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16/8-82

E2-82-364

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GEOMETRICAL THEORY OF THE RELATIVISTIC STRING IN $t = \tau$ GAUGE

Submitted to "TMФ" and to the
V International Seminar on High Energy Physics
and Field Theory (Protvino, 1982)

1982

1. INTRODUCTION

In papers[1-5] the classical theory of the relativistic string was considered from the pure geometrical point of view. In this approach the role of the dynamical variables was played not by the string coordinates but by the differential forms defined on the string world-surface. The coefficients of these forms are determined by the solutions of the nonlinear equations for the scalar functions. The well known, for example, is the connection of the nonlinear Liouville equation with dynamics of the free relativistic string[1,3-7].

If in the string theory the Lorentz-invariant gauge is used, then the evolution variable is one of curvilinear coordinates τ on the world-surface of the string. This variable is connected in a very complicated way with the time t in Minkowski space-time. Therefore many attempts were undertaken to construct the relativistic string theory so that the parameter of the time evolution τ be equal to time t (the so-called $t=\tau$ gauge in the string theory[2,5]).

In $t=\tau$ gauge the corresponding nonlinear equations for differential forms of the world-surface of the relativistic string interacting with an external field were obtained by F.Lund and T.Regge[2]. For the free string these equations are simplified and have the form[5,8]

$$\theta_{,11} - \theta_{,22} + \frac{\cos\theta}{\sin^3\theta}\left(\kappa_{,1}^2 - \kappa_{,2}^2\right) = 0,$$

$$\left(\kappa_{,1}\cdot \mathrm{ctg}^2\theta\right)_{,1} = \left(\kappa_{,2}\cdot \mathrm{ctg}^2\theta\right)_{,2}. \qquad (1.1)$$

In paper[8] the general solution for this system was obtained in which the functions $\theta(u^1,u^2)$ and $\kappa(u^1,u^2)$ were expressed in terms of four arbitrary functions of one variable $u^{\pm} = u^1 \pm u^2$.

In the present paper it will be shown that the world-surface of the free relativistic string in $t=\tau$ gauge can be described by one linear equation on the scalar function $\psi(t,\sigma)$

$$\psi_{,11} - \psi_{,22} = 0. \qquad (1.2)$$

The paper is organized as follows: In the second section we formulate the classical theory of the relativistic string in the $t=\tau$ gauge using the method of the co-moving frame and the exterior differential forms in the surface theory. In the third section the moving frame on the string world surface will be chosen in a special form. As a result, the theory of the free relativistic string in the $t=\tau$ gauge is reduced to one equation (1.2). We shall prove that this choice of the moving basis is possible always by virtue of the gauge freedom in the geometric theory of the string. Other applications of the proposed method are discussed in the Conclusion.

2. THE MOVING FRAME ON THE WORLD-SURFACE OF THE STRING AND THE BASIC EQUATIONS IN THE SURFACE THEORY

In the four-dimensional space-time the string coordinates $x^\mu(\tau,\sigma)$ obey the equations[9,10]

$$x^\mu_{,11} - x^\mu_{,22} = 0, \tag{2.1}$$

$$(x^\mu_{,1})^2 + (x^\mu_{,2})^2 = 0, \qquad x^\mu_{,1}\, x_{\mu,2} = 0, \tag{2.2}$$

where $x^\mu = (x^0, x^1, x^2, x^3) = (x^0, \vec{x})$, $x^\mu_{,i} = \partial x^\mu/\partial u^i$, $i=1,2$; $\mu=0,1,2,3$, $u^1=\tau$, $u^2=\sigma$.

In the gauge $x^0 = t = \tau = u^1$ the vector $\vec{x}(t,\sigma)$ describes the surface in the three-dimensional Euclidean space $\{x^1, x^2, x^3\}$ and eqs. (2.1), (2.2) take the form

$$\vec{x}_{,11} - \vec{x}_{,22} = 0, \tag{2.3}$$

$$\vec{x}^{\,2}_{,1} + \vec{x}^{\,2}_{,2} = 1, \qquad \vec{x}_{,1}\, \vec{x}_{,2} = 0. \tag{2.4}$$

