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RELATIVISTIC STRING THEORY AND NONLINEAR TWO-DIMENSIONAL SIGMA-MODEL

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1. INTRODUCTION

In many fields of the theoretical physics there arises in the last time the notion of the one-dimensional spatially extended relativistic object, the relativistic string. One may mention here the nonlinear two-dimensional Born-Infeld type field models[1], the dual-resonance approach to the hadron physics[2], the problem of the quark confinement into hadrons[3,4], the clarification of the galaxy formation mechanism in cosmology[5,6].

Recently new formulations of the relativistic string model have been proposed which use either unfamiliar mathematical methods[7-9], or new physical ideas[10]. It is very interesting also to determine the connection of this model with the well investigated quantum field models, for example, with two-dimensional gauge models[11].

It will be shown in this paper in what way the nonlinear two-dimensional sigma model[12] with the SO(1,2)/SO(1,1) symmetry group appears in the theory of the relativistic string. For this purpose we derive equations which determine the unit normal $m^\mu(u^1,u^2)$ at any point of the world sheet of the string. These equations are

$$\nabla_i \nabla^i m^\mu - (m^\nu_{,i}\, m^{,i}_{\nu})\, m^\mu = 0, \tag{1.1}$$

where ∇_i is the covariant differentiation with respect to the inner metric on the world sheet of the string which is the minimal surface in the three-dimensional Minkowski space. If in the relativistic strong theory the orthonormal gauge is used, then Eq. (1.1) is reduced to the usual n-field equation[12,13] the n-field taking value on the hyperboloid of one sheet $(x^0)^2-(x^1)^2-(x^2)^2 = -1$.

2. THE RELATIVISTIC STRING DYNAMICS AND THE SURFACE THEORY

Let us recall the basic geometric ideas in the relativistic string theory[2,14]. This theory describes the one-dimensional spatially extended object, the action of which is proportional to the area of its world sheet in the Minkowski space. Let $x^\mu(u^1,u^2)$, $\mu = 0,1,2,\ldots$ be the parametric representation of this sheet. Its intrinsic geometry is defined, as is well known[15,16] by the metric tensor

$$g_{ij}(u^1,u^2) = \frac{\partial x^\mu}{\partial u^i}\frac{\partial x_\mu}{\partial u^j} = x^\mu_{,i}\, x_{\mu,j}\,, \quad x^\mu_{,i} = \frac{\partial x^\mu}{\partial u^i}, \tag{2.1}$$

$$i,j = 1,2, \quad \mu = 0,1,\ldots .$$

In space-time we shall use the following metric signature (+−−...).

From the physical point of view one demands in the relativistic string theory that one tangent vector, for example $x^\mu_{,1}$, has to be a time-like vector and the second tangent vector $x^\mu_{,2}$ must be space-like.

$$(x^\mu_{,1})^2 = g_{11} > 0, \qquad (x^\mu_{,2})^2 = g_{22} < 0. \tag{2.2}$$

In this case the parameter $u^1 = \tau$ is the evolution parameter and the second parameter $u^2 = \sigma$ specifies the points along the string. By virtue of (2.2) $g = \det\|g_{ij}\| < 0$ and the differential element of the string world surface area is $\sqrt{-g}\,du^1du^2$. The action of the relativistic string is defined by

$$S = -\gamma \iint d^2u \sqrt{-g}\,, \tag{2.3}$$

where γ is a dimensional constant. The principle of least action, as applied to the functional S,

$$\frac{\delta\sqrt{-g}}{\delta x_\mu} = 0, \tag{2.4}$$

leads to the problem of determining in the space-time the two-dimensional minimal area surface.

In addition to the description of the surface by its radius-vector $x^\mu(u^1,u^2)$ in the differential geometry/15,16/ there is the possibility to use for this purpose the basic squared differential forms of the surface. This description of the surface arises naturally by consideration of the moving frame on the surface. There are various methods of the choice of this frame. It is convenient to take a basis formed by two tangent vectors to the surface $x^\mu_{,1}$ and $x^\mu_{,2}$ and by the unit normal.

