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THE WAHLQUIST-ESTABROOK METHOD FOR THE INVESTIGATION OF THE NONLINEAR MODELS OF CLASSICAL FIELD THEORY

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The Wahlquist-Estabrook method[1-6] which yields the Inverse Scattering Method equations[7,8] is considered. A short review is presented on the interpretation of the prolongation forms as connection forms[2,3] and also on a possible extention of the method for the case of two spatial dimensions[5,6]. The prolongation structure approach of Wahlquist and Estabrook is illustrated by its application to the sine-Gordon model[3] and to the nonlinear Schrödinger equation in two spatial dimensions[5,6].

We share the belief of the authors of the method that the differential-geometric methods must lead to greater understanding of a broad spectrum of nonlinear phenomena, including completely integrable models of the classical field theory, which have the solitary wave solutions (solitons).

Section 1 provides a brief introduction to the prolongation structure approach of the Wahlquist and Estabrook studying the nonlinear evolution equations with two independent variables[1,2]. In Sec. 2 it is shown how the prolongation structure for the generalized sine-Gordon equation can be found[3]. The generalization of the prolongation structure approach of Wahlquist and Estabrook to two spatial dimensions[5,6] is discussed in Sec.3 for the generalized nonlinear Schrödinger equation.

1. Let us discuss the main ideas of Wahlquist and Estabrook method using a few nonlinear equations as an example. It is well known that useful information about the given equation can be obtained if it can be presented as complete integrability condition for some system of equations. For example, the Burger's equations

$$u_t + u_{xx} + (u^2)_x = 0 \tag{1.1}$$

can be presented as the integrability condition for the system of equations

$$\xi_x = u\,\xi, \qquad \xi_t = -(u_x + u^2)\xi, \tag{1.2}$$

$$\xi_{xt} = \xi_{tx} \Rightarrow u_t + u_{xx} + (u^2)_x = 0. \tag{1.3}$$

Then we come to the Koul-Hopf substitution

$$u = \frac{\xi_x}{\xi} \Rightarrow \xi_t + \xi_{xx} = 0. \tag{1.4}$$

Recall also the Korteweg-de Vries equation

$$u_t + 12uu_x + u_{xxx} = 0. \tag{1.5}$$

Just the representation of this equation as integrability condition for the system of equations

$$\xi_x = \lambda - 2u - \xi^2,$$
$$\xi_t = 4[(u+\lambda)(2u+\xi^2-\lambda) + \frac{1}{2}u_{xx} - 2u_x\xi] \tag{1.6}$$

made it possible to develop the Inverse Scattering Method for solving a nonlinear evolution equations[9].

It can be shown that another system of equations

$$\xi_x = \begin{pmatrix} 0 & \lambda-2u \\ 1 & 0 \end{pmatrix}\xi, \quad \xi_t = \begin{pmatrix} -2u_x & 4(u+\lambda)(2u-\lambda)+2u_{xx} \\ -4(u+\lambda) & 2u_x \end{pmatrix} \tag{1.7}$$

also satisfies the same conditions, where ξ is a column-vector $\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$.

Both systems contain the arbitrary parameter λ, which appears as eigen-value parameter of the inverse scattering problem. Notice that system (1.7) is yet linear one.

Returning to the idea of finding the system for which the given nonlinear equation (or system of equations) represents the integrability condition we come to the conclusion that solution of the problem may be non-unique. Hence we must bear in mind this uncertainty of the Wahlquist-Estabrook method.

Let us describe the basic idea of the method using differential form language. In the Wahlquist-Estabrook method the language of differential forms, Pfaffian integrable systems and the theory of Cartan-Ehresmann connections[10-13] are widely used. Every nonlinear evolution equation with two independent variables x and t may be expressed in terms of second-rank differential forms (2-forms) $\{\alpha_i\}$, $i = 1,\ldots,N$, such that there exists the unique correspondence between the solutions of nonlinear equation and the solution manyfold for the forms $\{\alpha_i\}$. It is supposed that the set of forms $\{\alpha_i\}$ on the corresponding manyfold X is completely integrable, i.e., the set of forms constitutes a closed ideal of differential forms $\{\alpha_i\}$

$$dI \subset I. \tag{1.8}$$

A <u>prolongation structure</u> represents the set of 1-forms ω^β, $\beta = 1,2,\ldots,M$ on $X\times Y$, $Y = \{\xi^1, \xi^2, \ldots, \xi^M\}$, such that

$$d\omega \subset \pi * I + \omega, \tag{1.9}$$

where π is the Cartesian projection of $X\times Y$ onto X, i.e., $\pi(x,y) = x$. From (1.9) it follows that the system $\{\pi * \alpha_i, \omega^\beta\}$ is integrable one on $X\times Y$ and the projection of any solution manyfold of this system on X represents a solution manyfold of the set of forms $\{\alpha_i\}$. If 1-forms ω^β generate one-parametrical family, $\omega^\beta = \omega^\beta(\lambda)$, then the set of equations for ξ, which follows from the condition $\omega^\beta = 0$, gives the linear "inverse scattering" equations.

