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GROUP STRUCTURE OF SOME HIDDEN SYMMETRY TRANSFORMATIONS

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INTRODUCTION

For some two-dimensional field-theoretical models (complete-integrable models) there exists an infinite set of integrals of motion. In the relativistic invariant case these are, for instance, the sin-Gordon model, Thirring model, and chiral models. For the chiral models there exist local and nonlocal conserved quantities. In papers [1,2] it was shown that the nonlocal currents for the chiral models have the Noether character and the corresponding hidden symmetry group was found in ref. [3] (see also refs. [4,5]). In paper [6] it is shown that the higher local conserved energy-momentum tensors for the conformal invariant two-dimensional models and the higher conserved charges for models invariant with respect to global gauge and γ_5-gauge transformations also have the Noether character.

In the first part of the present report the group structure of hidden symmetry transformations giving the local conserved charges for the massless SU(N) Thirring model is investigated. It is shown that among the on-shell hidden symmetry transformations there exists a class of transformations which change the Lagrangian by the full derivative for arbitrary fields. There are found also generators of the hidden symmetry transformations which form an infinite-dimensional Lie algebra of the Kac-Moody type [7,8,3].

In the second part the group structure of hidden symmetry transformations for the supersymmetric chiral models is considered. This problem was solved in ref. [9] only for the on-shell case, and there was not given the explicit form of the corresponding generator functions. We show that the off-shell generators can be constructed from the on-shell ones found in ref. [10]. The method of deriving these generators as solutions of a system of differential equations, allows us to find an additional infinite series of generators [2,10]. It is shown that all these series are equivalent and the corresponding nonlocal equivalence operators are obtained. Such equivalent series of transformations and the corresponding conserved currents exist also in the case of ordinary chiral models. It is shown that the off-shell transformations survive after the quantization, like the on-shell transformations [11,12], and consequently, we are able to use them for deriving also the nonlocal Ward identity [13].

I. SU(N) THIRRING MODEL

1. Hidden Symmetry Transformations

Consider the massless SU(N) Thirring model for which the Lagrangian is given by

$$\mathcal{L}(x) = i\bar{\psi}_j(x)\partial\!\!\!/\psi_j(x) + g(\bar{\psi}_j(x)\gamma^\mu\psi_j(x))(\bar{\psi}_k(x)\gamma_\mu\psi_k(x)), \tag{1.1}$$

where $j,k=1,\dots,N$ are isotopic indices and g is a dimensionless coupling constant and consequently $\mathcal{L}(x)$ is conformal invariant. It is evident that $\mathcal{L}(x)$ is invariant also with respect to the global gauge transformations as well as to the global γ_5-gauge transformations. The corresponding conserved currents are given by

$$(j_\mu(x))_{jk} = \bar{\psi}_j(x)\gamma_\mu\psi_k(x),$$
$$(j^5_\mu(x))_{jk} = \bar{\psi}_j(x)\gamma_5\gamma_\mu\psi_k(x) = \epsilon_{\mu\nu} j^\nu_{jk}(x), \tag{1.2}$$

where $\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}$, $\epsilon_{10} = -\epsilon_{01} = 1$ and the following representations for the γ-matrices are used

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ i & 0 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & 1 \\ -i & 0 \end{pmatrix}, \quad \gamma_5 = \gamma_1\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}. \tag{1.3}$$

Then introducing the light-cone coordinates $x_\pm = 1/2(x_0 \mp x_1)$, the Lagrangian $\mathcal{L}(x)$ can be written in the form

$$\mathcal{L} = i\psi^*_{1,j}(x)\partial_-\psi_{1,j}(x) + i\psi^*_{2,j}\partial_+\psi_{2,j} + g(\psi^*_{1,j}\psi_{1,j})(\psi^*_{2,k}\psi_{2,k}), \tag{1.4}$$

where indices 1,2 denote spinor components of ψ. In terms of the eight-cone variables the currents (1,2) and the corresponding continuity equations are written

$$(j_+(x))_{jk} = \psi^*_{1,j}\,\psi_{1,k}(x), \quad (j_-(x))_{jk} = \psi^*_{2,j}\,\psi_{2,k}(x), \quad (j,k=1,\dots,N) \tag{1.5}$$

and

$$\partial_- j_+(x) = 0, \quad \partial_+ j_-(x) = 0. \tag{1.6}$$

From (1.4) it is evident that the Lagrangian (1.1) of the massless Thirring model obeys a more wide symmetry than global gauge and γ_5-gauge transformations [6]. These are the transformations [6]

$$\psi'_1(x) = \exp\{i\eta^{(k)}_{1,a}(x_+)\omega^{(k)}_{a,1}\}\psi_1(x),$$
$$\psi'_2(x) = \exp\{i\eta^{(k)}_{2,a}(x_-)\omega^{(k)}_{a,2}\}\psi_2(x), \tag{1.7}$$

where $\eta^{(k)}_{1,a}(x_-)$ and $\eta^{(k)}_{2,a}(x_+)$ $(a=1,\dots,N^2-1)$ are arbitrary functions only of x_+ and x_-, i.e., they are solutions of the equations

$$\mathrm{tr}\{j_+\partial_-\eta^{(k)}_{1,a}(x)\} = 0, \qquad \mathrm{tr}\{j_-\partial_+\eta^{(k)}_{2,a}(x)\} = 0, \tag{1.8}$$

and for convenience we choose $\eta^{(0)}_{1,a} = \eta^{(0)}_{2,a} = T_a$ which are generators of the isotopic group SU (N) and $\omega^{(k)}_{a,1(2)}$ are parameters of the transformations under considerations. It may be pointed out that the transformations (1.7) do not break the Lorentz invariance. However, the invariance with respect to the space-reflection, can, in original, be broken.

