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ON THE GEOMETRIC MEANING OF THE N=1 YANG-MILLS PREPOTENTIAL

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1. The geometric structures inherent in the superspace description of supersymmetric gauge and supergravity theories radically differ from those in ordinary space-time. Gauge potentials in superspace have no chance to be the fundamental quantities as they carry too much degrees of freedom even in the fixed gauge. Therefore, in any self-contained superfield gauge theory they are expected to be composed of a lesser number of unconstrained superfields directly related to the physical field content of the theory, prepotentials. These are known for the N=1 Yang-Mills [1] and supergravity [2-4] theories and, on the linearized level, for their N=2 counterparts [5,6] *.

The prepotentials provide an adequate realization of the minimal invariance group of a given superfield theory and hence can be regarded as natural carriers of its intrinsic geometry. But only in the case of N=1 supergravity this role of them was exposed quite clearly. Ogievetsky and Sokatchev have shown [3] that the prepotential of minimal N=1 supergravity $H^m(x,\theta,\bar\theta)$ specifies the position of a real hypersurface $R^{4,4} = \{ x^m, \theta^\alpha, \bar\theta^{\dot\alpha} \}$ in the complex superspace $C^{4,2} = \{ x_L^m, \theta_L^\alpha \}$:

$$H^m(x,\theta,\bar\theta), \quad \mathrm{Re}\, x_L^m = x^m, \quad \theta_L^\alpha = \theta^\alpha, \quad \bar\theta_R^{\dot\alpha} \equiv (\theta_L^\alpha)^+ = \bar\theta^{\dot\alpha} . \qquad (1)$$

The prepotentials of nonminimal N=1 supergravity have a similar interpretation [7] . It is desirable to understand in analogous terms the other cases listed above and, before all, the text-book case of N=1 Yang-Mills. A deeper insight into its group and geometric structure may help in achieving a complete unconstrained superfield formulation of gauge theorie and supergravities with higher N .

Actually, the underlying geometry of the N=1 Yang-Mills theory reveals a strong resemblance to that of minimal N=1 supergravity and it is the aim of the present note to explicitly demonstrate this. The N=1 Yang-Mills theory is shown to be

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* The objects suggested in ref. [7] as prepotentials of full N=2 supergravity seem not to be the true ones as they are still subjected to certain constraints.

associated with embedding of $R^{4,4}$ in the extended complex superspace $C^{4+M,2} = \{x^m_L, \phi^i_L, \theta^\alpha_L\}$ $(i=1,\dots,M)$, where ϕ^i_L are local coordinates on the group G^c, the complexification of the gauge group G, and $M = \dim G$. The N=1 Yang-Mills prepotential $V^i(x,\theta,\bar\theta)$ is nothing but $\mathrm{Im}\,\phi^i_L$ restricted to the hypersurface $R^{4,4}$. It specifies the position of this hypersurface with respect to G^c/G -directions in $C^{4+M,2}$.

2. Let G be a compact group of internal symmetry with M Hermitean generators T^i. Our basic idea is to extend it to complex noncompact group G^c with 2M generators T^i_L, $T^i_R \equiv (T^i_L)^+$ which satisfy the commutation relations:

$$[T^i_L, T^k_L] = ic^{ik\ell}\, T^\ell_L, \quad [T^i_R, T^k_R] = ic^{ik\ell}\, T^\ell_R, \tag{2.1a}$$

$$[T^i_L, T^k_R] = 0, \tag{2.1b}$$

where $c^{ik\ell}$ are totally skew-symmetric structure constants of G. The group G^c has the structure of direct product $G_L \times G_R$ with G_L and G_R generated, respectively, by T^i_L and T^i_R (e.g., if $G = SU(n)$, then $G^c = SL(n, C)$). The initial group G appears as a diagonal in this product. Its generators are identified with the sum $T^i_L + T^i_R = T^i$ while the remaining generators $i(T^k_L - T^k_R) = A^k$ span the M-dimensional coset space G^c/G. The latter is real and symmetric.