2. Let us discuss a variant of the techniques, developed by Wahlquist and Estabrook[1] for finding prolongation structure for generalized sine-Gordon equation. Below we follow to Hermann[3]. Let G be a Lie algebra of matrices; u, u_x are variables. Suppose that

$$\underset{\sim}{A} = \lambda A_1 + u_x A_2,$$
$$\underset{\sim}{B} = \lambda^{-1} B(u), \tag{2.1}$$

where λ is a constant parameter and A_1, A_2 are constant elements of G; $u \to B(u)$ is a map into G. Let $u = u(x,t)$ and $u_x = \frac{\partial u}{\partial x}$. The condition that maps $u \to (\underset{\sim}{A}, \underset{\sim}{B})$ satisfy the prolongation equation is

$$\underset{\sim}{A}_t - \underset{\sim}{B}_x - [\underset{\sim}{A}, \underset{\sim}{B}] = u_{xt} A_2 - \lambda^{-1} B_u u_x -$$
$$-[\lambda A_1 + u_x A_2, \lambda^{-1} B(u)] = u_{xt} A_2 - [A_1, B(u)] - \tag{2.2}$$
$$-\lambda^{-1}(B_u - [A_1, B(u)])u_x,$$

where

$$A_t = u_x t + A_2, \qquad \left(B_u = \frac{\partial B}{\partial u}\right).$$
$$B_x = \lambda^{-1} B_u u_x$$

We would like the right-hand side of (2.2) to vanish for each value of λ. Solution of the equation

$$B_u = [A_1, B(u)] \tag{2.3}$$

can be presented via a power series

$$B(u) = B_0 + B_1 u + \frac{B_2 u^2}{2!} + \ldots,$$
$$B_u = B_1 + B_2 u + \frac{B_3 u^2}{2} + \ldots,$$

$$[A_0, B(u)] = [A_0, B_0] + [A_0, B_1]u + \frac{[A_0,B_2]}{2!}u^2 + \ldots$$

Equating, we have:

$$[A_0,B_0] = B_1,$$

$$[A_0, B_1] = B_2 = [A_0, [A_0, B_0]]$$

and so forth. In other words

$$B_j = (A\,dA_0)^j, \quad j = 0,1,2,\ldots,$$

Hence,

$$B(u) = \sum_j \frac{B_j u^j}{j!} = \sum_j \frac{(Ad\,A_0)^j(B_0)}{j!}u^j =$$

$$= \sum_j \frac{Ad(A_0u)^j}{j!}(B_0) = \exp\{Ad(A_0u)\}(B_0) = \qquad (2.4)$$

$$= Ad\{\exp(A_0u)\}(B_0) = \exp(A_0u)B_0\exp(-A_0u).$$

This determined the λ^{-1} coefficient in (2.2) in the case of (2.2) is set equal to zero. Equating to zero the coefficient of λ^0, we obtain the following relation

$$u_{xt}A_2 = [A_1, \exp(A_0u)B_0\exp(-A_0u)]. \qquad (2.5)$$

The requirement that u satisfies an equation

$$u_{xt} = f(u) \qquad (2.6)$$

gives the condition to determine f:

$$f(u)A_2 = [A_1, \exp(A_0u)B\exp(-A_0u)], \qquad (2.7)$$

For example, if $G = SL(2,R)$, it is easy to choose the elements A_0, A_1, A_2 so that these equations reduce to the single sine-Gordon equation [4]

$$u_{xt} = \sin u.$$

Another possibility is to allow u to be a vectorial function of (x,t).