The Noether currents [6] corresponding to the transformations (1.7) have the form

$$J^{(k)}_{+,a} = \mathrm{tr}\{j_+\eta^{(k)}_{1,a}(x)\}, \quad J^{(k)}_{-,a} = \mathrm{tr}(j_-\eta^{(k)}_{2,a}(x)), \tag{1.9}$$

which are conserved. i.e.,

$$\partial_- J^{(k)}_{+,a} = 0, \quad \partial_+ J^{(k)}_{-,a} = 0, \tag{1.10}$$

when the equations of motion are satisfied. To provide existence of the conserved charges, corresponding to (1.8), the following boundary conditions are supposed

$$\lim_{x_1\to\pm\infty} |\eta^{(k)}_{1(2),a}(x)| \le M < \infty. \tag{1.11}$$

From the Lorentz invariance it follows that the Lorentz dimension of $\eta^{(k)}_{1,2}(x)$ is opposite in sign to the Lorentz dimension of the corresponding parameters $\omega^{(k)}_{1(2)}$. Consequently $\eta^{(k)}_{1,a}\omega^{(k)}_{1,a}$ and $\eta^{(k)}_{2,a}\omega^{(k)}_{2,a}$ are scalars. Note that if $\eta^{(k)}_{1,a}(x\ ) = \eta^{(k)}_{2,a}(x\ )$, then the corresponding transformations (1.7) do not break the invariance with respect to space-reflections.

Consider also the polynomial in the field ψ solutions of eqs. (1.8) (only on the on-shell case $\partial_\pm j\ = 0$)

$$\eta^{(m,n)}_{a,\pm} = \partial^{m_1}_\pm(j_\pm)^{n_1},\dots,\partial^{m_q}_\pm(j_\pm)^{n_q}\, T_a\, \partial^{m_{q+1}}_\pm(j_\pm)^{n_{q+1}}\dots\partial^{m_n}_\pm(j_\pm)^{n_n}, \tag{1.12}$$

where m_j and n_j are integer satisfying the following conditions

$$\sum_{j=1}^{m} m_j = m , \quad \sum_{j=1}^{n} n_j = n .$$

In this case the corresponding parameters of transformations (1.7) have the Lorentz dimensions (m+n).

To find the class of transformations (1.7) which preserve the Lagrangian (1,1) up to the full derivative, consider first the abelian case [6]. Then consider the transformations with generators $j_{\pm}^{(0,n)} = (j_{\pm})^n$. Inserting these generators into the invariance condition, we get

$$j_{\pm}\partial_{\mp} (j_{\pm})^n = \frac{n}{n+1} \partial_{\mp} (j_{\pm})^{n+1}, \tag{1.13}$$

i.e., the Lagrangian is changed by the full derivative for arbitrary fields configurations ($\partial_{\pm} j_{\mp} \neq 0$). On the nonabelian case, it can be checked that linear combinations of (1.12) with the form

$$\eta^{(0,n)}_{\pm,a} = (j_{\pm})^n T_a + T_a (j_{\pm})^n + (n-1) j_+ T_a j_+^{n-1} \tag{1.14}$$

also change the Lagrangian (1.1) by the full derivative.

It can be verified that the transformations (1.7) also are the symmetry of equations of motion for the Thirring model.

2. The Group Structure of the Hidden Symmetry Transformation for the Thirring Model

Now the group structure of the transformations (1.7) can be investigated. For this aim consider the commutator of two infinitesimal transformations of the kind (1.7), i.e.,

$$\begin{aligned}
(U_1 U_2 - U_2 U_1)_{jk} \psi_k(x) &= \{ \delta_{j\ell} + i \eta_a^{(p)} (\psi_m + i(\eta_b^{(q)})_{mn} \psi_n \, \delta\omega_b^{(q)})_{j\ell} \times \\
&\times \delta\omega_a^{(p)} \} \{ \delta_{\ell k} + i(\eta_b^{(q)})_{\ell k} \, \delta\omega_b^{(q)} \} \psi_k - \{ \delta_{j\ell} + i \eta_b^{(q)} [\psi_m + \\
&+ i(\eta_a^{(p)})_{mn} \psi_n \, \delta\omega_a^{(p)}]_{j\ell} \, \delta\omega_b^{(q)} \} \{ \delta_{\ell k} + i(\eta_a^{(p)})_{\ell k} \, \delta\omega_a^{(p)} \} \psi_k = \\
&= \{ [\eta_a^{(p)}(x), \eta_b^{(q)}(x)]_{jk} + \frac{\delta(\eta_a^{(p)})_{jk}}{\delta\psi_m} (\eta_b^{(q)} \psi)_m - \frac{\delta(\eta_b^{(q)})_{jk}}{\delta\psi_m} \\
&(\eta_a^{(p)} \psi)_m - (\psi^* \eta_b^{q})_n \frac{\delta(\eta_a^{p})_{jk}}{\delta\psi_n^*} + (\psi^* \eta_a^{(p)})_n \frac{\delta(\eta_b^{(q)})_{jk}}{\delta\psi_n^*} \} \psi_k \times \\
&\times \delta\omega_a^{(p)} \delta\omega_b^{(q)} = [\eta_a^{(p)}(x), \eta_b^{(q)}(x)]_{jk} \psi_k(x) \, \delta\omega_a^{(p)} \, \delta\omega_b^{(q)} ,
\end{aligned} \tag{2.1}$$

where the symbol $[\![\, , \,]\!]$ is introduced for the commutator of two generator functions and $[\, , \,]$ is the ordinary matrix commutator. It is evident that when the generators $\eta_a^{(k)}$ do not depend on fields ψ, the commutator (2,1) coincides with the ordinary matrix commutator. From (2.1) it also follows that when the symmetry group G is abelian, the corresponding hidden symmetry transformations also form an infinite-parameter abelian group.

