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QUANTUM-MECHANICAL OSCILLATOR WITH AN ARBITRARY ANHARMONICITY: 1/N-EXPANSION AND PERTURBATION THEORY

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In this paper we investigate the properties of the 1/N - expansion for the arbitrary N-dimensional oscillator with a power anharmonicity r^{2n}.

A lot of papers are devoted to the anharmonic oscillator which is caused both by different applications of this problem (e.g., in the molecular and solid state physics) and by its relation with some problems of quantum field theory. The utmost attention has been paid to the one-dimensional oscillator with nondegenerate minimum and quartic anharmonicity of the type x^4.

A number of results for more general oscillators are also obtained. One can find an ample bibliography in the review-papers by Hioe et al., Zinn-Justin, Simon [1], so we do not give here but some inevitable references.

Certain advancement in the standard perturbation theory for the nondegenerate oscillator has been achieved by Dolgov, Eletsky and Popov [2,3]. In ref. [2] the first 15 terms of the expansion of the ground-state energy were obtained in an analytical form for the quartic N-dimensional oscillator. In ref. [3] the first 3 terms of the perturbation theory were found for the N-dimensional oscillator with the arbitrary anharmonicity r^{2n}. The perturbation theory at large orders has been investigated a la Lipatov in the papers by Zinn-Justin et al. [4,5]: in ref. [4] it was discussed for the nondegenerate oscillator with arbitrary N and n and in ref. [5] for the one-dimensional oscillator with degenerate minima. Note that the perturbation theory in the case of double-well potentials has been investigated to a less extent in comparison with the nondegenerate case.

Among the methods beyond the perturbation theory in powers of the coupling constant the 1/N-expansion stands out. On the one hand, the properties of the 1/N-expansion permit us to reconstruct the standard perturbations, on the other hand, to advance towards the strong coupling. Besides, the 1/N -expansion can serve for the unified description of the one-minimum and double-well potentials.

The first three terms of the expansion up to the order N^{-1} have been obtained by the number of authors [2,6]. The 1/N - expansion at large orders has been studied in the paper by Brezin and Hikami [7]. In our paper [8] we have found the analytical expressions for the first seven terms of the expansion up to the order N^{-5} for the ground and first excited levels. These results were of help in obtaining also an analytical form of the first seven terms of the perturbation series for the oscillator with degenerate minima.

The 1/N-expansion in refs. [2,6-8] has been studied for the quartic oscillator. In this paper we generalize the results of ref. [8] to the case of the arbitrary isotropic oscillator with power anharmonicity. Note that Mlodinow and Papanicolaou [9] found the first three terms in this general case.

1. CONSTRUCTION OF THE 1/N-EXPANSION

So, let us consider the Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \frac{m^2}{2}\sum_{i=1}^{N} x_i^2 + \frac{g}{N^{n-1}}\left(\sum_{i=1}^{N} x_i^2\right)^n . \qquad (1.1)$$

Three are two parameters in it with the dimension of energy (m and $g^{1/n+1}$), which define the energy scale for different limits of weak ($g\to 0$) and strong ($g\to\infty$) couplings. It is convenient to introduce such a parameter with the dimension of energy (we denote it by ω) that would fix the energy scale for arbitrary values of g. Then the ratio E/ω will only be a function of the dimensionless coupling constant $\lambda = g/\omega^{n+1}$. We define ω by the following relations:

$$\frac{m^2}{\omega^2} = 1 - \frac{4n}{2^n}\lambda, \quad \lambda = g/\omega^{n+1}. \qquad (1.2)$$

Besides, it is rather convenient for the following to introduce an additional parameter connected with λ

$$\ell = \left[1+\lambda\frac{n(n-1)}{2^{n-1}}\right]^{1/2} . \qquad (1.3)$$

It may seem that the relations (1.2) were introduced ad hoc, but they were not. Further it will become clear, that ω is energy splitting between the ground and the first excited levels when N tends to infinity. So, by its physical meaning ω really fixes the energy scale in our problem. Moreover, the relations (1.2) arise quite naturally while constructing the 1/N-expansion by means of the path integral, being the generalization of the similar formulae of ref. [8].

Performing the change $H/\omega \to H$, $x_i \to x_i/\sqrt{\omega}$, we obtain

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \frac{1}{2}\left(1 - \frac{4n}{2^n}\lambda\right)\sum_{i=1}^{N} x_i^2 + \frac{\lambda}{N^{n-1}}\left(\sum_{i=1}^{N} x_i^2\right)^n .$$

Now it is clear that the parametrization (1.2) is really a convenience while investigating different limits in g. The one-minimum potential ($m^2>0$) corresponds to the λ, varying from 0 (when $g\to 0$) to $2^n/4n$ (when $g\to\infty$). The double-well potential ($m^2<0$) corresponds to the λ, varying from $2^n/4n$ to infinity (when $g\to 0$).