3. Below we will show how to generalize the prolongation structure approach of Wahlquist and Estabrook for an evolution equation in two spatial dimensions. This strategy was developed by Morris [5] for a generalization of nonlinear Schrödinger equation to two spatial dimensions:

$$i\frac{\partial}{\partial t}A = [(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})A - 2A(\Phi - \Psi)],$$

$$(\partial/\partial y - \partial/\partial x)\Phi = -\frac{1}{2}(\partial/\partial x + \partial/\partial y)(AA^*),$$

$$(\partial/\partial y + \partial/\partial x)\Psi = 1/2(\partial/\partial y - \partial/\partial x)(AA^*).$$

The first step is to construct prolongation structure for stationary Schrödinger equation and the second one is to extend that prolongation structure into one for the evolution Schrödinger equation.

As was shown by Morris [5], to determine a prolongation structure for the equations

$$(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})A = 2A(\Phi - \Psi), \qquad (3.1)$$

$$(\frac{\partial}{\partial y} - \frac{\partial}{\partial x})\Phi = -\frac{1}{2}(\frac{\partial}{\partial x} + \frac{\partial}{\partial y})(AA^*), \qquad (3.2)$$

$$(\frac{\partial}{\partial y} + \frac{\partial}{\partial x})\Psi = \frac{1}{2}(\frac{\partial}{\partial y} - \frac{\partial}{\partial x})(AA^*), \qquad (3.3)$$

we must first settle on an appropriate closed set of 2-forms. Let us introduce the variables R and L defined by

$$R = \frac{1}{2}(\frac{\partial}{\partial x} + \frac{\partial}{\partial y})A, \qquad (3.4)$$

$$L = \frac{1}{2}(\frac{\partial}{\partial y} - \frac{\partial}{\partial x})A, \qquad (3.5)$$

and then write down the Eqs. (3.1)-(3.3) in the form

$$(\frac{\partial}{\partial y} - \frac{\partial}{\partial x})\Phi = -(RA^* + AR^*), \qquad (3.6)$$

$$(\frac{\partial}{\partial y} + \frac{\partial}{\partial x})\Psi = (LA^* + AL^*), \qquad (3.7)$$

$$L_y = -R_x + A(\Phi - \Psi), \qquad (3.8)$$

$$R^*_y = L_x + A^*(\Phi - \Psi), \qquad (3.9)$$

where (3.9) is equivalent to the complex conjugate of (3.1).

The Eqs. (3.4)-(3.9) have an equivalent expression in terms of the closed ideal of 2-forms, α_i, $i = 1,\ldots,6$, defined by

$$\alpha_1 = dA\wedge dy - (R-L)dx\wedge dy, \quad (3.10)$$

$$\alpha_2 = dA\wedge dx + (R+L)dx\wedge dy, \quad (3.11)$$

$$\alpha_3 = d\Phi\wedge(dx+dy) - (RA^* + AR^*)dx\wedge dy, \quad (3.12)$$

$$\alpha_4 = d\Psi\wedge(dx-dy)(+(LA^* + AL^*)dx\wedge dy, \quad (3.13)$$

$$\alpha_5 = dL\wedge dx - dR\wedge dy + A(\Phi-\Psi)dx\wedge dy, \quad (3.14)$$

$$\alpha_6 = dR^*\wedge dx + dL^*\wedge dy + A^*(\Phi-\Psi)dx\wedge dy. \quad (3.15)$$

To find a prolongation structure

$$\Omega = d\xi + F(A, A^*, R, R^*, L, L^*, \Phi, \Psi, \xi)dx + G(A, A^*, R, R^*, L, L^*, \Phi, \Psi, \xi)dy \quad (3.16)$$

we can choose F and G to be as follows:

$$F = x_1 + x_2 A + x_3 A^* + L x_4 + R^* x_5 + \Phi x_6 + \Psi x_7, \quad (3.17)$$

$$G = x_8 + x_9 A + x_{10} A^* + L^* x_5 - R x_4 + \Phi x_6 - \Psi x_7, \quad (3.18)$$

and the Lie bracket must be

$$[F,G] = (R-L)G_A - (R+L)F_A + (R^*-L^*)G_{A^*} - (R^*+L^*)F_{A^*} - (LA^*+AL^*)F_\Psi + (RA^*+AR^*)F_\Phi - A^*(\Phi-\Psi)F_{R^*} - A(\Phi-\Psi)F_L, \quad (3.19)$$

where $H_p = \partial H/\partial p$ for partial derivatives. Substitution of (3.17) and (3.18) into (3-19) leads to bracket relations of the vector fields x_i.