Consider the following generator functions

$$\eta^{(\nu)}_{\pm,a}(x) = (x_{\pm})^{\nu} T_a , \tag{2.2}$$

where ν is an arbitrary (in general) complex number and T_a are generators of the group SU(N). Then substituting (2.2) into (2.1) we find

$$\begin{aligned}
[\![\eta^{(\nu)}_{\pm,a}(x), \eta^{(\nu')}_{\pm,b}(x)]\!] &= [\eta^{(\nu)}_{\pm,a}(x), \eta^{(\nu')}_{\pm,b}(x)] \\
&= (x_{\pm})^{\nu+\nu'} [T_a, T_b] = i C_{abc} \eta^{(\nu+\nu')}_{\pm,c}(x).
\end{aligned} \tag{2.3}$$

Here C_{abc} are the structure constants of the group SU(N). It may be pointed out that (2.2) with $Re\,\nu > 0$ does not satisfy the boundary condition (1.11).

Consider also the polynomial in ψ generators

$$(\eta^{(0,m)}_{a \, +(-)}(x))_{jk} = [(j_{+(-)})^m T_a]_{jk} = (\psi_{1(2)})_j (\psi^*_{1(2)} T_a)_k (\psi^*_{1(2)} \psi_{1(2)})^m \tag{2.4}$$

$$m = 1,2,\ldots$$

and

$$(\tilde{\eta}^{(0,m)}_a(x))_{jk} = tr(j_{+(-)})^m (T_a)_{jk} = (\psi^*_{1(2),\ell} \psi_{1(2),\ell})(T_a)_{jk} . \tag{2.5}$$

It can be checked that $\eta_a^{0,m}$ as well as $\tilde{\eta}_a^{0,m}$ obey the infinite-dimensional algebra

$$[\![\eta^{(0,m)}_{a,\pm}(x), \eta^{(0,n)}_{b,\pm}(x)]\!] = i C_{abc} \eta^{(0\, m+n)}_{c,\pm}, \tag{2.6}$$

i.e., the generators (2.4) and (2.5) satisfy the subalgebra (for integer ν) of algebra for generators (2.2). The last algebra coincides also with the hidden symmetry Lie algebra for the chiral models [3-5] and with those suggested earlier in [7,8].

And finally, we note that we have not found the closed Lie algebra of a gauge type for the generators with derivatives $\eta_a^{(m,n)}$(1,12), as well as for the polynomial generators (1.14) which change the Lagrangian by the full derivative for arbitrary fields ($\partial_{\pm} j_{\mp} \neq 0$). It may also be pointed out that our considerations are valid in both the cases of Thirring models with commuting and anticommuting classical spinor fields. However, in the second case the number of generators (2.4) and (2.5) is finite ($n=0,1,\ldots,N$).

II. SUPERSYMMETRIC CHIRAL MODELS

3. Off-Shell Transformations

The action for the supersymmetric two-dimensional chiral models is given by

$$S = \frac{1}{2} \int d^2x\, d^2\theta\, \mathrm{tr}\{D^2 \mathcal{G}^{-1}(x,\theta) D_\alpha \mathcal{G}(x,\theta)\}, \tag{3.1}$$

where $\mathcal{G}(x,\theta)$ is a superfield with values on some group G (principal chiral fields) or $\mathcal{G}^{-1} = \mathcal{G} = I - 2\mathcal{P}(x,\theta)$, where $\mathcal{P}^2 = \mathcal{P}$ is the projective field and

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i(\gamma^\mu\theta)_\alpha \partial_\mu, \quad (\alpha=1,2) \tag{3.2}$$

is the spinor covariant derivative. For the γ-matrices the representation (1.3) is used. From (3.1) we have the equations of motion, which for convenience are written in the form

$$D^\alpha A_\alpha(x,\theta) = 0, \tag{3.3}$$

where

$$A_\alpha(x,\theta) = \mathcal{G}^{-1}(x,\theta) D_\alpha \mathcal{G}(x,\theta). \tag{3.4}$$

From (3.4) and the zero curvature in superspace, i.e., $\{D_1, D_2\} = 0$ (which follows from (3.2)) we have

$$\mathcal{F}_{12} = \mathcal{F}_{21} = D_1 A_2 + D_2 A_1 + A_1 A_2 + A_2 A_1 = 0. \tag{3.5}$$

However, here $\mathcal{F}_{11} \neq 0$, $\mathcal{F}_{22} \neq 0$ because of the nonzero torsion in the superspace, i.e., $\{D_1, D_1\} \neq 0$, $\{D_2, D_2\} \neq 0$.