We define the complex superspace $C^{4+M,2} = \{x^m_L, \phi^i_L, \theta^\alpha_L\}$ as a direct sum of ordinary chiral $N=1$ superspace $C^{4,2} = \{x^m_L, \theta^\alpha_L\}$ and the group G_L considered as a complex manifold with local coordinates ϕ^i_L:

$$C^{4+M,2} = C^{4,2} \oplus G_L . \tag{2.2}$$

For convenience, we choose the exponential parametrization of group elements:

$$g^c(\phi_L, \phi_R) = g_L(\phi_L)\, g_R(\phi_R) = e^{i\phi^k_L T^k_L}\, e^{i\phi^k_R T^k_R} = e^{i(\mathrm{Re}\,\phi^k_L T^k + \mathrm{Im}\,\phi^k_L A^k)}, \quad (\phi^k_R \equiv (\phi^k_L)^+) . \tag{2.3}$$

Note that $\mathrm{Re}\,\phi^i_L$ and $\mathrm{Im}\,\phi^i_L$ provide, respectively, a particular parametrization of G and G^c/G. Since the coordinates x^m_L, θ^α_L are related to their conjugates $x^m_R, \bar\theta^{\dot\alpha}_R$ by P-parity, it is natural to accept the same convention for ϕ^i_L, ϕ^i_R:

$$\phi^i_L \overset{P}{\leftrightarrow} \phi^i_R , \quad \mathrm{Re}\,\phi^i_L \overset{P}{\rightarrow} \mathrm{Re}\,\phi^i_L , \quad \mathrm{Im}\,\phi^i_L \overset{P}{\rightarrow} -\mathrm{Im}\,\phi^i_L .$$

Correspondingly, if T^i are scalars, A^i should be pseudoscalars:

$$T^i_L \overset{P}{\leftrightarrow} T^i_R , \quad T^i \overset{P}{\rightarrow} T^i , \quad A^i \overset{P}{\rightarrow} -A^i . \tag{2.4}$$

Clearly, (2.4) is the automorphism of the algebra (2.1).

The group G^c has a natural realization as the group of left multiplications of $g^c(\phi_L, \phi_R)$:

$$\begin{aligned} e^{i\lambda^k_L T^k_L}\, e^{i\phi^k_L T^k_L} &= e^{i\phi'^k_L(\phi_L, \lambda_L) T^k_L} , \\ e^{i\lambda^k_R T^k_R}\, e^{i\phi^k_R T^k_R} &= e^{i\phi'^k_R(\phi_R, \lambda_R) T^k_R} , \end{aligned} \tag{2.5}$$

thus inducing nonlinear transformations for ϕ^i_L, ϕ^i_R. To promote the global G^c-transformations to the local ones, we assume that the group parameters λ^i_L in (2.5) are arbitrary analytic functions of ordinary superspace coordinates:

$$\lambda^i_L = \lambda^i_L(x_L, \theta_L), \quad \lambda^i_R = (\lambda^i_L)^+ = \lambda^i_R(x_R, \bar\theta_R) , \tag{2.6}$$

The gauge group thus defined forms a semi-direct product with supersymmetry realised on x^m_L, θ^{α}_L*. This product is contained as a subgroup in the supergroup of general analytic coordinate transformations of $C^{4+M,2}$ (to be more exact, in its "triangular" subgroup which leaves $C^{4,2}$ invariant). As implied by the relation (2.1b), the left and right components of the gauge group $G^c_{loc} = G_{L\,loc} \times G_{R,loc}$ commute with each other, so at the initial stage the "left" and "right" worlds are entirely disjoined (though conjugated).

