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SUMMATION OF ASYMPTOTIC EXPANSIONS IN CERTAIN PROBLEMS FROM QUANTUM MECHANICS AND QUANTUM FIELD THEORY

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1. INTRODUCTION

The problem of summing up divergent series is by no means a new one and at present there is a long list of various methods for the reconstruction of a function f(g) given its asymptotic (alternating) expansion

$$f(g) \sim \sum_{k=0}^{\infty} f_k g^k , \tag{1.1}$$

where

$$f_k \underset{k\to\infty}{\sim} c(-a)^k k^b (ak)! [1+O(\tfrac{1}{k})], \quad a>0 \tag{1.2}$$

(see for instance [1]).

Unfortunately, in all problems of interest the actual application of the summation methods is hampered by the humbleness of the information we have on f(g). That is, we seldom know the exact values of more than just a few of the first coefficients f_k and we know even less about the analytic properties of f(g).

In what will follow we shall bypass the problem of uniqueness when trying to reconstruct a function f(g) from its asymptotic expansion (1.1). Only alternating series will be considered.

One of the most popular methods for summing up divergent asymptotic series has been invented by Borel. Given the expansion (1.1) we define the Borel transform B(z) of f(g) as

$$B(z) = \sum_{k=0}^{\infty} \frac{f_k z^k}{k!}. \tag{1.3}$$

The series (1.3) may have a non-zero radius of convergence and, when continued analytically along the positive real axis, it may turn out that the integral

$$\int_0^{\infty} e^{-t} B(gt)\,dt \tag{1.4}$$

converges. The method of Borel for summing up the divergent series (1.1) consists in prescribing to it the value of the integral (1.4) when the latter exists.

It is known that if f(g) is analytic in a sufficiently large domain then the integral (1.4) will converge to the actual value of f(g) at least for values of g close to the positive real axis.

The formulae (1.3) and (1.4) can be modified:

$$B^{(\alpha,\gamma)}(z) = \sum_{k=0}^{\infty} \frac{f_k z^k}{\Gamma(k+1+\alpha-\gamma)\,k^{\gamma}} \tag{1.5}$$

together with

$$f(g) = \int_0^{\infty} e^{-t}\, t^{\alpha-\gamma} \left(t\frac{\partial}{\partial t}\right)^{\gamma} B^{(\alpha,\gamma)}(tg)\,dt \tag{1.6}$$

or

$$B_{\mu,\nu}(z) = \sum_{k=0}^{\infty} \frac{f_k z^k}{\Gamma(\nu k+\mu+1)} \tag{1.7}$$

and correspondingly

$$f(g) = \int_0^{\infty} e^{-t}\, t^{\mu} B_{\mu,\nu}(gt^{\nu})\,dt. \tag{1.8}$$

More about Borel's method and its applications can be found in the monograph [1] or in the review papers [2].

The series which defines the Borel transform converges in a circle of radius 1/a (the nearest to the origin singularity of B(z) is at $z=-1/a$ for alternating series and at $z=\frac{1}{a}$ for non-alternating series). The analytic continuation of B(z) beyond this circle can be performed by means of a conformal mapping $W=W(z)$ as a result of which the origin is mapped into itself, the interval $[0,\infty)$ on $[0,1)$ and the interval $(-\infty,-\frac{1}{a})$ on the unit circle $|w|=1$. Next, B(w) is expanded as a power series in w. Sometimes it is not the function B[z(w)] which is expanded in powers of w but rather $z^{-\lambda}B(z)$, i.e.,

$$B(z)\to\tilde{B}(w)=z^{\lambda}\sum_k \tilde{B}_k\cdot w^k . \tag{1.9}$$

The meaning of this trick is to adapt the large-z behaviour of B(z) to the large-g asymptotics of f(g) when the latter is a power-law.

Now let us consider the Borel transform as given by equation (1.7). We can choose the values of μ and ν in such a way that the series (1.7) will define an entire function $B_{\mu,\nu}(z)$. Then, from equation (1.8) it is obvious that if $B_{\mu,\nu}(z) \underset{z\to\infty}{\sim} z^{\rho}$, then the same power will dominate the large-g behaviour of f(g), i.e., $f(g) \underset{g\to\infty}{\sim} g^{\rho}$. A technique which enables us to determine the power ρ from the information supplied by the perturbation theory and from the large-order asymptotics (1.2) would essentially simplify the problem of reconstructing f(g).