As we take interest mainly in the ground-state energy, we shall limit ourselves to the radial part of the Hamiltonian, which standardly can be transferred to the form not containing the first derivative:

$$H = -\frac{1}{2}\frac{d^2}{dr^2} + \left[\frac{(N-1)(N-3)}{8r^2} + \frac{1}{2}\left(1 - \frac{4n}{2^n}\lambda\right)r^2 + \frac{\lambda}{N^{n-1}} r^{2n}\right] . \qquad (1.4)$$

At large N the asymptotics of the ground-state energy coincides with the minimum of the asymptotic potential of the Hamiltonian (1.4):

$$E_{as}/\omega = V_{as}(r_0), \quad dV_{as}(r_0)/dr = 0 , \qquad (1.5)$$

$$V_{as}(r) = \frac{N^2}{8r^2} + \frac{1}{2}\left(1 - \frac{4n}{2^n}\lambda\right)r^2 + \frac{\lambda}{N^{n-1}} r^{2n}.$$

From eq. (1.5) it is easy to find

$$r_0 = \sqrt{N/2}, \quad E_{as}/\omega = N\left(\frac{1}{2} - \lambda\frac{n-1}{2^n}\right). \qquad (1.6)$$

Replacing the origin of coordinates to the minimum of the asymptotic potential (1.5) $r = r_0 + x/\sqrt{2}$, we obtain the Hamiltonian in the representation which is suitable for the construction of the 1/N-expansion

$$H = N\Big\{-\frac{1}{N}\frac{d^2}{dx^2} + \frac{(1-1/N)(1-3/N)}{4(1+x/\sqrt{N})^2} + \frac{1}{4}\left(1 - \frac{4n}{2^n}\lambda\right)(1+x/\sqrt{N})^2 + \qquad (1.7)$$
$$+ \frac{\lambda}{2^n}(1+x/\sqrt{N})^{2n}\Big\}.$$

Our goal is to find coefficients ϵ_k of the expansion for the ground-state energy

$$E_0/\omega = N\epsilon_0 + \sum_{k=0}^{\infty}\frac{\epsilon_{k+1}}{N^k}, \quad \epsilon_0 = \frac{1}{2} - \frac{n-1}{2^n}\lambda \qquad (1.8)$$

by solving the Schrödinger equation with the Hamiltonian (1.7)

$$(H - E_0/\omega)\psi = 0 . \qquad (1.9)$$

In this case the Hamiltonian (1.7) and hence the wave function of eq. (1.9) expand in half-integer powers of 1/N. Therefore, the wave function can be written as

$$\psi(x) = \psi_0(x)\left[1 + \sum_{m=1}^{\infty}\left(\frac{\chi_m(x)}{N^{m-1/2}} + \frac{\eta_m(x)}{N^m}\right)\right] . \qquad (1.10)$$

The wave function in the first approximation takes the form

$$\psi_0(x) = (2\ell/\pi)^{1/4}\exp(-\ell x^2/2) .$$

χ_m and η_m are the polynomials in odd and even powers of x, respectively:

$$\chi_m(x) = \sum_{s=1}^{3m-1} a_m^s x^{2s-1},$$

$$\eta_m(x) = \sum_{s=1}^{3m+1} b_m^s x^{2s-2}.$$

Substituting eq. (1.10) into eq. (1.9), one can obtain the recurrent system of differential equations for χ_m and η_m, which should be solved to derive the coefficients ϵ_k of the 1/N - expansion.

2. 1/N-EXPANSION OF THE GROUND-STATE ENERGY

We give here the first six coefficients of the expansion (1.8) computed analytically by means of SCHOONSHIP.

$$\epsilon_0 = (n+1-\ell^2)/2n, \qquad (2.1)$$

$$\epsilon_1 = \ell - 1.$$

The next coefficients have the form

$$\epsilon_k = \sum_{m=k-2}^{5k-6} \frac{P_m^k}{\ell^m}, \qquad (2.2)$$

where P_m^k are the polynomials in powers of anharmonicity n.

$$\epsilon_2 = \sum_{m=0}^{4} \frac{P_m^2}{\ell^m},$$

$P_0^2 = (-2n^2 - 7n + 31)/36,$

$P_1^2 = -(n+1),$

$P_2^2 = (13n^2 + 29n - 20)/36,$

$P_3^2 = n+1,$

$P_4^2 = -11(n+1)^2/36.$

$$\epsilon_3 = \sum_{m=1}^{9} \frac{P_m^3}{\ell^m},$$

$P_1^3 = (n+1)(4n^3 - 75n + 361)/432,$

$P_2^3 = (-28n^2 - 65n - 1)/18,$

$P_3^3 = (n+1)(-2n^3 + 153n^2 + 321n - 914)/216,$

$P_4^3 = (13n^3 + 125n^2 + 208n + 60)/18,$

$P_5^3 = (n+1)(-41n^3 - 371n^2 - 388n + 518)/144,$

$P_6^3 = (n+1)(-35n^2 - 128n - 81)/18,$

$P_7^3 = (n+1)^2(139n^2 + 497n + 34)/216,$

$P_8^3 = 11(n+1)^3/9,$

$P_9^3 = -155(n-1)^4/432.$

$$\epsilon_4 = \sum_{m=2}^{14} \frac{P_m^4}{\ell^m},$$

$P_2^4 = (-2\,536n^6 + 3\,828n^5 + 78\,222n^4 - 137\,669n^3 + 907\,617n^2 +$
$+ 3\,128\,853n + 845\,309)/1\,166\,400,$