Now we wish to show that the N=1 Yang-Mills theory has G^c_{loc} as an invariance group and it naturally emerges upon extracting a special hypersuface in the superspace $C^{4+M,2}$. This hypersurface is the real superspace $R^{4,4} = \{x^m, \theta^\alpha, \bar\theta^{\dot\alpha}\}$, just as in the case of N=1 supergravity /3/. But it possesses now purely internal degrees of freedom besides those represented by $H^m(x,\theta,\bar\theta)$, because of additional bosonic dimensions in $C^{4+M,2}$. So, the embedding conditions (1.1) should be augmented with 2M conditions

$$\mathrm{Im}\,\phi^i_L = V^i(x,\theta,\bar\theta), \quad \mathrm{Re}\,\phi^i_L = U^i(x,\theta,\bar\theta) , \tag{2.7}$$

* Our consideration proceeds in the same way for rigid and local supersymmetries.

where V^i and U^i are real pseudoscalar and scalar superfields. Their transformation properties in G^c_{loc} are uniquely determined by those of ϕ^i_L, ϕ^k_R (2.5). These superfields parametrize, respectively, the coset space G^c/G and the subgroup G, hence they are of the Goldstone character with respect to corresponding transformations. We want G to be unbroken; then $U^i(x,\theta,\bar\theta)$ should be made to have no dynamical manifestations. To achieve this, one may proceed as in standard nonlinear σ-modles (see, e.g., ref. [8]) and arrange the theory to be invariant under the right gauge G-transformations:

$$e^{i(U^kT^k+V^kA^k)} \to e^{i(U^kT^k+V^kA^k)}\, e^{i\lambda^i T^i}, \qquad (2.8)$$

where $\lambda^i = \lambda^i(x,\theta,\bar\theta)$ are M real superparameters. Then $U^i(x,\theta,\bar\theta)$ represent purely gauge degrees of freedom. From the geometric standpoint, the invariance under (2.8) means that different G-directions in $\mathbf{C}^{4+M,2}$ are indistinguishable; the dynamics is required to depend only on the position of the hypersurface with respect to the directions spanning the coset space G^c/G.

On imposing the natural gauge condition

$$U^i(x,\theta,\bar\theta) = 0 \qquad (2.9)$$

we are left with M pseudoscalar superfields $V^i(x,\theta,\bar\theta)$ which live on the coset space G^c/G and transform under G^c_{loc} according to the generic formula of nonlinear realizations [9]

$$e^{i(Re\lambda^k_L T^k + Im\lambda^k_L A^k)}\, e^{iV^kA^k} = e^{iV^{k\prime}A^k}\, e^{iK^i(V,\lambda)T^i} \qquad (2.10)$$

with λ^i_L given by (2.6). The transformation rule of matter superfields $\Phi(x,\theta,\bar\theta)$ is then defined following general prescriptions of refs. [9]:

$$\Phi'(x,\theta,\bar\theta) = e^{iK^k(V,\lambda)\bar T^k}\,\Phi(x,\theta,\bar\theta), \qquad (2.11)$$

$\bar T^k$ being a proper matrix representation of T^k.

A simple consideration exploiting the automorphism (2.4) shows that the transformation law (2.10) admits the equivalent form:

$$e^{i\lambda^i_R T^i_L}\, e^{2V^iT^i_L}\, e^{-i\lambda^i_L T^i_L} = e^{2V^{i\prime}T^i_L} \qquad (2.12)$$

(and the conjugated one, with T^i_R instead of T^i_L). Since T^i_L satisfy the same commutation relations as T^i, eq. (2.12) coincides with the standard transformation law of N=1 Yang-Mills prepotential [1]. In fact, its conventional form is recovered when choosing a particular, non-self-conjugated representation for A^i

$$\bar A^i = i\bar T^i \quad (\bar T^i_L = \bar T^i,\ \bar T^i_R = 0) \qquad (2.13)$$