In the next paragraph we shall outline one such technique and in the third paragraph we shall apply it to the asymptotic expansions of the groundstate energy of the quantum-mechanical anharmonic oscillator gx^4, the critical exponents and the β-function of Gell-Mann-Low.

2. DESCRIPTION OF THE SUMMATION METHOD

Our objective is to sum up the asymptotic series (1.1) of which we know the exact values of the first $N+1$ coefficients $f_0, f_1, \ldots, f_N$ and the leading term of the large-k asymptotics of f_k, i.e., eq. (1.2). We shall suppose that the quantity we are interested in, that is the function $f(g)$, satisfies the asymptotic condition $f(g) \underset{g\to\infty}{\sim} \text{const}\, g^{\rho}$.

Let us denote by z_{max} the largest value of z up to which the truncated series

$$B^{(N)}_{\mu,\nu}(z) = \sum_{k=0}^{N} \frac{f_k z^k}{\Gamma(\nu k+\mu+1)} \tag{2.1}$$

provides a good approximation to the "true" Borel transform $B_{\mu,\nu}(z)$. The parameters μ and ν are chosen in accordance with eq. (1.2) to ensure the convergence of (1.7) everywhere in the finite z-plane.

The actual value of z_{max} will depend on the accuracy of approximation required as well as on N, ν and μ. In order to determine z_{max} we shall have to estimate the first of the neglected terms:

$$\frac{f_{N+1} z^{N+1}_{max}}{\Gamma(\nu N+\nu+\mu+1)}$$

which means that we shall need an estimation of the value of f_{N+1}. This uncertainty can be avoided if f_N is used instead of f_{N+1} but this will produce a lower bound on z_{max} which is unsatisfactory when N is relatively small ($N=3,4,5\ldots$). Therefore we shall use an extrapolation procedure based on eq. (1.2) with correction terms arranged to suit the last several of the exactly known coefficients. In other words, we define

$$R_L = \left|\frac{f_L}{f_{L+1}}\right|. \tag{2.2}$$

When L is sufficiently large, we have, because of (1.2),

$$R_L \approx aL + a\cdot b\,. \tag{2.3}$$

The approximation we seek is obtained by writing R_L in the form

$$R_L = aL + ab + \frac{\xi_1}{L} + \frac{\xi_2}{L^2} + \ldots + \frac{\xi_q}{L^q}, \quad q \le N-1 \tag{2.4}$$

and next determining $\xi_1, \xi_2, \ldots, \xi_q$ from the system of linear equations we get when substituting the exact values of f_{N-q}, $f_{N-q+1}, \ldots, f_N$.

In some cases for which the number of calculated coefficients is fairly large we have examined the reliability of the extrapolation (2.4) and found the results satisfying.

Now, let us consider the function

$$\Psi^{(N)}_{\mu,\nu}(z) = \frac{dB^{(N)}_{\mu,\nu}(z)}{dz} \cdot \frac{z}{B^{(N)}_{\mu,\nu}(z)} \tag{2.5}$$

which is approximately constant $\Psi^{(N)}_{\mu,\nu}(z) \approx \rho(\mu,\nu,N)$ whenever $B^{(N)}_{\mu,\nu}(z)$ satisfies a simple power-law $B^{(N)}_{\mu,\nu}(z) \approx Cz^{\rho}$ and vice versa.

Since $f(g)$ does not depend on μ and ν it is obvious that the dependence of ρ on μ and ν will tend to vanish with $N\to\infty$. Unfortunately, in all cases of interest the number of calculated coefficients N is rather small and we must pick up some "good" values for μ and ν. This is done in accordance with the principle of minimal sensitivity[3] which in our case ties the "good" values μ_0 and ν_0 to the minimum of the variation of $\rho(\mu,\nu;N)$ for a given N. Eventually the approximate Borel transform to be inserted in (1.8) takes the form

$$B_{\mu,\nu}(z) = \begin{cases} B^{(N)}_{\mu,\nu}(z), & z \le z_{max} \\ \left(\dfrac{z}{z_{max}}\right)^{\rho} B^{(N)}_{\mu,\nu}(z_{max}), & z \ge z_{max}. \end{cases} \tag{2.6}$$