Consider the following generalized supergauge transformations

$$\mathcal{G}'(x,\theta) = \mathcal{G}(x,\theta) U(x,\theta) = \mathcal{G}(x,\theta) \exp\{i\Omega_a^{(k)}(x,\theta)\omega_a^{(k)}\}, \tag{3.6}$$

where $\Omega_a^{(k)}(x;\theta)$ ($k=0,1,\ldots$, $a=1,\ldots,M$, where M is the number of independent parameters of G) are generators of the hidden symmetry group transformations, $\Omega_a^{(0)} = T_a$ the generators of G and $\omega_a^{(k)}$ the corresponding parameters (they are only even numbers). It can be pointed, that under the ordinary supertransformations the generators $\Omega_a^{(k)}$ are transformed as the fields $\mathcal{G}(x,\theta)$. The generators $\Omega_a^{(k)}$ can be found from the invariance (up to the full spinor divergence) of the Lagrangian [10]

$$\delta^{(k)}\mathcal{L} = \mathrm{tr}\{A^\alpha (D_\alpha \Omega_a^{(k)}(x,\theta)\}\delta\omega_a^k = \mathrm{tr}\{D^\alpha K_{\alpha,a}^{(k)}\}\delta\omega_\alpha^k. \tag{3.7}$$

To find $K_{\alpha,a}^{(k)}$ without using the equations of motion (3.2) (i.e., the transformations (3.8) are off-shell symmetry) we adopt the following ansatz for $\Omega_a^{(k)}$

Ansatz: there exists an infinite set of matrix functions $\Omega_a^{(k)}(x,\theta)$, which satisfy the equations (only for $\alpha=1$ or 2)

$$D_\alpha \Omega_a^{(k+1)}(x,\theta) = (\gamma_5 D)_\alpha \Omega_a^{'(k)}(x,\theta) + [(\gamma_5 A)_\alpha, \Omega_a^{(k)}](x,\theta), \tag{3.8}$$

where $\Omega_a^{(0)} = T_a$ and for $\Omega_a^{(k)}$ there exists the following representation

$$\Omega_a(x,\theta,\lambda) = X(x,\theta,\lambda) T_a X^{-1}(x,\theta,\lambda). \tag{3.9}$$

Here

$$\Omega_a(x,\theta,\lambda) = \sum_{k=0}^{\infty} \lambda^k \Omega_a^{(k)}(x,\theta), \tag{3.10}$$

λ is an arbitrary parameter and for the nonsingular matrix function $X(x,\theta,\lambda)$ the Taylor decomposition

$$X(x,\theta,\lambda) = \sum_{k=0}^{\infty} \lambda^k X^{(k)}(x;\theta) \tag{3.11}$$

is supposed also.

Then eqs. (3.8) in terms of $\Omega_a(x,\theta)$ and $X(x,\theta,\lambda)$ are written as

$$(I-\lambda\gamma_5)_\alpha^{\ \beta} D_\beta \Omega_a(x,\theta,\lambda) = \lambda[(\gamma_5 A)_\alpha, \Omega_a(x,\theta,\lambda)], \quad (\alpha=1,2) \tag{3.12}$$

$$(I-\lambda\gamma_5)_\alpha^{\ \beta} D_\beta X(x,\theta,\lambda) = (\gamma_5 A)_\alpha (x,\theta) X(x,\theta,\lambda) \quad (\alpha=1,2). \tag{3.13}$$

Substituting (3.11) in (3.13) we have also

$$D_\alpha X^{(k+1)}(x,\theta) = (\gamma_5 D)_\alpha X^{(k)}(x,\theta) + (\gamma_5 A)_\alpha X^{(k)}(x,\theta) \quad (x=0,1,\ldots). \tag{3.14}$$

The integrability conditions for eqs. (3.8), (3.12-3.13) (for $a=1$ and 2) are satisfied only in the on-shell case ($D^a A_a=0$) and they have the following form (see [10])

$$D^a D_a X + A^a D_a X = 0,$$
$$D^a D_a \Omega_a + [A^a, D_a \Omega_a] = 0. \qquad (3.15)$$

The on-shell solutions of eqs. (3.8) were found in paper [10]. To find the corresponding off-shell transformations we will prove that the transformations (3.10) with generators (3.9), where X is a solution only of one of equations (3.13) (e.g., for $a=1$), are invariance of the Lagrangian (up to the full divergence). Then, solutions of the first (for $a=1$) eqs. (3.14) can be found without using the equation of motion (3.2), i.e., off-shell. Then

$$\delta_a \mathcal{L} = \sum_{k=0}^{\infty} \lambda^k \delta_a^{(k)} \mathcal{L} = \mathrm{tr}\{A^a D_a \Omega_a(x,\theta,\lambda)\} = \mathrm{tr}\{A_2 D_1 \Omega_a -$$
$$- A_1 D_2 \Omega_a(x,\theta,\lambda)\} = \mathrm{tr}\{\lambda A_2 D_1 \Omega_a + \lambda A_2[A_1, A_a] - \qquad (3.16)$$
$$- A_1 D_2 \Omega_a\} = \mathrm{tr}\{\lambda D^a[(\gamma_5 A)_a \Omega_a] + (\frac{1}{\lambda} - \lambda) D^a [X^{-1}(\gamma_5 D)_a X T_a]\},$$

where eqs. (3.14-3.15), the representation (3.11) and the equality (3.7) are used.

Moreover, the transformations (3.8) are invariance also of the equations of motion (3.2). Indeed,

$$D^a \delta_a A_a = D^a (D_a D_a + [A_a, \Omega_a]) =$$
$$= D^a D_a \Omega_a - \frac{1+\lambda}{\lambda} D_1 D_2 \Omega_a - \frac{1-\lambda}{\lambda} D_2 D_1 \Omega_a = 0, \qquad (3.17)$$

where

$$\delta_a A_a = D_a \Omega_a + [A_a, \Omega_a] \qquad (3.18)$$

is substituted and eqs. (3.12) are used.