Suppose that a two-dimensional evolution equation, which can be expressed in terms of a closed set of 2-forms $\{\alpha_i\}$, $i = 1,\dots,N$, possesses a linear prolongation structure $\{\alpha_i, \Omega^\beta\}$, $i = 1,\dots,N$, $\beta = 1,\dots,M$, where the 1-forms Ω^β are expressed by

$$\Omega^\beta = \sum_{\alpha=1}^{N}(F^\beta_\alpha dx + G^\beta_\alpha dy)\xi^\alpha + d\xi^\beta \quad (3.20)$$

and suppose that

$$d\Omega^\beta = \sum_{i=1}^{N} f^{\beta i}\alpha_i + \sum_{\gamma=1}^{M}\eta^\beta_\gamma\wedge\Omega^\gamma. \quad (3.21)$$

The 2-forms $\{\bar\Omega^\beta\}$

$$\bar\Omega^\beta = \Omega^\beta\wedge dt + \sum_{\gamma=1}^{M}(GA-FB)^\beta_\gamma\xi^\gamma dx\wedge dy + (A^\beta_\alpha dx + B^\beta_\alpha dy)\wedge d\xi^\gamma \quad (3.22)$$

provide a prolongation structure for a three-dimensional evolution equation defined by a set of 3-forms $\{\bar\alpha_i\}$, $i = 1,\dots,N$.

$$\bar\alpha_i = \alpha_i\wedge dt, \quad i = 1,\dots,K, \quad (3.23)$$

$$\bar\alpha_j = \alpha_j\wedge dt + \beta_j, \quad j = K+1,\dots,N, \quad (3.24)$$

where β_j are defined by the equation

$$\sum_{\lambda=K+1}^{N} f^{\beta i}\beta_i = [(dGA - dFB)\xi]^\beta\wedge dx\wedge dy. \quad (3.25)$$

The matrices A and B in (3.22) satisfy the conditions

$$[A,B] = 0, \quad (3.26)$$

$$[G,A] + [B,F] = 0. \quad (3.27)$$

The forms $\{\alpha_i\}$, $i = 1,\dots,K$, which are basically unaltered, have been called the linearization forms [6] and the remaining forms $\{\alpha_j\}$, $j = K+1,\dots,N$ – the dynamic forms.

In our case the dynamic forms are $\alpha_3 \div \alpha_6$ and Eq. (3.27) can be rewriten as follows

$$-\left[\begin{array}{cc|c} 0 & & 0 \\ \hline \beta_3 & \beta_5 & \\ \beta_6 & \beta_4 & 0 \end{array}\right] = (dGA - dFB)\wedge dx\wedge dy, \quad (3\text{-}28)$$

where A and B satisfy the Eqs. (3.26) and (3.27).

The matrices

$$A = \frac{i}{2}\left[\begin{array}{cc|c} 0 & & 0 \\ \hline 1 & 0 & \\ 0 & 1 & 0 \end{array}\right] \quad \text{and} \quad B = \frac{i}{2}\left[\begin{array}{cc|c} 0 & & 0 \\ \hline 1 & 0 & \\ 0 & -1 & 0 \end{array}\right] \quad (3.29)$$

give

$$\beta_3 = 0, \quad \beta_4 = 0,$$
$$\beta_5 = -\frac{i}{2}dA\wedge dx\wedge dy, \quad \beta_6 = \frac{i}{2}dA^*\wedge dx\wedge dy. \quad (3.30)$$

The corresponding generalized dynamic forms

$$\tilde{\alpha}_5 = \alpha_5 \wedge dt - \frac{i}{2} dA \wedge dx \wedge dy, \qquad \tilde{\alpha}_6 = \alpha_6 \wedge dt + \frac{i}{2} dA^* \wedge dx \wedge dy \tag{3.31}$$

yield to the generalized Schrödinger equations.

The prolongation structure is given by

$$\begin{aligned}
\Omega^1 &= d\xi^1 \wedge dt - (\lambda\xi^4 A\xi^2 + \xi^3) dx \wedge dt - \xi^3 dy \wedge dt, \\
\Omega^2 &= d\xi^2 \wedge dt - (A^*\xi^1 + \lambda\xi^2 - \xi^4) dx \wedge dt - \xi^4 dy \wedge dt, \\
\Omega^3 &= d\xi^3 \wedge dt - (\Phi\xi^1 + L\xi^2 + \lambda\xi^3) dx \wedge dt - (\Phi\xi^1 - R\xi^2 - A\xi^4) dy \wedge dt \\
&+ \frac{i}{2}[(\lambda\xi^1 + A\xi^2) dx \wedge dy + (dx + dy) \wedge d\xi^1], \\
\Omega^4 &= d\xi^4 \wedge dt - (R^*\xi^1 + \Psi\xi^2 + \lambda\xi^4) dx \wedge dt \\
&- (L^*\xi^1 - \Psi\xi^2 + A^*\xi^3) dy \wedge dt + \\
&+ \frac{i}{2}[-(A^*\xi^1 + \lambda\xi^2) dx \wedge dy + (dx - dy) \wedge d\xi^2].
\end{aligned} \tag{3.32}$$