$P_3^4 = (n+1)(4n^4 + 36n^3 - 1\,299n^2 - 3\,446n + 1\,341)/432,$

$P_4^4 = (-2\,116n^6 - 44\,712n^5 + 525\,447n^4 + 2\,109\,481n^3 -$
$- 4\,729\,203n^2 - 13\,875\,957n - 5\,272\,516)/388\,800,$

$P_5^4 = (n+1)(-16n^4 + 1\,854n^3 + 14\,703n^2 + 21\,836n - 1\,365)/432,$

$P_6^4 = (13\,378n^6 - 600\,789n^5 - 6\,212\,496n^4 - 11\,021\,383n^3 +$
$+ 13\,819\,839n^2 + 37\,447\,656n + 16\,260\,763)/388\,800,$

$P_7^4 = (n+1)(-67n^4 - 1\,322n^3 - 5\,129n^2 - 5\,556n - 914)/48,$

$P_8^4 = (n+1)(106\,219n^5 + 2\,290\,532n^4 + 9\,505\,621n^3 +$
$+ 6\,793\,426n^2 - 11\,405\,218n - 9\,866\,456)/233\,280,$

$P_9^4 = (n+1)^2(2\,561n^3 + 21\,397n^2 + 40\,022n + 14\,850)/432,$

$P_{10}^4 = (n+1)^2(-140\,863n^4 - 1\,323\,091n^3 - 2\,755\,776n^2 - 498\,691n +$
$+ 997\,097)/77\,760,$

$P_{11}^4 = (n+1)^3(-3\,341n^2 - 12\,848n - 7\,995)/432,$

$P_{12}^4 = (n+1)^4(33\,911n^2 + 130\,723n + 20\,780)/15\,552,$

$P_{13}^4 = 155(n+1)^5/48,$

$P_{14}^4 = -39\,709(n+1)^6/46\,656.$

$$\epsilon_5 = \sum_{m=3}^{19} \frac{P_m^5}{\ell^m},$$

$P_3^5 = (n+1)(34\,832n^7 - 243\,952n^6 - 1\,473\,816n^5 + 8\,027\,440n^4 -$
$- 1\,427\,183n^3 + 80\,852\,451n^2 + 341\,859\,515n + 26\,408\,609)/55\,987\,200,$

$P_4^5 = (-2\,536n^7 + 6\,380n^6 + 361\,806n^5 - 2\,977\,463n^4 -$

$$- 19\,298\,345n^3 - 20\,204\,661n^2 + 5\,766\,863n + 6\,504\,332)/583\,200,$$

$$P_5^5 = (n+1)(53\,192n^7 + 221\,516n^6 - 8\,068\,710n^5 + 31\,594\,309n^4 + 142\,913\,479n^3 - 910\,497\,945n^2 - 2\,195\,432\,869n - 454\,019\,188)/13\,996\,800,$$

$$P_6^5 = (-10\,160n^7 - 474\,536n^6 + 9\,341\,148n^5 + 108\,582\,272n^4 + 310\,576\,931n^3 + 279\,097\,257n^2 + 12\,671\,773n - 42\,961\,301)/583\,200,$$

$$P_7^5 = (n+1)(59\,164n^7 + 14\,593\,504n^6 - 141\,205\,281n^5 - 1\,840\,760\,161n^4 - 1\,520\,102\,779n^3 + 13\,292\,397\,831n^2 + 22\,698\,732\,316n + 6\,435\,969\,046)/27\,993\,600,$$

$$P_8^5 = (44\,366n^7 - 2\,637\,121n^6 - 50\,697\,558n^5 - 258\,913\,295n^4 - 527\,721\,659n^3 - 441\,769\,122n^2 - 103\,972\,585n + 18\,250\,502)/194\,400,$$

$$P_9^5 = (n+1)(-1\,762\,642n^7 + 52\,485\,065n^6 + 1\,188\,742\,542n^5 + 5\,027\,567\,779n^4 + 2\,130\,271\,105n^3 - 17\,646\,398\,652n^2 - 24\,754\,360\,129n - 7\,974\,968\,116)/13\,996\,800,$$

$$P_{10}^5 = (n+1)(2\,003\,978n^6 + 75\,025\,536n^5 + 596\,590\,344n^4 + 1\,749\,809\,572n^3 + 2\,126\,055\,579n^2 + 954\,439\,686n + 67\,222\,013)/583\,200,$$

$$P_{11}^5 = (n+1)^2(-10\,026\,275n^6 - 431\,897\,298n^5 - 3\,327\,578\,433n^4 - 6\,878\,325\,460n^3 + 1\,993\,319\,040n^2 + 14\,203\,470\,528n + 7\,102\,859\,518)/11\,197\,440,$$

$$P_{12}^5 = (n+1)^2(-2\,537\,821n^5 - 40\,856\,567n^4 - 185\,514\,202n^3 - 320\,710\,267n^2 - 211\,155\,251n - 38\,962\,360)/116\,640,$$