Let us introduce the co-moving frame on the world surface of the string $\vec{x}(u^1,u^2)$ using two unit tangent vectors $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_1\vec{e}_2=0$ and unit normal $\vec{e}_3$. The origin of this frame is defined by vector $\vec{x}(u^1,u^2)$. It appears to be very convenient to use here the Cartan exterior differential forms. We have the following equation for the infinitesimal displacement of the basis $\{\vec{x}; \vec{e}_1, \vec{e}_2, \vec{e}_3\}$ on the surface[11,12]

$$d\vec{x} = \omega^i \vec{e}_i, \tag{2.5}$$

$$d\vec{e}_a = \Omega_{ab}\vec{e}_b, \quad \Omega_{ab} = -\Omega_{ba}, \qquad \vec{e}_a\vec{e}_b = \delta_{ab}, \quad i,j,k,\ldots=1,2; \quad a,b,c,\ldots=1,2,3, \tag{2.6}$$

where ω^i and Ω_{ab} are linear differential forms ("one-forms") in the basis $\{du^1, du^2\}$: $\omega^i = \omega^i_j du^j$, $\Omega_{ab} = \Omega_{ab|i}\, du^i$. The exterior differentiation of the linear eqs. (2.5) and (2.6) $d\wedge d\vec{x}=0$, $d\wedge d\vec{e}_a = 0$ gives the conditions of their integrability

$$\omega^j \wedge \Omega_{j3} = 0, \tag{2.7}$$

$$d\omega^i = \omega^j \wedge \Omega_{ji}, \tag{2.8}$$

$$d\Omega_{ab} = \Omega_{ac} \wedge \Omega_{cb}. \tag{2.9}$$

Eqs. (2.7)-(2.9) are the basic equations in the surface theory because to any their solution ω^i, Ω_{ab} there corresponds unique surface $\vec{x}(u^1,u^2)$ up to its motion in a space as a whole.

The quadratic differential forms of the surface[13,14], that were introduced in the geometry by Gauss, can be expressed in terms of ω^i and Ω_{ab}. For the first fundamental quadratic form of the surface we have

$$(d\vec{x})^2 = \sum_{i,j=1}^{2} g_{ij}\, du^i du^j, \qquad g_{ij} = \vec{x}_{,i}\cdot\vec{x}_{,j}. \tag{2.10}$$

This form defines the intrinsic geometry of the surface. Substituting (2.5) into (2.10) we obtain the relation

$$g_{ij} = \omega^k_i\, \omega^k_j. \tag{2.11}$$

The second quadratic form[13] determines the external curvature of the surface. The coefficients of this form b_{ij} give us the projection of the second derivatives of the surface radius vector $\vec{x}(u^1,u^2)$ on the normal e_3

$$\vec{x}_{;ij} = b_{ij}\vec{e}_3. \tag{2.12}$$

The semicolon means the covariant differentiation with respect to the metric g_{ij} (2.10). From eqs. (2.5), (2.6) and (2.12) it follows that

$$b_{ij} = (\vec{x}_{;ij}\ \vec{e}_3) = (\vec{x}_{,ij}\ \vec{e}_3) = -(\vec{x}_{,i}\ \vec{e}_{3,j}) = -\omega_i^k\,\Omega_{3k|j} \quad . \tag{2.13}$$

The quadratic forms g_{ij} and b_{ij} determine the surface also uniquely if they satisfy the Gauss equation

$$R_{ijk\ell} = b_{ik}b_{j\ell} - b_{i\ell}b_{jk} \tag{2.14}$$

and the Peterson-Codazzi equation

$$b_{ij;k} = b_{ik;j} \quad , \quad i,j,k,\ell = 1,2, \tag{2.15}$$

where $R_{ijk\ell}$ is the curvature tensor for the metric g_{ij} [13,14]. These equations are equivalent actually to the integrability conditions (2.7)-(2.9).