Now we solve the first ($a=1$) of eqs. (3.13). In terms of the components of $X(x,\theta)$

$$X(x,\theta) = \chi(x) + \theta^a \kappa_a(x) + \theta^1 \theta^2 \xi(x) \qquad (3.19)$$

we have

$$\chi^{(k+1)}(x) = \chi^{(k)}(x) + \int_{-\infty}^{x_+} dy^2 [V_+ \chi^{(k)} - a_1 \kappa_1^{(k)}](y_+, x_-),$$
$$\kappa_1^{k+1}(x) = \kappa_1^{(k)} - a_1(x) \chi^{(k)}(x),$$
$$\kappa_2^{(k+1)}(x) = \kappa_2^{(k)} + \int_{-\infty}^{x_+} dy^+ (a_1 \xi^{(k)} + b_1 \chi^{(k)})(y_+, x_-), \qquad (3.20)$$
$$\xi^{(k+1)}(x) = \xi^{(k)}(x) + i(\rho(x) + r(x)) \chi^{(k)}(x) + i a_1(x, \kappa_2^{(k)}(x), \quad (k=0,1,\dots).$$

Here a, v, r, ρ and b are components of the spinor supercurrent (3.4), i.e.,

$$A_a(x,\theta) = a_a(x) + \theta_a p(x) + (\theta\gamma_5)_a r(x) + (\gamma^\mu \theta)_a V_\mu +$$
$$+ \theta^1 \theta^2 b_a(x) \qquad (a=1,2). \qquad (3.21)$$

and we are starting from one trivial solution $\chi^{(0)} = T$, $\kappa^{(0)} = \xi^{(0)} = 0$, of the equation

$$(D_1 + A_1(x,\theta)) X^{(0)}(x,\theta) = 0. \qquad (3.22)$$

Note that $\chi^{(k)}$ and $\kappa_2^{(k)}$ are determined up to functions of x. It can be checked that in the on-shell case ($D^a A_a = 0$) which in terms of components (3.21) reads $j\partial a(x) = b(x)$, $\partial^\mu V_\mu(x) = 0$, $\rho = 0$, r - arbitrary the functions $X^{(k)}$ with components (3.20) satisfy the integrability condition (2.15), and consequently, the second eqs. (3.14).

In paper [10] it was shown that the solutions of eqs. (3.14) can be found starting from some nontrivial solution of eq. (3.22). There were found two such solutions:

$$X_2^{(0)}(x,\theta) = \mathcal{G}^{-1}(x,\theta) \qquad (3.23)$$

and $X_3^{(0)}$ is given with the following components

$$\chi_3^{(0)}(x) = P\exp\{-\int_{-\infty}^{x_+} dy^+ V_+(y_+, x_-)\} V(x_-),$$
$$(\kappa_3^{(0)})_1 = a_1(x) \chi_3^{(0)}(x),$$

$$(\kappa_3^{(0)})_2 = \tilde\kappa_2^{(0)}(x_-) P \exp\{ i \int_{-\infty}^{x_+} dy^+ a_1 a_1 (y_+, x_-)\} \times$$

$$\times \{ \int_{-\infty}^{x_+} dy_+ (b_1 - j a_1 (r+\rho)) P \exp\{-i \int_{-\infty}^{y_+} dz^+ a_1 a_1\}\}, \qquad (3.24)$$

$$\xi_3^{(0)}(x) = -i(\rho(x) + r(x)) \chi_3^{(0)} + i a_1(x) (\kappa_3^{(0)}(x))_2 ,$$

where the Lorentz scalar $V(x_-)$ and second component of spinor $\tilde\kappa_2(x_-)$ are arbitrary functions of x_- only. The last two functions can be determined in the on-shell case from the second eq. (3.22), i.e., $(D_2 + A_2) X_3^{(0)} = 0$. Then as has been shown in [10] there exist two additional sequences satisfying eqs. (3.16)

$$X_2^{(-k)} = X_2^{(0)} \tilde X_2^{(k)}, \qquad X_3^{(-k)} = X_3^{(0)} \tilde X_3^{(k)}, \qquad (3.25)$$

where $\tilde X_r^{(k)} (r=2,3)$ can be found from (2.20) by the substitution $A_\alpha \to \tilde A_\alpha^{(r)} = (X_r^{(0)})^{-1} A_\alpha X_r^{(0)}$ $(r=2,3)$. The corresponding inverse functions $(X_r^{(0)})^{-1}$ can be found from

$$X_r(x,\theta,\lambda) X_r^{-1}(x,\theta,\lambda) = X_r^{-1}(x,\theta,\lambda) X_r(x,\theta,\lambda)$$
$$= (X_r^{(0)} + \lambda X_r^{(1)} + \ldots)(Y_r^{(0)} + \lambda Y_r^{(1)}, + \ldots) = I. \qquad (3.26)$$

Then we have

$$Y_r^{(0)} = (X_r^{(0)})^{-1}, \quad Y_r^{(1)} = -(X_r^{(0)})^{-1} X_r^{(1)} (X_r^{(0)})^{-1}$$
$$Y_r^{(2)} = -(X_r^{(0)})^{-1} X_r^{(2)} (X_r^{(0)})^{-1} + (X_r^{(0)})^{-1} X_r^{(1)} (X_r^{(0)})^{-1} X_r^{(1)} (X_r^{(0)})^{-1}, \qquad (3.27)$$

where $X_r^{(0)} = I$ and $X_{2,3}^{(0)}$ are given by (2.24) and (2.25). Substituting (3.27) into (3.9) we have

$$\Omega_{1,a}^{(0)} = T_a, \quad \Omega_{1,a}^{(1)} = [X^{(1)}, T_a], \quad \Omega_a^{(2)} = [X^2, T_a] + [T_a X^1] X^1, \qquad (3.28)$$

$$\Omega_{2,a}^{(0)} = \mathcal{G}^{-1} T_a \mathcal{G}, \quad \Omega_{2,a}^{(1)} = \mathcal{G}^{-1} [\tilde\chi^{(1)}, T_a] \mathcal{G}, \qquad (3.29)$$

$$\Omega_{3,a}^{(0)} = X_3^{(0)} T_a (X_3^{(0)})^{-1}, \quad \Omega_{3,a}^{(1)} = X_3^{(0)} [\tilde X_3^{(1)}, T_a] (X_3^{(0)})^{-1}, \ldots \qquad (3.30)$$