3. APPLICATION AND NUMERICAL RESULTS

A. The Anharmonic Oscillator

We shall consider the groundstate energy $E(g)$ of the quantum-mechanical anharmonic oscillator $V(x) = \frac{1}{2}x^2 + gx^4$. The perturbation theory has been carried out to very high order[4] and the coefficients in the asymptotic series

$$E(g) \approx \sum_{k=0}^{\infty} E_k g^k \tag{3.1}$$

are shown to satisfy

$$E_k \underset{k\to\infty}{\sim} -(-1)^k \sqrt{\frac{6}{\pi^3}}\, 3^k\, \Gamma(k\nu + \tfrac{1}{2}). \tag{3.2}$$

On the other hand it is known that

$$E(g) \underset{g\to\infty}{\sim} \text{const}\, g^{1/3}. \tag{3.3}$$

Finally, the non-perturbative calculation of E(g) from the Schroedinger equation can be performed numerically for any value of g. All this makes the quantum-mechanical anharmonic oscillator an almost perfect testground for any new summation technique.

First we shall illustrate the reliability of our extrapolation procedure (2.4). In Table 1 we have given the extrapolated coefficients $\tilde{E}_k$ against the exact values E_k and the ratios $E_k/\tilde{E}_k$ for extrapolations based on 4-6 terms of the series (3.1).

Table 1

The extrapolated coefficients E_k in the case of the anharmonic oscillator. $R_k = E_k/\tilde{E}_k$, N is the number of the coefficients considered as an input and in our notations (2)2.419=241.9, etc.

k	$\lvert E_k \rvert$	N=4		N=5		N=6	
		$\lvert\tilde{E}_k\rvert$	R_k	$\lvert\tilde{E}_k\rvert$	R_k	$\lvert\tilde{E}_k\rvert$	R_k
5	(2)2.419	(2)2.457	0.98	–	–	–	–
6	(3)3.581	(3)3.750	0.95	(3)3.593	0.99	–	–
7	(4)6.398	(4)6.930	0.92	(4)6.474	0.99	(4)6.416	0.99
8	(6)1.330	(6)1.495	0.89	(6)1.365	0.97	(6)1.342	0.99
9	(7)3.145	(7)3.675	0.85	(7)3.276	0.96	(7)3.206	0.98
10	(8)8.335	(9)1.013	0.82	(8)8.846	0.94	(8)8.605	0.97

From Table 1 we see that the extrapolation procedure works even for relatively small values of N and one is tempted to add some of the extrapolated coefficients to the N we consider as an input. In other words we propose to investigate the functions $\tilde{\Psi}^{(N+M)}_{\mu,\nu}$ rather than $\Psi^{(N)}_{\mu,\nu}$, the former being based on:

$$\tilde{B}^{(N+M)}_{\mu,\nu}(z) = B^{(N)}_{\mu,\nu}(z) + \sum_{k=N+1}^{N+M} \frac{\tilde{f}_k z^k}{\Gamma(\nu k+\mu+1)}. \tag{3.4}$$

The effect of adding new terms to the ones we have had from the start is shown in fig.1.

The evidence of a section of the $\tilde{\Psi}$-curve which runs approximately parallel to the z-axis for $z > z_1$ is interpreted as indicative for a power-law behaviour of the Borel transform:

$$\tilde{B}^{(N+M)}_{\mu,\nu}(z) \sim z^{\rho}, \quad z > z_1 . \tag{3.5}$$

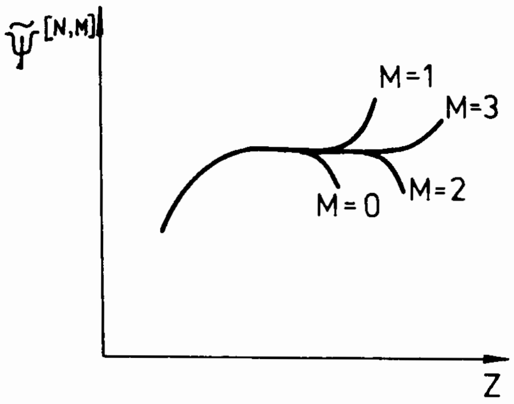

Fig.1. A typical behaviour of $\tilde{\Psi}^{(N+M)}_{\mu,\nu}$ as a function of M.