$$P_{13}^5 = (n+1)^3(16\,725\,127n^5 + 278\,381\,729n^4 + 1\,145\,245\,954n^3 + 1\,098\,022\,729n^2 - 848\,560\,363n - 911\,828\,300)/2\,799\,360,$$

$$P_{14}^5 = (n+1)^3(344\,329n^4 + 3\,114\,040n^3 + 8\,129\,316n^2 + 7\,428\,676n + 2\,084\,623)/7\,776,$$

$$P_{15}^5 = (n+1)^4(-13\,054\,217n^4 - 120\,010\,112n^3 - 263\,137\,935n^2 - 79\,938\,146n + 65\,466\,358)/1\,119\,744,$$

$$P_{16}^5 = -(n+1)^5(888\,361n^2 + 3\,504\,575n + 2\,160\,022)/23\,328,$$

$$P_{17}^5 = (n+1)^6(5\,354\,477n^2 + 21\,375\,721n + 4\,269\,764)/559\,872,$$

$$P_{18}^5 = 277\,963(n+1)^7/23\,328,$$

$$P_{19}^5 = -6\,416\,935(n+1)^8/2\,239\,488.$$

3. 1/N -EXPANSION FOR THE ENERGY OF THE FIRST EXCITED LEVEL

Taking into account the relation between the ground and first excited energies

$$E_1(N;g) = E_0(N+2;\, g(1+2/N)^{n-1}), \tag{3.1}$$

the formulae (2.1)-(2.6) lead to the expansion for E_1:

$$E_1/\omega = N\epsilon_0' + \sum_{k=0}^{\infty} \frac{\epsilon_{k+1}'}{N^k}, \tag{3.2}$$

$$\epsilon_0' = (n+1-\ell^2)/2n, \qquad \epsilon_1' = \ell. \tag{3.3}$$

While comparing eqs. (3.3) and eqs. (2.1), one may be convinced of $\epsilon_0' = \epsilon_0$ and $\epsilon_1' = \epsilon_1 + 1$, so that

$$(E_1 - E_0)/\omega = 1 + O(1/N)$$

and ω, as has been mentioned above, has really the physical meaning of the level's splitting in the limit of large N.

The other coefficients ϵ_k' have the form

$$\epsilon_k' = \sum_{m=k-2}^{5k-6}{}' \frac{Q_m^k}{\ell^m}. \tag{3.4}$$

Here Q_m^k are the polynomials in n; capital sigma primed denotes that summing is performed only over even or odd values of m so that parity of m and k is the same.

$$\epsilon_2' = \sum_{m=0}^{4}{}' \frac{Q_m^2}{\ell^m}, \tag{3.5}$$

$$Q_0^2 = -(n+1)(2n+5)/36,$$

$$Q_2^2 = (n+1)(13n+16)/36,$$

$$Q_4^2 = -11(n+1)^2/36,$$

$$\epsilon_3' = \sum_{m=1}^{9}{}' \frac{Q_m^3}{\ell^m},$$

$$Q_1^3 = (n+1)^2(4n^2 - 4n - 71)/432,$$

$$Q_3^3 = (n+1)^2(-2n^2 + 155n + 274)/216,$$

$$Q_5^3 = (n+1)^2(-41n^2 - 330n - 346)/144, \tag{3.6}$$

$Q_7^3 = (n+1)^3(139n+358)/216,$

$Q_9^3 = -155(n+1)^4/432,$

We omit the expressions for ϵ'_4 and ϵ'_5 though they are less complicated than (2.5), (2.6), and qui peut le plus peut le moins. Nevertheless we give here just the values of ϵ'_k in a particular case of the quartic anharmonicity. For n =2, $\ell = (1+\lambda)^{1/2}$, and ϵ'_k can be transferred to the form

$$\epsilon'_0 = \frac{1}{2} - \frac{\lambda}{4},$$

$$\epsilon'_1 = (1+\lambda)^{1/2},$$

$$\epsilon'_2 = \frac{\lambda}{(1+\lambda)^2}\left(-\frac{3}{4}\lambda + 2\right),$$

$$\epsilon'_3 = \frac{\lambda^2}{(1+\lambda)^{9/2}}\left(-\frac{21}{16}\lambda^2 + \frac{75}{4}\lambda - 9\right), \tag{3.7}$$

$$\epsilon'_4 = \frac{\lambda^3}{(1+\lambda)^7}\left(-\frac{333}{64}\lambda^3 + \frac{2707}{16}\lambda^2 - \frac{5713}{16}\lambda + 89\right),$$

$$\epsilon'_5 = \frac{\lambda^4}{(1+\lambda)^{19/2}}\left(-\frac{30885}{1024}\lambda^4 - \frac{108585}{64}\lambda^3 - \frac{32745}{4}\lambda^2 - \frac{122\,133}{16}\lambda - \frac{5\,013}{4}\right),$$

$$\epsilon'_6 = \frac{\lambda^5}{(1+\lambda)^{12}}\left(-\frac{916\,731}{4\,096}\lambda^5 + \frac{19\,485\,563}{1\,024}\lambda^4 - \frac{2\,606\,457}{16}\lambda^3 + \frac{5\,299\,749}{16}\lambda^2 - \frac{5\,869\,337}{32}\lambda + \frac{88\,251}{4}\right).$$

The formulae (3.7) correct the error of ref. (8), which has been committed while calculating ϵ'_2, and thus affected all the next ϵ'_k.