Consequently, $\Omega_{2,a}$ and $\Omega_{3,a}$ are coupled with $\Omega_{1,a}$ by similarity transformations, where the spinor supercurrent A_α is given by (3.28). To find other representations for G consider ordinary chiral models.

4. Additional Symmetry Transformations for Ordinary Chiral Models

All results of the previous paragraph can be obtained for ordinary chiral models, for which there exist the transformations (3.23) and (3.24) (see ref. [2]). Moreover, we shall show that there exist additional transformations. In the ordinary case eqs. (3.14) are written [2] as

$$\partial_\pm \eta^{(k+1)}(x) = \pm \partial_\pm \eta^{(k)} \pm A_\pm(x) \eta^{(k)}(x), \quad (k = -1, 0, \ldots), \qquad (4.1)$$

where $\eta^{(-1)}$ is one of solutions of the equation

$$\partial_\pm \eta^{(-1)}(x) + A_\pm \eta^{(-1)}(x) = 0 \qquad (4.2)$$

and consequently, $\partial_\pm \eta^{(0)} = 0$. However, if there exists the conserved current $\mathcal{J}_{\mu,\ldots}$ $(\partial^\mu \mathcal{J}_{\mu,\ldots} = 0)$ for which the condition

$$\epsilon^{\mu\nu} \partial_\mu \mathcal{J}_{\nu,\ldots} = 0 \qquad (4.3)$$

is satisfied also, then as a starting function $\eta^{(0)}(x)$, the solution of the equation

$$\partial_\pm \eta^{(0)}_{,\ldots} = \mathcal{J}_{\pm,\ldots} \qquad (4.4)$$

can be used. The integrability condition for (4.4) is satisfied because of (4.3), which is the case of (higher) energy-momentum tensors $(T_{\underset{(-)(-)}{++}})^n$ $n=1,2..$and $T_{\underset{--}{++}} = Tr(\partial_{\underset{(-)}{+}} g^{-1} \partial_{\underset{(-)}{+}} g)$ $T_{+-} = T_{-+} = 0$. Then substituting T_{++} into (4.4), we have

$$\eta^{(k+1)}_{\underbrace{\{+\ldots+\}}_{2n-1}}(x) = \eta^{(k)}_{\{+\}}(x) + \int_{-\infty}^{x_+} dy_+ (A_+ \chi^{(k)}_{\{+\}})(y_+, x_-), \qquad (4.5)$$

where

$$\eta^{(0)}_{\{+\}}(x) = \int_{-\infty}^{x_+} dy^+ (T_{++})^n. \qquad (4.6)$$

We point out that the Lorentz weight of $\eta^{(k)}_{\{+\}}$ is equal to 2n-1. We introduce a nonsingular one-parameter matrix function

$$Z_{\{+\}}(x,t) = \sum_{k=0}^{\infty} t^k \eta^{(k)}_{\{+\}}(x) \qquad (4.7)$$

which has the same Lorentz weight as $\eta^{(k)}_{\{+\}}$ given by (3.5). We denote by $\zeta^{(k)}_{\{-\}}$ the coefficients of the Taylor decomposition of $\chi^{-1}_{\{+\}}$. for which from (2.27) we have

$$\zeta^{(0)}_{\{-\}}(x) = (\eta^{(0)}_{\{+\}}(x))^{-1}, \quad \zeta^{(1)}_{\{-\}} = -\eta^{(1)}_{\{+\}}(\chi^{(0)}_{\{+\}})^{-1}, \ldots, \qquad (4.8)$$

The Lorentz weight of $\zeta^{(k)}_{\{-\}}$ is $-(2n-1)$. From (4.1), (4.4) and (4.7) it follows that $Z_{\{+\}}$ satisfies the following equations

$$(1-t)\partial_+ Z_{\{+\}}(x,t) = t A_+ Z_{\{+\}} + (T_{++})^n,$$
$$(1+t)\partial_- Z_{\{+\}}(x,t) = -tA_- Z_{\{+\}} , \qquad (4.9)$$

Then the generators of the transformations (3.8)

$$S^{(n)}_a(x,t) = Z_{\{+\}} T_a Z^{-1}_{\{+\}}(x,t) \qquad (4.10)$$

satisfy the equations

$$(1-t)\partial_+ S_a = t[A_+, S_a] + (T_{++})^n [Z^{-1}_{\{+\}}, S_a],$$
$$(1+t)\partial_- S_a = -t[A_-, S_a]. \qquad (4.11)$$

Note that the second eqs. in (4.9) and (4.11) are satisfied only in the on-shell case. Then as in the supersymmetric case (see also ref.[2]) we have

$$\delta_a \mathcal{L} = \sum_{k=0}^{\infty} t^k \delta^{(k)}_a \mathcal{L} = \mathrm{tr}(A^\mu \partial_\mu S_a) = \epsilon^{\mu\nu}\,\mathrm{tr}\,\{ i\partial_\mu (tA_\nu S_a) +$$
$$+ (\frac{1}{t} - t) Z^{-1}_{\{+\}} \partial_\nu Z_{\{+\}} T_a)\}, \qquad (4.12)$$

where the zero curvature $\epsilon^{\mu\nu}(\partial_\mu A_\nu + A_\mu A_\nu) = 0$ and both eqs. (4.9) and (4.11) are used, i.e., the transformations with generators (4.10) are symmetry of the action only in the on-shell case.

