosen only in the case of the pseudo-orthogonal symmetry groups, and in the usual sigma model on the sphere S^n it can not be taken. An analogous gauge was considered in the geometrical theory of the relativistic string /13/, and it is this theory in which the nonlinear sigma model with the pseudo-orthogonal symmetry group arises /7,8/. The final result of this note is sets of nonlinear equations (2.22), (2.23) and (3.8) describing the sigma models with the pseudo-orthogonal symmetry group.

2. SO(1,2)/SO(1,1) - sigma model

Following the paper[12] we represent the field variables in terms of the matrices $G_i(u^1,u^2)$, $i=1,2$ from the Lie algebra of the SO(1,2)-group. These matrices are splitted into the abelian gauge field A_i with values in Lie algebra of the gauge group SO(1,1) and into the vector field B_i

$$G_i = A_i + B_i\,, \quad i=1,2, \tag{2.1}$$

$$A_i = \begin{vmatrix} 0 & a_i & 0 \\ a_i & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}, \qquad B_i = \begin{vmatrix} 0 & 0 & b_i \\ 0 & 0 & c_i \\ b_i & -c_i & 0 \end{vmatrix}.$$

The gauge transformations of the field variables G_i are carried out by means of the matrix

$$g[\lambda(u^1,u^2)] = \begin{vmatrix} ch\lambda & sh\lambda & 0 \\ sh\lambda & ch\lambda & 0 \\ 0 & 0 & 1 \end{vmatrix} \tag{2.2}$$

in the following way

$$\bar{G}_i = g^{-1} G_i g - g^{-1}\partial_i g, \quad i=1,2, \tag{2.3}$$

the field A_i being transformed as the abelian gauge field

$$\bar{A}_i = g^{-1} A_i g - g^{-1}\partial_i g,$$
$$\bar{a}_i = a_i - \partial_i \lambda, \quad i=1,2 \tag{2.4}$$

and the field B_i as the SO(1,1)- doublet of the massless fields

$$\bar{B}_i = g^{-1} B_i g,$$
$$\bar{b}_i = b_i\, ch\lambda - c_i\, sh\lambda, \qquad \bar{c}_i = -b_i\, sh\lambda + c_i\, ch\lambda, \quad i=1,2. \tag{2.5}$$

In terms of these variables the nonlinear sigma model on the symmetric space SO(1,2)/SO(1,1) is given by the compatibility condition

$$G_{1,2} - G_{2,1} + [G_1, G_2] = 0 \tag{2.6}$$

and by the equations of motion[12]

$$D_i B^i = 0, \quad i=1,2, \tag{2.7}$$

where D_i denotes the covariant derivative with respect to the gauge group SO(1,1) [14]

$$D_i = \partial_i - ad A_i\,, \qquad ad A_i(B_j) = [A_i, B_j].$$

We choose the following metric signature in the coordinate space $\{u^1, u^2\}: (+,-)$. It is easy to verify the covariance of eqs. (2.6) and (2.7) under the transformations (2.3) and (2.5).

Before investigating eqs. (2.6) and (2.7), we fix the gauge in the theory. We choose the gauge condition in an unusual form. We demand that the gauge field a_i be equal to one of the vector fields, for example, to c_i

$$a_i(u^1,u^2) = c_i(u^1,u^2). \tag{2.8}$$

Let us prove that these conditions can be always satisfied by means of the gauge transformation (2.3) with the parameter $\lambda(u^1,u^2)$ chosen in an appropriate way. For this purpose it is convenient to introduce the matrix exterior differential forms

$$G = G_i\, du^i; \quad A = A_i\, du^i; \quad B = B_i\, du^i \tag{2.9}$$

and the differential forms for the field variables

$$a = a_i\, du^i; \quad b = b_i\, du^i; \quad c = c_i\, du^i$$

and to use the exterior differentiation[15]. If the transformed matrices G_i (2.3) obey the conditions (2.8), then we can write

$$a - d\lambda = -b\, sh\lambda + c\, ch\lambda. \tag{2.10}$$

This equation enables us to determine the function $\lambda(u^1,u^2)$ when its integrability condition

$$d^2\lambda = d\wedge d\lambda = 0 \tag{2.11}$$

is fulfilled. It is easy to show that this condition is satisfied indeed if one takes into account that the linear forms a, b, c obey the compatibility conditions (2.6) which are written in terms of the differential forms as follows

$$dG = G \wedge G. \tag{2.12}$$

Thus the gauge condition (2.8) can be chosen always.

It should be noted that the integrability condition (2.11) is fulfilled by virtue of (2.10) and (2.12) for the matrices G_i from the Lie algebra of the pseudo-orthogonal groups only. For orthogonal groups with real matrices G_i the gauge condition (2.8) cannot be imposed.