4. LARGE ORDERS OF THE 1/N-EXPANSION

Studying the asymptotic behaviour of the coefficients of the 1/N -expansion we use the conventional method for investigating the standard perturbation theory at large orders. Some of its peculiarities in this very problem have been discussed by

Brezin and Hikami in ref. /7/. To calculate the contribution of the complex saddle points, the potential must be transferred to the standard form $V(\tilde{g}r)/\tilde{g}^2$, where the parameter of the expansion $\tilde{g}$ in our case equals $1/\sqrt{N}$. This can be done with the centrifugal term in the Hamiltonian (1.4) having the form $N^2/8r^2$. Thus, we make the following change of variables:

$$N'^2 = (N-1)(N-3), \quad g' = g(N'/N)^{n-1}. \tag{4.1}$$

Finally, the large order behaviour will be described by the expression

$$E_0/\omega \sim N' \sum_k \frac{\Gamma(k+b)}{N'^k}[a(g')]^k \tilde{C}(g')$$

and we shall have to return from N′ and g′ to the former parameters N and g. It can be shown that such a return leads to the appearance but of the additional factor, so that

$$E_0/\omega \sim N \sum_k \frac{\Gamma(k+b)}{N^k}[a(g)]^k \tilde{C}(g)B(g), \tag{4.2}$$

$$B(g) = \exp\{\frac{2}{a}[1-(n-1)g\frac{\partial}{\partial g}\ln a]\}.$$

Being ascertain of the fact, we can perform the change (4.1) and omit the prime for the present. Then, it remains to include the main asymptotics of the energy (see (1.5), (1.6)) into the potential and transfer it to the standard form near the minimum $V(r) \simeq (r-r_0)^2/2$.

Finally we have the Schrödinger equation

$$-\frac{1}{2}\frac{d^2\psi}{dr^2} + [\frac{1}{\tilde{g}^2}V(\tilde{g}r) - \mathcal{E}]\psi = 0,$$

$$V(r) = \frac{(r^2-r_0^2)^2}{8r^2r_0^4}[1+\frac{4\lambda}{2^n}\sum_{m=1}^{n-1}(n-m)(r^2/r_0^2)^m], \tag{4.3}$$

$$r_0^2 = \ell, \quad \mathcal{E} = (E_0 - E_{as})/2\omega\ell,$$

where ℓ, we would like to remind, is defined by eq. (1.3).

The real turning point r_t exists when λ is negative: $\lambda = -\Lambda$, $\Lambda > 0$. In this region

$$\rho_n = 2\int_{r_0}^{r_t} dr\sqrt{2V(r)} = \tag{4.4}$$

$$= \frac{1}{2}\int_1^\beta dx\frac{x-1}{x}\sqrt{1-\frac{4\Lambda}{x^n}\sum_{m=1}^{n-1}(n-m)x^m},$$

with β being the solution of the equation

$$\sum_{m=1}^{n-1}\beta^{m}(n-m) = \frac{2^{n}}{4\Lambda}, \qquad 0<\Lambda<\frac{2^{n-1}}{n(n-1)}. \tag{4.5}$$

As is known, the quantity $a(g)$ in the formula (4.2) is nothing but $1/\rho_n$, and the total factor $\tilde{C}(g)$ can be represented as

$$C = 2\ell^{3/2}(\sqrt{\beta}-1)\exp\mathcal{A},$$

$$\mathcal{A} = \int_{r_0}^{r_t} dr\left(\frac{1}{\sqrt{2V(r)}}-\frac{1}{r-r_0}\right) = \tag{4.6}$$

$$= \lim_{\delta\to 0}\left\{-\ln\frac{\sqrt{\beta}-1}{\delta}+\sqrt{1-\Lambda\frac{n(n-1)}{2^{n-1}}}\int_{1+2\delta}^{\beta}\frac{dx}{x-1}\left[1-\frac{4\Lambda}{2^{n}}\sum_{m=1}^{n-1}(n-m)x^{m}\right]^{-1/2}\right\}.$$

Taking into account the connection between λ and g, the factor $B(g)$ of eq. (4.2) can be written out as follows:

$$B(g) = \exp\left\{2\rho_n + 2(n-1)\frac{\partial\rho_n}{\partial\Lambda}\,\frac{\Lambda(1+\Lambda 4n/2^{n})}{1-\Lambda n(n-1)/2^{n-1}}\right\}. \tag{4.7}$$