Sectioning into a solution manyfold of nonlinear Schrödinger equations gives

$$\xi^1_x = \lambda\xi^1 + A\xi^2 + \xi^3, \tag{3.33}$$

$$\xi^1_y = \xi^3, \tag{3.34}$$

$$\xi^2_x = A^*\xi^1 + \lambda\xi^2 - \xi^4, \tag{3.35}$$

$$\xi^2_y = \xi^4, \tag{3.36}$$

$$\xi^3_x = \Phi\xi^1 + L\xi^2 + \lambda\xi^3 - \frac{i}{2}\xi^1_t, \tag{3.37}$$

$$\xi^3_y = \Phi\xi^1 - R\xi^2 - A\xi^4 - \frac{i}{2}\xi^1_t, \tag{3.38}$$

$$\xi^4_x = R^*\xi^1 + \Psi\xi^2 + \lambda\xi^4 - \frac{i}{2}\xi^2_t, \tag{3.39}$$

$$\xi^4_y = L^*\xi^1 - \Psi\xi^2 + A^*\xi^3 + \frac{i}{2}\xi^2_t. \tag{3.40}$$

If $\xi^i_t = 0$, than (3.33)-(3.40) reduce to the linear inverse scattering equations

$$\underline{\xi}_x = \left[\begin{array}{cc|cc} \lambda & 0 & \Psi & R^* \\ 0 & \lambda & L & \Phi \\ \hline -1 & 0 & \lambda & A^* \\ 0 & 1 & A & \lambda \end{array}\right] \underline{\xi}, \tag{3.41}$$

$$\underline{\xi}_y = \left[\begin{array}{cc|cc} 0 & A^* & -\Psi & L^* \\ -A & 0 & -R & \Phi \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \underline{\xi}.$$

If $\xi^i_x = 0$ the Eqs. (3.33)-(3.36) become

$$\xi^1_y = -\lambda\xi^1 - A\xi^2, \quad \xi^2_y = A^*\xi^1 + \lambda\xi^2 \tag{3.42}$$

and Eqs. (3.37)-(3-38) become

$$\begin{aligned}
\frac{i}{2}\xi^1_t &= -(\frac{1}{2}AA^* + \lambda^2)\xi^1 + (\frac{1}{2}A_y - \lambda A)\xi^2, \\
\frac{i}{2}\xi^2_t &= (\frac{1}{2}A^*_y + \lambda A^*)\xi^1 + (\frac{1}{2}AA^* + \lambda^2)\xi^2,
\end{aligned} \tag{3.43}$$

These equations are equivalent to the standard Zakharov and Shabat [7,8] form of inverse scattering problem for the nonlinear Schrödinger equation

$$i\frac{\partial}{\partial t}A + \frac{\partial^2}{\partial y^2}A + 2|A|^2 A = 0. \tag{3.44}$$

The case $\xi^i_y = 0$ can be treated by the same method.

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Шпачка А.Б., Яновски А.Б. E2-82-546

Метод Уолквиста-Эстабрука и его применение к исследованию нелинейных моделей теории поля

Рассмотрен предложенный Уолквистом и Эстабруком /1/ метод получения уравнений вспомогательной линейной задачи рассеяния для нелинейных дифференциальных уравнений. Используя аппарат дифференциальных форм, метод позволяет находить солитонные решения нелинейных эволюционных уравнений, законы сохранения, а также преобразования Беклунда. На примере двух нелинейных моделей теории поля изложена процедура Уолквиста-Эстабрука нахождения структуры продолжения в случае как одной, так и двух пространственных переменных.

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The Wahlquits-Estabrook Method for the Investigation of the Nonlinear Models of Classical Field Theory

The Wahlquist-Estabrook method which yields the Inverse Scattering Method equations is considered. A short review is presented on the interpretation of the prolongation forms as connection forms and also on a possible extention of the method for the case of two spatial dimensions. The prolongation structure approach of Wahlquist and Estabrook is illustrated by its application to the sine-Gordon model and to the nonlinear Schrödinger equation in two spatial dimensions.

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