Further we shall consider the theory of the relativistic string in the three-dimensional Minkowski space. By virtue of (2.2) the normal $m^\mu(u^1,u^2)$ is the unit space-like vector

$$m^2 = -1. \tag{2.5}$$

The motion of the basis $\{x^\mu_{,1}, x^\mu_{,2}, m^\mu\}$ along the surface is described by the well-known equations/15/ of Gauss

$$\nabla_i x^\mu_{,j} = -b_{ij}\, m^\mu \tag{2.6}$$

and Weingarten

$$m^\mu_{,i} = -b^j_i\, x^\mu_{,j}\,, \tag{2.7}$$

$$\mu = 0,1,2, \quad i,j = 1,2.$$

Here ∇_i denotes the covariant differentiation with respect to the metric tensor g_{ij} (2.1). This tensor is used also for raising and lowering the Latin indices. In Eq. (2.6) $b_{ij}(u^1,u^2)$ is a tensor of the second squared form of the surface. Actually, equation (2.6) can be considered as the definition of the tensor b_{ij}. Thus, the motion of the basis $\{x^\mu; x^\mu_{,1}, x^\mu_{,2}, m^\mu\}$ along the surface will be defined by Eqs. (2.6) and (2.7) completely if the first (g_{ij}) and the second (b_{ij}) fundamental forms of the surface are given. These forms cannot be arbitrary but they have to obey the integrability conditions of the linear Eqs. (2.6) and (2.7). These conditions are partial nonlinear equations for g_{ij} and b_{ij}. In the differential geometry/15/ they are called the Gauss equation

$$R_{ijk\ell} = b_{i\ell}\, b_{jk} - b_{ik}\, b_{j\ell} \tag{2.8}$$

and the Peterson-Codazzi equations

$$\nabla_k b_{ij} = \nabla_j b_{ik}\,, \tag{2.9}$$

$$i,j,k = 1,2.$$

Here $R_{ijk\ell}$ is the curvature tensor for the metric g_{ij}/15/. The knowledge of the basis $\{x^\mu_{,1}, x^\mu_{,2}, m^\mu\}$ at any point of the surface enables us to reconstruct by further integration the surface $x^\mu(u^1,u^2)$ itself. This reasoning is formulated in the differential geometry/16/ as the basic theorem of the surface theory that reads: the symmetric tensors g_{ij} and b_{ij} which satisfy the Gauss and Peterson-Codazzi equations (2.8), (2.9) determine the surface up to its motion as a whole in space. Thus, for the determination of the surface one can use instead of its radius-vector $x^\mu(u^1,u^2)$ the tensory g_{ij} and b_{ij}, which satisfy Eqs. (2.8) and (2.9).

The world sheet of the relativistic string is a minimal surface/2,14/. In terms of the squared differential forms g_{ij} and b_{ij} this condition is written as

$$g^{ij}\, b_{ij} = b^i_i = 0. \tag{2.10}$$

This equation is in fact another form of string equations of motion (2.4)/14,15/.

3. EQUATIONS FOR THE NORMAL TO THE WORLD SHEET OF THE STRING

As was mentioned above a natural candidate for the field variable in the nonlinear sigma-model connected with the relativistic string is the normal to the world sheet of the string $m^{\mu}(u^1,u^2)$, $\mu=0,1,2$. This normal maps the minimal surface which is the world sheet of the string onto the hyperboloid of one sheet

$$(x^0)^2-(x^1)^2-(x^2)^2=-1. \qquad (3.1)$$

Let us obtain a closed set of equations for the normal $m^{\mu}(u^1,u^2)$. For this purpose we act by the operator ∇_i on the left and right-hand sides of Eq. (2.6) and sum over $i=1,2$

$$\nabla_i\nabla^i m^{\mu}=-\nabla^i(b^j_i\, x^{\mu}_{,j})=-(\nabla^i b^j_i)x^{\mu}_{,j}-b^j_i\nabla^i x^{\mu}_{,j}. \qquad (3.2)$$

Here we have used the obvious equality $\nabla_i m^{\mu}=m^{\mu}_{,i}$, as the normal $m^{\mu}(u^1,u^2)$ and the radius vector of the surface $x^{\mu}(u^1,u^2)$ are scalars under the transformation of the curvilinear coordinates on the surface u^1,u^2. We transform the first term in the right-hand side of (3.2) making use of Peterson-Codazzi equations (2.9) and the second term with the Gauss derivative formulas (2.6). As a result, we get

$$\nabla_i\nabla^i m^{\mu}=-(\nabla^j b^i_i)x^{\mu}_{,j}+b^j_i b^i_j m^{\mu}. \qquad (3.3)$$