With all the integrals calculated, we must perform the analytical continuation to the positive λ, after which we obtain the final expression

$$E_0/\omega = N\epsilon_0 + \sum_k \frac{\epsilon_{k+1}}{N^k},$$

$$\epsilon_{k+1} \simeq 2\,\mathrm{Re}\left\{-\frac{1}{2\pi^{3/2}}\,\Gamma(k+1/2)\,\frac{C_n}{\rho_n^{k+1/2}}\right\}, \tag{4.8}$$

$$C_n = \tilde{C}\,B(g).$$

To our regret, the integrals in eqs. (4.4, 4.6) cannot be calculated in a general form. In particular cases, for example, we have

$n=2$

$$\rho_2 = -\sqrt{1+\lambda}\,\frac{1-2\lambda}{3\lambda} - \ln\frac{\sqrt{1+\lambda}+1}{\sqrt{\lambda}} - i\frac{\pi}{2},$$

$$c_2 = 4\,\frac{(1+\lambda)^{7/4}}{2+\lambda+2\sqrt{1+\lambda}}; \tag{4.9}$$

$n=3$

$$\rho_3 = -\frac{1}{2}\ln\frac{\lambda+2+\sqrt{2(3\lambda+2)}}{2\lambda+\sqrt{\lambda(3\lambda+2)}} + \frac{1}{4}(3+\sqrt{2/\lambda})(1-\sqrt{\lambda/2})\ln\frac{\sqrt{|\lambda-2|}}{2\sqrt{\lambda}+\sqrt{3\lambda+2}} -$$

$$- i\frac{\pi}{2}\left[1-\frac{1}{4}\theta(2-\lambda)(3+\sqrt{2/\lambda})(1-\sqrt{\lambda/2})\right], \tag{4.10}$$

$$c_3 = 4\,\frac{(1+3\lambda/2)^{7/4}}{2+\lambda+2\sqrt{1+3\lambda/2}}.$$

The validity of the results obtained was verified by comparison with numerical calculations for different values of the coupling constant.

5. PERTURBATION THEORY

Knowing the $1/N$-expansion coefficients, in general we can reconstruct the standard perturbation series. Let us consider first of all the nondegenerate potential, i.e., the case when $m^2>0$ and $g\to 0$. Then, λ will also tend to zero, and thus can be presented as a series in powers of g. An expansion parameter will then be the dimensionless quantity g/m^{n+1}. The first three terms /3/ of the perturbation series in our notation are as follows:

$$E_0/m = \frac{N}{2} + \sum_{k=1}^{\infty} A_k(n;N)\,(g/m^{n+1}\cdot N^{n-1})^k,$$

$$A_1(n;N) = \frac{\Gamma(n+N/2)}{\Gamma(N/2)},$$

$$A_2(n;N) = -\frac{\Gamma^2(n+N/2)}{2\Gamma(N/2)}\sum_{i,j=0}^{n-1}\frac{\Gamma(i+j+1+N/2)}{\Gamma(i+1+N/2)\Gamma(j+1+N/2)}, \tag{5.1}$$

$$A_3(n;N) = \frac{\Gamma^3(n+N/2)}{2\Gamma(N/2)}\sum_{i,j,k=0}^{n-1}\sum_{\ell=0}^{i+j}\times$$

$$\times\frac{\Gamma(i+j+1+N/2)\,\Gamma(k+\ell+1+N/2)}{\Gamma(i+1+N/2)\Gamma(j+1+N/2)\Gamma(k+1+N/2)\Gamma(\ell+1+N/2)}.$$

Consider as an example the first correction to the energy, which can be written in the form

$$\mathcal{E}_1 = A_1/N^{n-1} = \frac{N}{2^n}\prod_{i=1}^{n-1}(1+2i/N),$$

from which it follows that all the terms of the $1/N$-expansion beginning from ϵ_0 up to ϵ_{n-1} contribute to $\mathcal{E}_1$. In the same way one can get convinced that the second correction $\mathcal{E}_2$ contains N in powers from +1 up to $[1-2(n-1)]$, i.e., to reproduce $\mathcal{E}_2$ it is necessary to know all the $1/N$-expansion terms from ϵ_0 to $\epsilon_{2(n-1)}$. And in general for the reconstruction of the k-th term of the perturbation series one ought to take into account all the terms of the $1/N$-expansion beginning from ϵ_0 up to $\epsilon_{k(n-1)}$. In the particular case of quartic oscillator $n-1=1$, so that the knowledge of the 1/N -coefficients from ϵ_0 to ϵ_k allows us to reconstruct all the $\mathcal{E}_1, \dots, \mathcal{E}_k$. But it is necessary to know more terms of the 1/N-expansion in proportion on order to reconstruct the perturbation series for the potentials with a larger power of anharmonicity.