(this choice is quite permissible because the structure of $V^{i\prime}$, $K^i(V,\lambda)$ in (2.10), (2.12) is determined solely by the commutation relations of generators). Using the general connection between representations and nonlinear realizations [9] one may always pass from superfields with the standard nonlinear transformation rule (2.11) to those transforming in G^c_{loc} linearly, according to the representation (2.13) or to the conjugated one:

$$\Phi_L = e^{iV^k\bar A^k}\,\Phi = e^{-V^k\bar T^k}\,\Phi = e^{-2V^k\bar T^k}\,\Phi_R,$$
$$\Phi'_L = e^{i\lambda^k_L\bar T^k}\,\Phi_L, \quad \Phi'_R = e^{i\lambda^i_R\bar T^i}\,\Phi_R. \qquad (2.14)$$

The relations (2.14) can be regarded as describing the transition from the real basis in the group space of G^c to its complex left and right bases, in a perfect analogy with the connection between real and chiral bases in superspace. These formulas were known earlier [4,10], but our approach renders to them the clear group meaning. Note that the substitution of (2.13) into (2.10) yields the transformation of N=1 prepotential in the form given by Siegel and Gates [4]. Also, the invariance under the right gauge group (2.8) reduces to the well-known freedom of complexifying the prepotential.

To summarize, we have derived the N=1 Yang-Mills prepotential from simple geometric and group principles similar to those constituting the basis of the Ogievetsky-Sokatchev formulation of minimal N=1 supergravity [3].

3. It follows from the above analysis that the N=1 gauge theories can be thought of as a kind of generalized nonlinear σ-models* ($\exp\{2iV^kA^k\}$ is nothing but a "chiral field" on

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*An analogous fact for ordinary gauge theories has been established in ref. [11].

the coset space G^c/G). So, relevant invariants and other geometric characteristics should have an adequate expression in the universal language of Cartan differential forms which is of a common use in nonlinear realizations [9]. In the present case the basic forms are spinorial ones, they are introduced by the relation:

$$e^{-iV^k A^k}[D_\alpha + i\mho^L_\alpha]\, e^{iV^k A^k} = i(\omega^\ell_\alpha A^\ell + \Omega^\ell_\alpha T^\ell) \equiv i\Omega_\alpha \qquad (3.1)$$

and by the conjugated one. Here D_α is an ordinary covariant spinor derivative (it may correspond to the flat as well as curved geometries on $R^{4,4}$) and $\mho^L_\alpha \equiv \mho^{Lk}_\alpha T^k_L$ is the spinor connection on the group G_L:

$$\mho^{L\,\prime}_\alpha = e^{i\lambda^i_L T^i_L}\, \mho^L_\alpha\, e^{-i\lambda^i_L T^i_L} + \frac{1}{i}\, e^{i\lambda^k_L T^k_L}\, D_\alpha\, e^{-i\lambda^k_L T^k_L} \quad . \qquad (3.2)$$

One easily checks that under the nonlinear realization (2.10) the objects ω^ℓ_α, Ω^i_α possess the standard transformation properties of Cartan forms.

It is essential that the gauge superpotentials $\mho^L_\alpha$, $\bar{\mho}^R_{\dot\alpha} = (v^L_\alpha)^+$ are purely subsidiary as they can be covariantly expressed in terms of $V^i(x, \theta, \bar\theta)$ by imposing the manifestly invariant conditions

$$\omega^\ell_\alpha = \bar\omega^\ell_{\dot\alpha} = 0 \qquad (3.3)$$

that is a particular case of inverse Higgs phenomenon [12]. The constraints (3.3) are algebraic with respect to $\mho^L_\alpha$, $\bar{\mho}^R_{\dot\alpha}$, so they can easily be solved to give

$$\mho^L_\alpha = \frac{1}{i}\, e^{-2V^i T^i_L}\, D_\alpha\, e^{2V^i T^i_L} \quad , \quad \bar{\mho}^R_{\dot\alpha} = (\mho^L_\alpha)^+ \quad . \qquad (3.4)$$