As the string world sheet is a minimal surface, then by virtue of (2.10) $b^i_i=0$. Eq. (3.3) becomes now

$$\nabla_i\nabla^i m^{\mu}=b^j_i b^i_j m^{\mu}. \qquad (3.4)$$

Making use of the Weingarten equations, (2.7) and Eq. (2.1) one obtains easily

$$m^{\mu}_{,i}\, m^{,i}_{\mu}=b^j_i b^l_k x^{\mu}_{,j} x^{,k}_{\mu}=b^j_i b^i_k g^k_j=b^j_i b^i_j. \qquad (3.5)$$

Finally we have the following equations for the normal $m^{\mu}(u^1,u^2)$

$$\nabla_i\nabla^i m^{\mu}-(m^{\nu}_{,i}\, m^{,i}_{\nu})m^{\mu}=0. \qquad (3.6)$$

Recall, that the covariant differentiation ∇_i is made here with respect to the inner metric on the world sheet of string (2.1). Therefore Eqs. (3.6) will be a closed set of equations provided the metric of the minimal surface (2.1) will be given. However, the dependence of Eqs. (3.6) on the metric tensor (2.1) is formal because on the world sheet of the string the conformally-flat coordinate set

$$g_{11}=-g_{22}, \quad g_{12}=g_{21}=0 \qquad (3.7)$$

can be chosen always[2,14].

As a result, Eq. (3.6) takes the form

$$m^{\mu}_{,11}-m^{\mu}_{,22}-[(m^{\nu}_{,1}\, m_{\nu,1})-(m^{\nu}_{,2}m_{\nu,2})]\, m^{\mu}=0. \qquad (3.8)$$

This is exactly the equation of motion for the n-field with the symmetry group SO(1,2)/SO(1,1).

In the n-field theory Eq. (3.8) is supplemented usually by the conditions[12]

$$(m^{\mu}_{,1}\pm m^{\mu}_{,2})^2=1. \qquad (3.9)$$

That can be made always by virtue of the conformal invariance of (3.8), i.e., the invariance under the transformations $\bar u^1\pm\bar u^2=f_{\pm}(u^1\pm u^2)$. In the relativistic string model[2,14] we have other conditions on the first derivatives of the normal $m^{\mu}(u^1,u^2)$

$$\bar g_{11}=m^2_{,1}=-\bar g_{22}=-m^2_{,2}, \quad \bar g_{12}=\bar g_{21}=(m_{,1}m_{,2})=0, \qquad (3.10)$$

where $\bar g_{ij}$ is the metric tensor on the hyperboloid of one sheet $m^2=-1$, on which the normal $m^{\mu}(u^1,u^2)$ maps the world sheet of the string. Conditions (3.10) follow directly from the derivative formulas of Weingarten (2.7) in the conformally-flat metric (3.7) in the string theory.

Thus, in the theory of the relativistic string moving in the three-dimensional space-time there arises naturally the nonlinear two-dimensional sigma model with the SO(1,2)/SO(1,1)-symmetry given by Eq. (3.8) and subsidiary conditions (3.10).

4. SIGMA MODEL IN THE STRING THEORY AND THE NONLINEAR LIOUVILLE EQUATION

The connection of the usual SO(3)/SO(2)-nonlinear sigma model with the sine-Gordon equation integrable by the inverse scattering method is well-known[12]. The sigma model defined by (3.9), (3.10) encountered in the relativistic string theory is related closely with another nonlinear equation, namely, with the nonlinear Liouville equation, the general solution of which is well-known[17]. Let us establish this relation. For this aim we turn to the Gauss equation (2.8) and to the Peterson-Codazzi equations (2.9) and consider them for the hyperboloid

$$m^{\mu} m_{\mu} = -1. \quad (4.1)$$

The coefficients of the second quadratic form $\bar{b}_{ij}$ for the hyperboloid (4.1) are defined by Eqs. (2.6) that in the case under consideration have the form

$$\bar{\nabla}_j m^{\mu}_{,i} = -\bar{b}_{ij} m^{\mu} , \quad (4.2)$$

where $\bar{\nabla}_j$ means the covariant differentiation with respect to the metric*

$$\bar{g}_{ij} = m^{\mu}_{,i} m_{\mu,j} . \quad (4.3)$$

In Eqs. (4.2) we take into account that the unit normal to the hyperboloid (4.1) is the vector m^{μ} itself. From (4.2) and (4.3) we obtain

$$\bar{b}_{ij} = (\bar{\nabla}_j m^{\mu}_{,i} m_{\mu}) = (m^{\mu}_{,ij} m_{\mu}) = -(m^{\mu}_{,i} m_{\mu,j}) = -\bar{g}_{ij} , \quad (4.4)$$

where $\bar{g}_{ij}$ is the metric tensor (3.10) on the hyperboloid (4.1).