By adding new terms to $\tilde{B}$ we widen the flat part of the $\tilde{\Psi}$-curve and thus increase the confidence with which ρ is determined. There are, of course, limits to this process and after some M the addition of new extrapolated coefficients does not result in widening the plateau.

A convenient way of studying $\tilde{\Psi}^{(N+M)}_{\mu,\nu}(z)$ is this: instead of drawing Ψ as a continuous curve we plot it as a set of points taken at equidistant values of z between 0 and z_{max}, and look at the density of their projections on the ordinate. A plateau in $\tilde{\Psi}$ will cause a peak in the density and similarly a peak in the density $P(\tilde{\Psi})$ at $\tilde{\Psi}=\rho$ is a signal for a plateau in $\tilde{\Psi}$ of height ρ.

The width of the peak measures the error in determining ρ for the given values of μ and ν. To this we must add the error due to the variation of $\rho(\mu,\nu;N)$ at (μ_0,ν_0).

In fig.2a,b we have shown the actual curves corresponding to N=5 and M=10 at $\mu_0=-0.2$ and $\nu_0=2.1$.

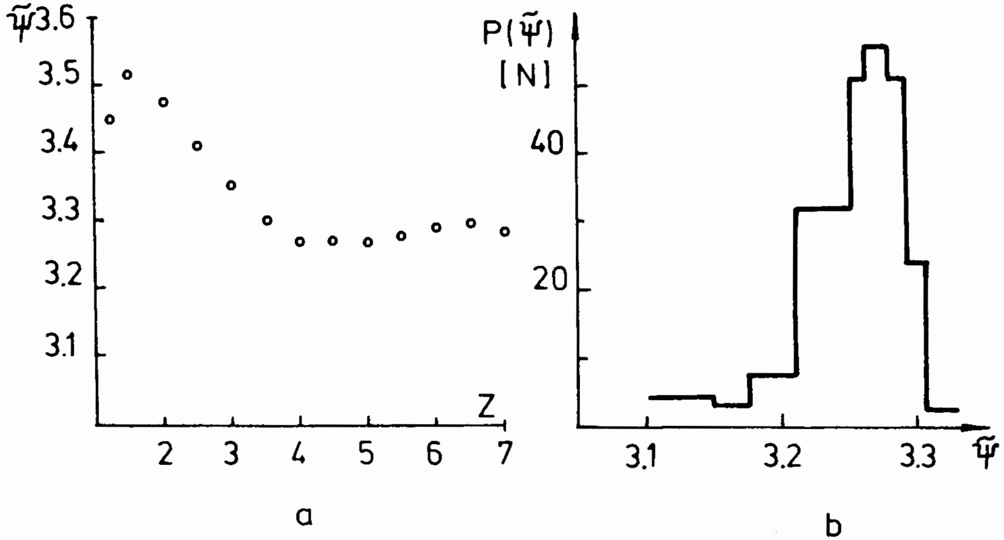

Fig.2

Table 2

The anharmonic oscillator: power evaluation (a) and summation-results (b); N is the number of input-coefficients

	N	μ_0	ν_0	$\rho(\mu_0,\nu_0;N)$	$\Delta\rho$
(a)	4	0.1	2.3	0.298	0.01
	5	-0.2	2.1	0.330	0.008
	6	0.1	2.3	0.318	0.007
	7	-0.2	2.1	0.330	0.005
	8	-0.4	2.0	0.337	0.005
	9	-0.4	2.0	0.336	0.005
	10	-0.3	2.0	0.333	0.005

	g	N=4	N=6	N=8	N=10	the exact results
(b)	10	1.44	1.48	1.50	1.50	1.505
	100	2.80	3.01	3.10	3.11	3.131
	1000	5.56	6.25	6.76	6.66	6.694

The values of $\rho(\mu_0,\nu_0,N)$ and of the groundstate energy for some values of g and for $N=4,5,\dots 10$ are listed in Table 2a,b. It is evident that even for $N=4$ and $N=5$ we can determine ρ with good accuracy.

B. The Critical Exponents

The critical exponents characterize the behaviour of various statistical quantities near the critical temperature Θ_c. They can be expressed through the anomalous dimensions of the Green's functions in the ϕ^4-quantum field theory at the infrared-stable point /7–9/.