Thus, we are finally left with the forms Ω^i_α, $\bar\Omega^i_{\dot\alpha}$ which are the connections on the coset space G^c/G:

$$\Omega_\alpha \equiv \Omega^i_\alpha T^i = \frac{1}{i}\, e^{-V^k T^k}\, D_\alpha\, e^{V^k T^k} \quad ,$$
$$\bar\Omega_{\dot\alpha} \equiv \bar\Omega^i_{\dot\alpha} T^i = \frac{1}{i}\, e^{V^k T^k}\, \bar D_{\dot\alpha}\, e^{-V^k T^k} \quad . \qquad (3.5)$$

All other quantities of the theory: the vector Cartan form, covariant derivatives and strengths are built up from Ω_α, $\bar\Omega_{\dot\alpha}$

following the standard procedure of refs. [13] . They have a familiar appearance, for this reason we do not present them here. The point to be emphasized is that our approach allows us to obtain all necessary quantities starting solely with the structure relations (2.1) and the standard nonlinear realization formula (3.1) supplemented by the constraints (3.3). Perhaps, it would be of interest to relate this formalism to the Levi superform approach advocated by Schwarz [14] as the adequate geometric language for treating real hypersurfaces in complex superspaces.

4. The most interesting question following from the above consideration is how to generalize the proposed construction to higher N Yang-Mills theories, at least to the case $N=2$. The necessity to complexify G in the $N=1$ case can be traced to the fact that the fundamental superspace of $N=1$ sypersymmetry is complex superspace $C^{4,2}$. Its true analog in the $N=2$ case seems to be a superspace the bosonic coordinates of which form a quaternion [15]. So in the $N=2$ case one may, instead of the extension $T^i \to \{T^i, iT^k\}$, try the extension $T^k \to \{T^k, q^i \otimes T^k, \ldots\}$, where q^i ($i = 1,2,3$) are imaginary quaternion units. Then the corresponding prepotential should acquire an additional triplet index. That is just what occurs in the $N=2$ electrodynamics [5] . A work along this line is now in progress.

Other possible applications concern the $N=1$ Yang-Mills theory itself. Once it is a kind of the nonlinear σ-model on the coset G^c/G, the idea arises that the corresponding linear σ-models may exist, with G^c as the vacuum symmetry group. In the conventional $N=1$ Yang-Mills theory, the noncompactness of G^c has no dynamical manifestations since the vacuum stability subgroup is compact and, besides, the Goldstone fields associated with spontaneous breaking G^c/G are purely gauge degrees of freedom (they are contained in the superspin zero part of $V^i(x, \theta, \bar\theta)$). In the relevant linear σ-model, the noncompactness would result in appearance of infinite-dimensional field multiplets, for any unitary representation of G^c is infinite-dimensional. We also note that the geometric analogy between the $N=1$ Yang-Mills and $N=1$ supergravity naturally suggests their unification within a larger theory of the Kaluza-Klein type. One may treat $\mathrm{Re}\,\phi^i_L \equiv \phi^i$ as an independent coordinate like x^m in eq. (1.1), choose the base superspace to be $R^{4+M,4} = \{x^m, \phi^i, \theta^\alpha, \bar\theta^{\dot\alpha}\}$ rather than $R^{4,4}$, and construct a $4+M$-dimensional extension of minimal $N=1$ supergravity through embedding $R^{4+M,4}$ into

$C^{4+M,2}$. The standard theory is expected to be recovered as the lowest order in a proper expansion in ϕ^1.

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Иванов Е.А. О геометрическом смысле препотенциала суперсимметричной N=1 -теории Янга-Миллса E2-82-427

Выявлена общность внутренних геометрий суперсимметричной N=1-теории Янга-Миллса и минимальной N=1-супергравитации. Показано, что препотенциал N=1 -теории Янга-Миллса параметризует фактор-пространство G^c/G, где G^c-комплексное расширение калибровочной группы G .

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Ivanov E.A. On the Geometric Meaning of the N=1 Yang-Mills Prepotential E2-82-427

A deep similarity between intrinsic superspace geometries of the N=1 Yang-Mills theory and minimal N=1 supergravity is established. The N=1 Yang-Mills prepotential is shown to take values in the coset G^c/G , G^c being the complexification of gauge group G .

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