It may be pointed out that (4.10) are representations of the same infinite Lie algebra, which is considered in ref.[3]. Moreover the representations (4.10), with $Z_{\{+\}}$ given by

(4.5-4.7) and the representation found in ref.[3] are equivalent. It can be proved that if we have two representations of the hidden symmetry transformations, which can be written in the form (4.10) (with the same T_a), they are equivalent. Indeed, suppose that we have two one-parametric functions Z_1 and Z_2 (with the same matrix dimensions), then we can construct the following matrix

$$U_{12}(x,\tau,\lambda) = Z_1(x,\lambda) Z_2^{-1}(x,\tau), \qquad (4.13)$$

which couples the representations $S^{(1)}_a$ and $S^{(2)}_a$, according to

$$S^{(1)}_a(x,\lambda) = U_{12} S_a U_{12}^{-1}. \qquad (4.14)$$

It may be pointed out that all results of this paragraph can be generalized to the supersymmetric case also.

5. Group Structure of the Hidden Symmetry Transformations in the Supersymmetric Case

To find the group structure of the supergauge transformations (3.6) consider the commutator

$$\mathcal{G}\cdot(U_1 U_2 - U_2 U_1) = \mathcal{G}\{[\Omega_a(x,\lambda), \Omega_b(x,\tau)] +$$
$$+ \delta_a \Omega_b(x,\tau) - \delta_b \Omega_a(x,\lambda)\}\delta\omega^1_a \delta\omega^2_b , \qquad (5.1)$$

where $\delta_a \Omega_b$ is the change of the generator function Ω_b from the transformation $U_1 = I + \Omega_a \delta\omega_a^1$, i.e.,

$$\delta_a \Omega_b = \Omega_b(\mathcal{G} + \mathcal{G}\Omega_a \delta\omega^1_a) - \Omega_b(\mathcal{G}). \qquad (5.2)$$

Following paper[3] the generators of transformations (3.6) are written in the form

$$\mathcal{T}_a(\lambda) = \int d^2x\, d^2\theta\, \mathcal{G}(x,\theta)\Omega_a(x,\theta,\lambda) \frac{\delta}{\delta\mathcal{G}(x,\theta)}. \qquad (5.3)$$

After integration over θ, we have also

$$\mathcal{T}_a(\lambda) = \int d^2x \{ \mathcal{G}(x)\chi_a(x,\lambda)\frac{\delta}{\delta\mathcal{G}} + h(x)\chi_a(x,\lambda)\frac{\delta}{\delta h(x)}$$
$$+ \mathcal{G}(x)\xi_a(x,\lambda)\frac{\delta}{\delta h(x)} + \mathcal{G}(x)\kappa^2_a(x,\lambda)\frac{\delta}{\delta\psi^\alpha(x)} \qquad (5.4)$$
$$+ \psi^\alpha(x)\kappa_{a,\alpha}(x,\lambda)\frac{\delta}{\delta\xi(x)} + \bar\psi^\alpha(x)\chi_a(x,\lambda)\frac{\delta}{\delta\psi^\alpha(x)}\},$$

where the following notation is used

$$\mathcal{G}(\mathbf{x},\theta) = \mathcal{G}(\mathbf{x}) + \theta^{\alpha}\psi_{\alpha}(\mathbf{x}) + \theta^{1}\theta^{2}h(\mathbf{x}).$$

Note that to compute the commutator (5.1), it is necessary to find the change $\delta_a\Omega_b$. For this purpose /3/, the equation

$$(1-r)D_1\delta_a\Omega_b(\mathbf{x},\theta,r) = r\delta_a[A_1,\Omega_b(\mathbf{x},\theta,r)] =$$

$$= r[\delta_a A_1,\Omega_b] + r[A_1,\delta_a\Omega_b] \tag{5.5}$$

$$= \frac{r}{1-\lambda}[[A_1,\Omega_a(\mathbf{x},\theta,\lambda)],\Omega_b(\mathbf{x},\theta,r)] + r[A_1,\delta_a\Omega_b],$$

which follows from (5.2), (3.12), (3.18) and (3.5), is used. The solution of (5.5) satisfying the boundary condition $\delta_a\Omega_b(0) = \delta_a T_b = 0$ has the form

$$\delta_a\Omega_b(r) = \frac{\lambda}{\lambda-r}[\Omega_a(\lambda)-\Omega_a(r),\Omega_b(r)] =$$
$$= \frac{\lambda}{\lambda-r}\{[\Omega_a(\lambda),\Omega_b(r)] - C_{abc}\Omega_c(r)\}. \tag{5.6}$$

Here

$$[\Omega_a(\lambda),\Omega_b(\lambda)] = X(\lambda)[T_a,T_b]X^{-1}(\lambda) = C_{abc}\Omega_c(\lambda), \tag{5.7}$$

where C_{abc} are structure constants of the group G which are substituted in (4.6). Then inserting (5.6) into (5.1) we find

$$[\mathcal{J}_a(\lambda),\mathcal{J}_b(r)] = C_{abc}\int d^2x\, d^2\theta\; \mathcal{G}\,\frac{r\Omega_c(r)-\lambda\Omega_c(\lambda)}{\lambda-r}\,\frac{\delta}{\delta\mathcal{G}}. \tag{5.8}$$