This is not the case when we study the double-well potential (that is, when $m^2 < 0,\ g \to 0$). Then λ tends to infinity and ϵ_k behaves as

$$\epsilon_k = \beta_k \cdot \lambda^{1-k/2}[1 + O(1/\sqrt{\lambda})] .$$

Such an asymptotics leads to the equality of the number of the 1/N -expansion terms to the number of perturbation terms reconstructed with its help. In the case of degenerate oscillator the perturbation series takes the form

$$E_0/m = -N\frac{n-1}{4n}\left(\frac{2^n}{4n}\right)^{1/(n-1)}\frac{1}{\Delta} + \sum_{k=0}^{\infty} B_k(n;N)(\Delta/N)^k , \qquad (5.2)$$

$$\Delta = (g/m^{n+1})^{1/(n-1)},$$

where for convenience we changed m^2 to $-m^2$. From eqs, (2.1-2.6) we know ϵ_k with $k = 0,1,\dots,5$, which allows us to define the first term in eq. (5.2) and the next coefficients $B_0, \dots, B_4$:

$$B_0(n;N) = \frac{1}{2}[2(n-1)]^{1/2} ,$$

$$B_1(n;N) = \left(\frac{4n}{2^n}\right)^{1/(n-1)} [N^2/4 - N + (-2n^2 - 7n + 31)/36],$$

$$B_2(n;N) = \left(\frac{4n}{2^n}\right)^{2/(n-1)} \frac{n+1}{[2(n-1)]^{1/2}}[N^2/2 - 2N + (4n^3 - 75n + 361)/216],$$

$$B_3(n;N) = \left(\frac{4n}{2^n}\right)^{3/(n-1)} \frac{2}{[2(n-1)]}[-N^4/8 + N^3 + N^2(28n^2 + 65n - 71)/36 -$$

$$- N(28n^2 + 65n + 1)/9 + (-2\,536n^6 + 3\,828n^5 +$$

$$+ 78\,222n^4 - 137\,669n^3 + 907\,617n^2 + 3\,128\,853n +$$

$$+ 845\,309)/583\,200] ,$$

$$B_4(n;N) = \left(\frac{4n}{2^n}\right)^{4/(n-1)} \frac{n+1}{[2(n-1)]^{3/2}}[-N^4(n+9)/4 + 2N^3(n+9) +$$

$$+ N^2(-4n^4 - 36n^3 + 1\,299n^2 + 2\,582n - 9\,117)/216 +$$

$$+ N(4n^4 + 36n^3 - 1\,299n^2 - 3\,446n + 1\,341)/54 +$$

$$+ (34\,832n^7 - 243\,952n^6 - 1\,473\,816n^5 + 8\,027\,440n^4 -$$

$$- 1\,427\,183n^3 + 80\,852\,451n^2 + 341\,859\,515n +$$

$$+ 26\,408\,609)/6\,998\,400]. \qquad (5.3)$$

Now one can verify that $B_n(n;N)$ can be written as polynomials in even powers of $(N-2)$, from which immediately follows the relation:

$$B_k(n;N) = B_k(n;4-N)$$

and particularly

$$B_k(n;1) = B_k(n;3), \qquad (5.4)$$
$$B_k(n;0) = B_k(n;4).$$

And lastly, the relation (3.1) between the energies of the ground and first excited states results in the existence of the expansion for E_1 similar to (5.2) with the coefficients B'_k, instead of B_k, where

$$B'_k(n;N) = B_k(n;N+2) . \qquad (5.5)$$

From eqs. (5.4, 5.5) it follows, for example, that in the case of one-dimensional degenerate oscillator $B'_k(n;1) = B_k(n;1)$. In other words the perturbation series for E_0 and E_1 coincide so far as they do not reproduce the exponentially small splitting of the levels due to the quantum-mechanical tunneling.

Eqs. (5.3, 5.5) allow us to find the coefficients B'_k:

$$B'_0(n;N) = \frac{1}{2}[2(n-1)]^{1/2} ,$$

$$B'_1(n;N) = \left(\frac{4n}{2^n}\right)^{1/(n-1)} [N^2/4 - (n+1)(2n+5)/36],$$

$$B'_2(n;N) = \left(\frac{4n}{2^n}\right)^{2/(n-1)} \frac{n+1}{[2(n-1)]^{1/2}}[N^2/2 + (n+1)(4n^2 - 4n - 71)/216],$$

$$B'_3(n;N) = \left(\frac{4n}{2}\right)^{3/(n-1)} \frac{1}{2(n-1)}[-N^4/4 + N^2(n+1)(28n+37)/18 +$$

$$+ (n+1)(-2\,536n^5 + 6\,364n^4 + 71\,858n^3 - 209\,527n^2 -$$

$$- 697\,256n - 385\,891)/291\,600] ,$$

$$B_4'(n;N) = \left(\frac{4n}{2^n}\right)^{4/(n-1)} \frac{n+1}{[2(n-1)]^{3/2}} [-N^4(n+9)/4 - N^2(n+1)(4n^3+32n^2 -$$

$$- 1\,331n - 2\,547)/216 + (n+1)(34\,832n^6 - \tag{5.6}$$

$$- 278\,784n^5 - 1\,195\,032n^4 + 9\,740\,872n^3 -$$

$$- 6\,502\,455n^2 - 80\,995\,494n - 51\,740\,191)/6\,998\,400].$$

For $n=2$ formulae (5.3) and (5.6) coincide with the similar expressions of ref. (8). Remind that in the case of the quartic oscillator there is the relation between the perturbation coefficients of degenerate and nondegenerate oscillators in one and two dimensions