Let us show that from the geometrical point of view Eqs. (3.8) and (3.10) defining the sigma model are equivalent to one nonlinear Liouville equation for a scalar function. For this purpose we show that the Gauss equation (2.8) and the Peterson-Codazzi equations (2.9) for the hyperboloid defined by (3.1), (3.8) and (3.10) are reduced to the Liouville equation.

The Peterson-Codazzi equations (2.9) for $\bar{b}_{ij}$ from (4.4)

$$\bar{\nabla}_k \bar{b}_{ij} = \bar{\nabla}_j \bar{b}_{ik} \quad (4.5)$$

are satisfied in virtue of the Ricci lemma, $\nabla_i \bar{g}_{k\ell} = 0$, identically.

The only nontrivial equation in this case is the Gauss equation (2.8)

$$\bar{R}_{1212} = \bar{b}_{12}\bar{b}_{12} - \bar{b}_{11}\bar{b}_{22} , \quad (4.6)$$

where $\bar{R}_{1212}$ is the curvature tensor for the metric $\bar{g}_{ij}$. Taking into account (4.3) and (3.10) one transforms Equation (4.6) to the form

$$\bar{R}_{1212} = -\bar{g}_{11}\bar{g}_{22} = (\bar{g}_{11})^2 . \quad (4.7)$$

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* In differential geometry/16/ $\bar{g}_{ij}$ is called the tensor of the third squared form of the surface $x^{\mu}(u^1,u^2)$, i.e., the metric tensor of Gauss's map.

Introducing the notation

$$\bar{g}_{11} = e^{\phi} \quad (4.8)$$

and making use of the obvious expression for $\bar{R}_{ijk\ell}$ in terms of $\bar{g}_{ij}$ and its derivatives/15,18/ we get from (4.7) the nonlinear Liouville equation

$$\phi_{,11} - \phi_{,22} = 2e^{\phi} . \quad (4.9)$$

4. CONCLUSION

The above results give rise to a natural question: is the relativistic string theory in the three-dimensional space-time equivalent to the nonlinear sigma model (3.8), (3.10)? At first sight this equivalence has to take place, because the string theory, as is well-known/7,9/, is reduced in the three-dimensional space-time to the Liouville equation also. However, this is not so. In contrast to the sigma model (3.8) and (3.10) where the knowledge of solution of the Liouville equation allows us to obtain both the quadratic forms of the hyperboloid (4.1) ($\bar{g}_{ij}$ and $\bar{b}_{ij}$), in the string theory only the first quadratic form g_{ij} (3.7) can be reconstructed by this solution. In the coordinate set (2.7) the coefficients of the second quadratic form of the string world sheet b_{ij} are expressed in terms of two arbitrary functions of one variable $q_{\pm}(u^1 \pm u^2)$. Without loss of generality these functions can be taken as constants/9/.

If the string is moving in the four-dimensional space-time, then at any point of its world sheet there are two unit spacelike normals. These normals and two unit tangent vectors to the string world sheet form a moving basis. It is important that the string theory admits the SO(1,1)xSO(2)-rotations of this basis, the tangent space and the normal space being not mixed. Therefore the nonlinear two-dimensional sigma model on the symmetric space SO(1,3)/SO(1,1)xSO(2) has to appear here.

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Релятивистская струна и нелинейная двумерная сигма-модель

Показано, как в теории релятивистской струны, движущейся в 3-мерном пространстве-времени, возникает двумерная нелинейная сигма-модель с группой симметрии SO(1,2)/SO(1,1).

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Relativistic String Theory and Nonlinear Two-Dimensional Sigma-Model

It is shown in what way the nonlinear two-dimensional SO(1,2)/SO(1,1) sigma-model appears in the theory of the relativistic string moving in three-dimensional space-time.

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