Substituting the power decomposition (2.11) into (5.8), we get

$$[\mathcal{J}_a^{(m)},\mathcal{J}_b^{(n)}] = C_{abc}\mathcal{J}_c^{(m+n)}, \tag{5.9}$$

which coincides in from with the hidden symmetry Lie algebra for ordinary chiral models /3/. However, according to (4.4) the transformations (3.7) are supertransformations of the gauge type. With respect to these transformations the scalar components $\mathcal{G}(\mathbf{x})$ of $\mathcal{G}(\mathbf{x},\theta)$ form an invariant subspace. Moreover,all parameters of these transformations are commuting numbers.

6. Quantum Transformation Laws

Because of nonlinearity of the transformations (2.8) for classical fields in the quantum case these transformations should be correctly determined. This concerns also the generators of the transformations (2.11) as well as the corresponding conserved currents. First consider the quantum transformation laws. In an infinitesimal form the transformation law (2.8) can be written as

$$\mathcal{G}'(\mathbf{x},\theta) = \mathcal{G}(I + \Omega_a^{(k)}\delta\omega_a^{(k)}), \tag{6.1}$$

where $\Omega_a^{(k)}$ is given by (3.11). In the quantum case we have, by definition

$$\delta\hat{\mathcal{G}}_r = \lim_{\substack{r\to 0\\ r^2<0}} \delta_r\hat{\mathcal{G}} = \delta\omega_a^k[\mathcal{G}\hat{\Omega}_a^{(k)}(\mathbf{x},\theta) - \text{sing. term OPE}], \tag{6.2}$$

where $\hat{\mathcal{G}}$ are the corresponding quantum (renormalized) fields, $\hat{\Omega}_a$ are quantum generators, which will be determined later, and the singular terms of the operator product expansion at short distances of operators $\hat{\mathcal{G}}$ and $\hat{\Omega}_a$ can be determined in a nonperturbative way from the dimensional consideration only.

In the case of scalar fields with the zero scale dimension we have

$$\hat{\Omega}(\mathbf{x},\theta)\,\hat{\mathcal{G}}(\mathbf{x},\theta) \approx c\,\ln[\mu^2(x_1-x_2)^2 - i\epsilon]\,\mathcal{G} + \text{reg. terms}, \tag{6.3}$$

which is the case of an ordinary chiral field, as well as the supersymmetric case. Consider also the product of one conserved vector current and one scalar field

$$j_\mu(x_1)S_a(x_2) \approx \frac{c_1 x_\mu^{12}}{x_{12}^2 - i\epsilon}S_a(x_2) + c_2\frac{\epsilon_{\mu\nu}x_{12}^{\nu}}{x_{12}^2 - i\epsilon}\tilde{S}_a(x_2) + \tag{6.4}$$
$$+ \text{reg. term.}$$

in the ordinary case and one spinor conserved supercurrent and one scalar superfield with the zero scale dimension,i.e.,

$$A_\alpha(x_1,\theta)\Omega_a(x_2,\theta) \approx C(\gamma_5 D)_\alpha \ln(\mu^2 x_{12}^2 - i\epsilon)\Omega_a + \text{reg.terms}. \tag{6.5}$$

Here μ is a parameter with the dimension of mass, and the normalization constant C can be determined from equal-time commutators.

The quantum generator of the transformation (6.2) can be determined from the corresponding quantum equations (3.12) and (3.13).

$$(I-\lambda\gamma_5)_\alpha{}^\beta D_\beta \hat{\Omega}^\delta_a(x,\theta,\lambda) = \lambda[(\gamma_5\hat{A})_\alpha(x,\theta)\hat{\Omega}_a(x+\delta,\theta,\lambda) - C(\gamma_5 D)_\alpha \ln(\mu^2 r^2 - i\epsilon)\hat{\Omega}_a(x+\delta,\theta,\lambda), \qquad (6.6)$$

$$(I-\lambda\gamma_5)_\alpha{}^\beta D_\beta X(x,\partial\theta,\lambda) = (\gamma_5 A)_\alpha(x,\theta)X(x+\delta,\theta,\lambda) - C(\gamma_5 D)_\alpha \ln(\mu^2 r^2 - i\epsilon)X(x+\delta,\lambda), \qquad (6.7)$$

where

$$\hat{\Omega}^\delta_a = \hat{X}(x)\, T_a \hat{X}^{-1}(x+\delta) \ - \text{sing terms} . \qquad (6.8)$$

It can be checked that $\hat{\Omega}_a$ form the same infinite Lie algebra as the corresponding classical generators.

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Групповая структура некоторых преобразований скрытых симметрий

Обсуждается вопрос о существовании скрытых симметрий для классических двумерных моделей Тирринга и суперсимметричных киральных моделей. Показано, что генераторы, порождающие высшие локальные сохраняющиеся заряды, образуют бесконечномерную алгебру Ли. То же самое показано и для нелокальных генераторов в суперсимметричных киральных моделях.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

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Group Structure of Some Hidden Symmetry Transformations

The question about the existence of some hidden symmetries for the classical two-dimensional Thirring models and for the supersymmetric chiral models is discussed. It is shown that the generators creating the higher local conserved charges, form an infinite-dimensional Lie algebra. The same is proved about the nonlocal generators for the supersymmetric chiral models.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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