$$B_k(2;1) = (-1)^k 2^{-(k+1)/2} A_k(2;2),$$

$$B_k(2;2) = (-1)^k 2^{(k+1)/2} A_k(2;1),$$

which allows one to obtain the asymptotics of $B_k(2;1)$, $B_k(2;2)$ and $B_k(2;3)$ at large k. The first of these remarkable coincidences has been noticed by Zinn-Justin [10], Avron and Seiler [11] have verified it by analytical computations up to the 11-th order of perturbation theory. The asymptotics of $B_k(2;0)$ and $B_k(2;4)$ were derived from the asymptotic formulae (4.9) for $\epsilon_k(\lambda)$. When λ is large

$$\epsilon_k(\lambda) \simeq \beta_k(n)\lambda^{1-k/2} \tag{5.7}$$

and substitution of eq. (5.7) into the 1/N-expansion (1-8) leads to the relation between B_k and β_k:

$$B_k(n;0) = \beta_{k+1}(n)(4n/2^n)^{k(n+1)/2(n-1)-1/2}. \tag{5.8}$$

Taking the limit of large λ in eq. (4.10) we obtain the asymptotics of $\beta_k(3)$, hence that of $B_k(3;0)$:

$$B_k(3;0) = B_k(3;4) = \frac{4\cdot 3^{3/4}}{\pi^{3/2}} \Gamma(k+½) \frac{(2\sqrt{2})^k}{[\ln(2+\sqrt{3})]^{k+1/2}}. \tag{5.9}$$

Taking the same limit $\lambda \to \infty$ in the general eqs. (4.4-4.6), we can derive the asymptotics $B_k(n;0)$ for arbitrary n:

$$B_k(n;0) = B_k(n;4) =$$

$$= \frac{b_n}{\pi^{3/2}} (n/2)^{1/4} (n-1)^{3/4} [2^{1/(n-1)} n^{(n+1)/2(n-1)}]^k \frac{\Gamma(k+1/2)}{a_n^{k+1/2}},$$

$$a_n = \int_0^1 dx \sqrt{(x^n - nx + n - 1)/x}, \tag{5.10}$$

$$b_n = \lim_{\delta\to 0} \exp\{\ln\delta + \sqrt{n(n-1)/2} \int_0^{1-\delta} \frac{dx}{\sqrt{x(x^n - nx + n - 1)}}\}$$

in addition to the asymptotics obtained in ref. (8):

$$B_k(2;0) = B_k(2;4) = \frac{2\sqrt{3}}{\pi^{3/2}} (3\sqrt{2})^k \Gamma(k+1/2),$$

$$B_k(2;1) = B_k(2;3) = -\frac{3\sqrt{2}}{\pi} (3/\sqrt{2})^k \Gamma(k+1), \tag{5.11}$$

$$B_k(2;2) = -\frac{2\sqrt{3}}{\pi^{3/2}} (3\sqrt{2})^k \Gamma(k+1/2).$$

There are also known the asymptotics of $B_k(n;1)$ in the case of one-dimensional double-well potential with arbitrary anharmonicity obtained by Brezin, Parisi and Zinn-Justin [5]. In our notation their result takes the form

$$B_k(n;1) = B_k(n;3) = -\frac{3-(-1)^n}{2\pi} \sqrt{n+1}\, (\sqrt{2(n+1)}/(n-1))^{1/(n-1)} \times$$

$$\times \left[\frac{n+1}{\sqrt{2}(n-1)^{(3n-1)/2(n-1)}}\right]^k \frac{\Gamma(k+n/2(n-1))}{\Gamma(n/2(m-1))}. \tag{5.12}$$

As long as we know, formulae (5.9-5.12) are the complete set of results for degenerate oscillators, concerning the leading asymptotics in the large order behaviour. Some applications of the results obtained in the present paper will be given elsewhere.

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Кудинов А.В., Смондырев М.А. E2-82-705
Квантовомеханический осциллятор с произвольной ангармоничностью: 1/N - разложение и теория возмущений

Исследованы свойства 1/N -разложения для задачи об N - мерном ангармоническом осцилляторе с произвольной степенной ангармоничностью. Первые шесть членов разложения энергий основного и первого возбужденного уровней получены в аналитическом виде. Исследовано асимптотическое поведение коэффициентов при больших порядках 1/N -разложения. Выведенные формулы использованы при определении точных выражений для первых шести коэффициентов стандартной теории возмущений по степеням константы связи в случае N-мерного потенциала с двумя вырожденными минимумами. Обсуждается асимптотическое поведение этих коэффициентов при больших порядках теории возмущений.

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Koudinov A.V., Smondyrev M.A. E2-82-705
Quantum-Mechanical Oscillator with an Arbitrary Anharmonicity: 1/N -Expansion and Perturbation Theory

We investigate the properties of the 1/N-expansion for the N-dimensional anharmonic oscillator with the arbitrary power anharmonicity. The first six terms of the expansion for the energy of the ground and first excited states are obtained in an analytical form. We study also the large-order behaviour of the 1/N-expansion. We use the formulae derived to find the exact analytical expressions for the first six coefficients of the standard perturbation theory in powers of the coupling constant in the case of the N-dimensional double-well potential. The large-order behaviour of these coefficients is discussed.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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