We shall concentrate our attention on the critical exponents η, ν and ω which control the behaviour of the correlation function $\Gamma(\vec{x})$ and the correlation length ξ in the neighbourhood of a phase transition (D is the number of space-time dimensions):

$$\Gamma(\vec{x}) \underset{|\vec{x}|\to\infty}{\sim} |\vec{x}|^{2-D-\eta} \qquad \Theta \approx \Theta_c$$
$$\xi \underset{t\to 0}{\sim} t^{-\nu}(1+\text{const.}\,t^{\omega\nu}+\dots), \quad t=\Theta-\Theta_c \tag{3.6}$$

One of the approaches to the problem of determining the critical exponents is the Wilson's ϵ-expansion /5/, where $2\epsilon=4-D$. The series generated by the ϵ-expansion technique are of the type (1.1) with the parameters in the asymptotic formula (1.2) as follows /8/:

$$\alpha=1, \quad a=3/(n+8)$$

$$b=\begin{cases} 3+\frac{n}{2} & \text{for } \eta \\ 4+\frac{n}{2} & \text{for } \frac{1}{\nu} \\ 5+\frac{n}{2} & \text{for } \omega. \end{cases} \tag{3.7}$$

Here and in what will follow n comes from a global O(n)-symmetry of the Lagrangian.

The first several coefficients in the expansions in powers of 2ϵ for η, $1/\nu$ and ω have been calculated for various values of n /7/.

The results obtained with our summation technique are given in Table 3.

Table 3

The critical exponents: power-evaluation and summation-results for $D=3$ $(2\epsilon=1)$; n is the number of components of the field

critical exponent	n	μ_0	ν_0	ρ_0	results: summation	results: experiment /9/
η	1	0.00	1.65	2.65	0.0355	0.016 ± 0.014
	2	-0.05	1.65	2.66	0.0374	-
ν	1	-0.05	1.75	1.32	0.6272	0.625 ± 0.005
	2	0.05	1.65	1.42	0.6658	0.675 ± 0.001
ω	1	-0.05	1.65	0.83	0.7867	-
	2	0.00	1.65	0.81	0.7849	-

In the case $D=2$ exact results are available for η and ν from the Ising model: $\eta(2\epsilon=2)=0.25$ and $\nu(2\epsilon=2)=1.0$. The results we have obtained are respectively $\eta=0.205$ and $\nu=0.91$.

C. The β-Function of Gell-Mann and Low

We have used our method to determine the large-g behaviour of the β-function of Gell-Mann and Low in the model with interaction $\mathcal{L}_{int} = -\frac{16\pi^2}{4!} g\phi^4$ in four-dimensional space-time. The first four coefficients of the perturbation theory expansion of $\beta(g)$ are well-known [11] as are the values of the parameters in (1.2).

Our result is $\rho = 1.86$.

In conclusion, we would like to point out that the success or failure of any summation method which is based on the Borel transformation depends on how well it approximates the "true" Borel-Transform not only near the origin but also along the whole interval of integration. A better approximation of the Borel transform for larger values of the argument leads, as we have demonstrated in this paper, to a reconstruction of the objective function which is fairly accurate even for very large values of g.

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Илчев А.С., Митрюшкин В.К. E2-82-451
Суммирование асимптотических рядов в некоторых задачах квантовой механики и теории поля

Предлагается метод суммирования знакопеременных асимптотических рядов, базирующийся на модифицированном преобразовании Бореля, в котором борелевский образ $B(z)$ есть целая функция z. В качестве примеров даны результаты суммирования рядов теории возмущений для основного уровня gx^4 энергии ангармонического осциллятора, критических индексов и β-функции Гелл-Манна-Лоу в модели теории поля $g(\vec{\phi}^2)^2$.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

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Ilchev A.S., Mitrjushkin V.K. E2-82-451
Summation of Asymptotic Expansions in Certain Problems from Quantum Mechanics and Quantum Field Theory

We propose a method for summing up alternating asymptotic series which is based on a modified Borel transformation with a Borel transform $B(z)$ which is an entire function of z. The method is designed to produce results when $B(z)$ exhibits a power-law asymptotic for z large and positive. The method is illustrated on the examples of the groundstate energy level of the quantum-mechanical anharmonic oscillator gx^4, the critical exponents and the Gell-Mann-Low function in $g(\vec{\phi}^2)^2$-field theory.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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