The world-surface of the string defined by eqs. (2.3), (2.4) on the radius vector $\vec{x}\,(u^1,u^2)$ can be specified in terms of the quadratic forms g_{ij} and b_{ij} as follows

$$g_{11} + g_{22} = 1, \quad g_{12} = g_{21} = 0, \quad g_{11} = \sin^2\theta, \quad g_{22} = \cos^2\theta, \tag{2.16}$$

$$b_{11} = b_{22}. \tag{2.17}$$

To obtain eq. (2.17) we have to substitute eq. (2.12) into (2.3) and to use the Christoffel symbols for metric tensor (2.16)

$$\begin{aligned} &\Gamma^1_{11} = \Gamma^1_{22} = \operatorname{ctg}\theta\cdot\theta_{,1}, \qquad \Gamma^1_{12} = \Gamma^1_{21} = \operatorname{ctg}\theta\cdot\theta_{,2}, \\ &\Gamma^2_{11} = \Gamma^2_{22} = -\operatorname{tg}\theta\cdot\theta_{,2}, \qquad \Gamma^2_{12} = \Gamma^2_{21} = -\operatorname{tg}\theta\cdot\theta_{,1} \ . \end{aligned} \tag{2.18}$$

Eqs. (2.14) and (2.15) are reduced to the system (1.1) [5] in which

$$b_{11} = b_{22} = \operatorname{ctg}\theta\cdot\kappa_{,2}\ , \qquad b_{12} = b_{21} = \operatorname{ctg}\theta\cdot\kappa_{,1}\ . \tag{2.19}$$

We can obtain the system (1.1) using the technique of the moving frame also. For this purpose we have to impose on ω^i and $\Omega_{ab|j}$ conditions (2.16) and (2.17)

$$\begin{aligned} &\omega_1^1\omega_1^1 + \omega_1^2\omega_1^2 + \omega_2^1\omega_2^1 + \omega_2^2\omega_2^2 = 1, \\ &\omega_1^1\omega_2^1 + \omega_1^2\omega_2^2 = 0, \end{aligned} \tag{2.20}$$

$$\omega_1^k\Omega_{3k|1} = \omega_2^k\Omega_{3k|2}\ . \tag{2.21}$$

But on this way there is another possibility which leads to the more simple eq. (1.2).

3. THE ROTATION OF THE MOVING FRAME AND THE INTEGRATION OF THE NONLINEAR EQUATION DETERMINING THE STRING DYNAMICS IN THE $t = \tau$ GAUGE

At any point of the surface the moving frame $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ can be rotated around the normal $\vec{e}_3$ on the arbitrary angle $\lambda(u^1,u^2)$. By this rotation the basic equations (2.7)-(2.9) determining the surface keep obviously their form. We shall use this freedom and choose the angle $\lambda(u^1,u^2)$ so that the matrix elements $\Omega_{ab|j}$, $j = 1,2$ will satisfy some conditions. It appears to be very convenient to take this condition in the following form

$$\Omega_{12|j} = i\Omega_{23|j}\ , \quad j = 1,2. \tag{3.1}$$

The appearance here of the imaginary unit should not confuse us because the rotation of the moving frame is the auxiliary mathematical method that simplifies eqs. (2.7)-(2.9) only.

Let us prove the possibility to impose conditions (3.1). By the transition from the basis $\{\vec{e}_a\}$ to the new one $\{\vec{e}_a'\}$

$$\vec{e}_a' = R_{ab}\vec{e}_b\ , \quad a,b = 1,2,3 \tag{3.2}$$

with the matrix

$$R[\lambda(u^1,u^2)] = \begin{Vmatrix} \cos\lambda & \sin\lambda & 0 \\ -\sin\lambda & \cos\lambda & 0 \\ 0 & 0 & 1 \end{Vmatrix} \tag{3.3}$$

the differential form Ω is transformed as follows

$$\Omega \to \bar{\Omega} = R\Omega R^{-1} + dR\cdot R^{-1}. \tag{3.4}$$

Imposing the condition

$$\bar{\Omega}_{12|j} = i\bar{\Omega}_{23|j}\ , \quad j = 1,2 \tag{3.5}$$

we get from (3.4)

$$d\lambda = -\Omega_{12} + i(\Omega_{23}\cdot\cos\lambda - \Omega_{13}\cdot\sin\lambda). \tag{3.6}$$

This equation will define the angle $\lambda(u^1,u^2)$ if its integrability condition will be fulfilled

$$d^2\lambda \equiv d\wedge d\lambda = 0. \tag{3.7}$$

Taking the exterior differential of the right-hand side of eq. (3.6) and using eq. (2.9) we are convinced of the validity of eq. (3.5). The change of the basis (3.2) leads to the following transformations of the differential forms ω^i

$$\bar{\omega}^k = \omega^i (R^{-1})_{ik}, \quad i,k = 1,2. \tag{3.8}$$

Without loss of generality we can put

$$\bar{\omega}_1^1 = \sin\phi, \quad \bar{\omega}_2^1 = 0, \quad \bar{\omega}_1^2 = 0, \quad \bar{\omega}_2^2 = \cos\phi. \tag{3.9}$$