In the gauge (2.8) the compatibility conditions (2.6) are reduced to two equations

$$a_{1,2} - a_{2,1} = a_2 b_1 - a_1 b_2 , \tag{2.13}$$

$$b_{1,2} - b_{2,1} = 0, \tag{2.14}$$

where $f_{j,k} = \partial f_j / \partial u^k$, $j,k = 1,2$. The equations of motion (2.7) become

$$b_{1,1} - b_{2,2} = a_1^2 - a_2^2 , \tag{2.15}$$

$$a_{1,1} - a_{2,2} = a_1 b_1 - a_2 b_2 . \tag{2.16}$$

From (2.14) it follows that the fields b_1 and b_2 are expressed in terms of partial derivatives of one function, $\Theta(u^1,u^2)$

$$b_i = \Theta_{,i} , \quad i = 1,2. \tag{2.17}$$

Taking into account (2.7) we can write eqs. (2.13)-(2.16) in the form

$$\Theta_{,11} - \Theta_{,22} = a_1^2 - a_2^2 , \tag{2.18}$$

$$a_{1,2} - a_{2,1} = a_2 \Theta_{,1} - a_1 \Theta_{,2} , \tag{2.19}$$

$$a_{1,1} - a_{2,2} = a_1 \Theta_{,1} - a_2 \Theta_{,2} . \tag{2.20}$$

The substitution

$$a_1 = e^{\Theta} \cdot \varkappa_{,2} , \qquad a_2 = e^{\Theta} \cdot \varkappa_{,1} \tag{2.21}$$

satisfies eq. (2.20) identically. As a result, we obtain the set of two nonlinear equations for two field variables $\Theta(u^1,u^2)$ and $\varkappa(u^1,u^2)$

$$\Theta_{,11} - \Theta_{,22} + e^{2\Theta}(\varkappa_{,1}^2 - \varkappa_{,2}^2) = 0, \tag{2.22}$$

$$\varkappa_{,11} - \varkappa_{,22} - 2(\varkappa_{,1}\Theta_{,1} - \varkappa_{,2}\Theta_{,2}) = 0. \tag{2.23}$$

So, the nonlinear SO(1,2)/SO(1,1) - sigma model is described in the gauge (2.8) by two equations (2.22) and (2.23).

3. SO(1,3)/[SO(1,1)xSO(2)] -sigma model

In this case there are two gauge abelian fields a_i^1 and a_i^2, and the matrices A and B in (2.1) have the form

$$A_i = \begin{vmatrix} 0 & a_i^1 & 0 & 0 \\ a_i^1 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_i^2 \\ 0 & 0 & -a_i^2 & 0 \end{vmatrix}, \qquad B_i = \begin{vmatrix} 0 & 0 & b_i^1 & b_i^2 \\ 0 & 0 & c_i^1 & c_i^2 \\ b_i^1 & -c_i^1 & 0 & 0 \\ b_i^2 & -c_i^2 & 0 & 0 \end{vmatrix}. \tag{3.1}$$

The gauge transformations (2.3) are carried out with the matrix

$$g[\lambda(u^1,u^2),\varphi(u^1,u^2)] = \begin{vmatrix} ch\lambda & sh\lambda & 0 & 0 \\ sh\lambda & ch\lambda & 0 & 0 \\ 0 & 0 & \cos\varphi & -\sin\varphi \\ 0 & 0 & \sin\varphi & \cos\varphi \end{vmatrix}. \tag{3.2}$$

As we have now two gauge functions $\lambda(u^1,u^2)$ and $\varphi(u^1,u^2)$, then on the field variables (3.1) we can impose two gauge conditions. We take these conditions in the form

$$a_i^1 = c_i^1 , \qquad a_i^2 = -b_i^2 , \quad i = 1,2. \tag{3.3}$$

For conditions (3.3) to take place, the gauge functions λ and φ have to obey the following equations

$$d\lambda = a^1 + \cos\varphi(sh\lambda \cdot b^1 - ch\lambda \cdot c^1) + \sin\varphi(sh\lambda \cdot b^2 - ch\lambda \cdot c^2), \tag{3.4}$$

$$d\varphi = -a^2 + \sin\varphi(ch\lambda\, b^1 - sh\lambda \cdot c^1) - \cos\varphi(ch\lambda \cdot b^2 - sh\lambda \cdot c^2).$$

It can be verified directly that the integribility conditions of these equations

$$d^2\lambda = 0, \qquad d^2\varphi = 0 \tag{3.4}$$

are fulfilled by virtue of (2.12) and (3.1).

Thus, the gauge (3.3) is acceptable. In the gauge (3.3) the compatibility conditions (2.6) are written as

$$
\begin{aligned}
& a_{1,2} - a_{2,1} = a_2 b_1 + c_2 d_1 - a_1 b_2 - c_1 d_2, \\
& d_{1,2} - d_{2,1} = a_2 c_1 - a_1 c_2 + b_1 d_2 - b_2 d_1, \\
& b_{1,2} - b_{2,1} = 0, \qquad c_{1,2} - c_{2,1} = 0,
\end{aligned} \tag{3.5}
$$

where

$$
a_i = a_i^1, \quad b_i = b_i^1, \quad d_i = b_i^2, \quad c_i = c_i^2, \quad i = 1, 2.
$$

The Euler equations $D_i B^i = 0$ have the form

$$
\begin{aligned}
& a_{1,1} - a_{2,2} = a_1 b_1 - c_1 d_1 - a_2 b_2 + c_2 d_2, \\
& d_{1,1} - d_{2,2} = a_1 c_1 + b_1 d_1 - a_2 c_2 - b_2 d_2, \\
& b_{1,1} - b_{2,2} = (a_1)^2 - (a_2)^2 - (d_1)^2 + (d_2)^2, \\
& c_{1,1} - c_{2,2} = 2(a_1 d_1 - a_2 d_2).
\end{aligned} \tag{3.6}
$$

From the last two equations of the set (3.6) it follows that

$$
b_i = \Theta_{,i}, \qquad c_i = \varkappa_{,i}. \tag{3.7}
$$

Finally, the nonlinear SO(1,3)/[SO(1,1)xSO(2)] -sigma model in the gauge (3.3) is described by the following set of nonlinear equations

$$
\begin{aligned}
& \Theta_{,11} - \Theta_{,22} = a_1^2 - a_2^2 - d_1^2 + d_2^2, \qquad \varkappa_{,11} - \varkappa_{,22} = 2(a_1 d_1 - a_2 d_2), \\
& a_{1,1} - a_{2,2} = a_1 \Theta_{,1} - a_2 \Theta_{,2} - d_1 \varkappa_{,1} + d_2 \varkappa_{,2}, \\
& a_{1,2} - a_{2,1} = a_2 \Theta_{,1} - a_1 \Theta_{,2} - d_2 \varkappa_{,1} + d_1 \varkappa_{,2}, \\
& d_{1,1} - d_{2,2} = d_1 \Theta_{,1} - d_2 \Theta_{,2} + a_1 \varkappa_{,1} - a_2 \varkappa_{,2}, \\
& d_{1,2} - d_{2,1} = -d_1 \Theta_{,1} + d_2 \Theta_{,1} + a_2 \varkappa_{,1} - a_1 \varkappa_{,2}.
\end{aligned} \tag{3.8}
$$

Probably the substitutions analogous to (2.21) have to be here which reduce the last four equations in (3.8) to two equations of the second order.

4. Conclusion

For the physical applications of the nonlinear sigma models considered here (for example, in the relativistic string theory /7,8/) it is interesting to outline what is a consequence of the noncompact nature of the symmetry group in terms of the nonlinear

equations (2.22), (2.23) and (3.8), and are these equations completely integrable from the view point of the inverse scattering method? For this aim the Lax representations for these equations must be found. Unfortunately, the method used in the sigma models on spheres /1,9,11/ for this purpose cannot be applied here.

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Received by Publishing Department on November 3 1982.

Нестеренко В.В. E2-82-761

Нелинейная двумерная сигма-модель с псевдоортогональной группой симметрии

Нелинейная двумерная сигма-модель на симметрических пространствах $SO(1,2)/SO(1,1)$ и $SO(1,3)/[SO(1,1)\times SO(2)]$ формулируется как теория калибровочных полей, взаимодействующих с векторными безмассовыми полями. Предлагается специальная калибровка, в которой калибровочные поля оказываются связанными с векторными полями. В этой калибровке явно выписаны системы нелинейных уравнений, к которым сводятся уравнения движения в рассматриваемой сигма-модели.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1982

Nesterenko V.V. E2-82-761

Nonlinear Two-Dimensional Sigma Model with the Pseudoorthogonal Symmetry Group

Nonlonear two-dimensional sigma model on the symmetric spaces $SO(1,2)/SO(1,1)$ and $SO(1,3)/[SO(1,1)\times SO(2)]$ is formulated in terms of the gauge fields interacting with the vector massless fields. A special gauge is proposed in which the gauge fields are related with the vector fields. In this gauge the equations of motion are reduced to the systems of nonlinear equations on the scalar functions. These equations are written obviously.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1982