Indeed it is easy to prove using eq. (3.3) that the condition (2.20) are satisfied for ω^i obtained from (3.8) and (3.9). Equations (2.7)-(2.9) and conditions (2.21) keep their form in new variables $\bar{\omega}^i$, $\bar{\Omega}$ obviously.

From eqs. (2.7), (2.8), (2.21), (3.5) and (3.9) it follows that the matrices $\bar{\Omega}_{ab|j}$, $j=1,2$ have the form

$$\bar{\Omega}_{ab|1} = \begin{vmatrix} 0 & -\phi_{,2} & i\phi_{,1}\cdot \mathrm{ctg}\phi \\ \phi_{,2} & 0 & i\phi_{,2} \\ -i\phi_{,1}\cdot\mathrm{ctg}\phi & -i\phi_{,2} & 0 \end{vmatrix}, \quad \bar{\Omega}_{ab|2} = \begin{vmatrix} 0 & -\phi_{,1} & i\phi_{,2}\,\mathrm{ctg}\phi \\ \phi_{,1} & 0 & i\phi_{,1} \\ -i\phi_{,2}\cdot\mathrm{ctg}\phi & -i\phi_{,1} & 0 \end{vmatrix}. \tag{3.10}$$

The compatibility conditions (2.9) for matrices (3.10) result in one equation which we write here in variables $\xi = u^1 + u^2$, $\eta = u^1 - u^2$

$$\phi_{,\xi\eta} + \mathrm{ctg}\phi \cdot \phi_{,\xi} \cdot \phi_{,\eta} = 0. \tag{3.11}$$

On substituting $\psi = \cos\phi$ Eq. (3.11) reduces to the D'Alembert equation

$$\psi_{,11} - \psi_{,22} = 0. \tag{3.12}$$

So, the classical dynamics of the relativistic string in the $t=\tau$ gauge is described by one equation (3.12) for the scalar function $\psi(t,\sigma)$. If the string is of finite extension, or closed, or it has the point mass at the ends, then eq. (3.11) must be supplemented by the corresponding boundary conditions. We shall not consider these possibilities here believing for simplicity that the string is infinite $-\infty < u^i < +\infty$, $i=1,2$.

4. CONCLUSION

The reduction of the relativistic string theory to one equation (3.12) for the scalar function $\psi(t,\sigma)$ can be useful not only on the classical level but also for the construction of the quantum theory of this object. An important advantage of this approach as compared with the Liouville equation in quantum theory of the relativistic string /6,7/ is the well defined meaning of the evolution parameter $\tau = u^1$ which is equal to the time t of the Minkowski space. As a consequence the Hamiltonian theory of the relativistic string in our approach can be formulated straightway.

The possibility of describing the classical dynamics of the string by the D'Alembert equation (3.12) does not mean that the quantum theory of this object is trivial. The problem arising here is the following. The commutation relations imposed on the function $\psi(t,\sigma)$ must be coordinated with the commutators of the string coordinates $x^\mu(\tau,\sigma)$.

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Received by Publishing Department
on May 20 1982.

Барбашов Б.М., Нестеренко В.В. E2-82-364

Геометрическая теория релятивистской струны в калибровке $t=\tau$

Строится классическая теория релятивистской струны в калибровке $t=\tau$ с использованием формализма подвижного репера и внешних дифференциальных форм в теории поверхностей. Подвижный базис на мировой поверхности струны выбирается специальным образом. В результате теория струны в 4-мерном пространстве-времени сводится к уравнению д'Аламбера на одну скалярную функцию.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1982

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Geometrical Theory of the Relativistic String in $t=\tau$ Gauge

By making use of the co-moving frame method and the exterior differential forms in the surface theory the classical theory of the relativistic string in the gauge is constructed. The moving frame on the string world-sheet is chosen in a special form. As a result, the theory of the free relativistic string in the four-dimensional space-time is reduced to the D'Alembert equation for one scalar function.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1982