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# PECULIARITIES OF N=1 SUPERGRAVITY WITH LOCAL U(1) INVARIANCE

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## 1. INTRODUCTION

At present it is becoming clear that the number N of gravitinos does not specify the kind of extended supergravity completely. Even in the simplest case $N=1$ we are aware of, at least, three kinds of supergravities. Two $N=2$ versions are already known. For higher N one may expect even greater diversity. The versions differ by the content of auxiliary fields. Correspondingly, differences occur in the interactions with matter fields, in the mechanism of spontaneous symmtery breaking (when auxiliary fields get nonzero vacuum expectations); also, in some versions important additional local symmetries appear, etc. In view of all that it seems instructive to study the simplest case $N=1$ in detail. This explains the appearance of a number of papers devoted to the new minimal version of $N=1$ supergravity with local U(1) symmetry /1-6/.

In the present paper we reveal some new and unique features of this model using the complex superspace approach to supergravity. In particular, we show the existence of a new geometric invariant which is not present in the framework of real superspace. We also give an example of successful implementation of a geometric constraint in the action by means of a Lagrange multiplier. At the end we consider another version of the model with 16+16 fields. Our hope is that some of the above features will reappear in the more interesting case of $N=2$ supergravity.

The paper is planned as follows. First, a framework * for the description of the various $N=1$ models is introduced. A complex superspace $\mathbf{C}^{4,4}$ ** is considered with coordinate transformations leaving invariant the chiral $\mathbf{C}^{4,2}$ subspace. The physical real superspace $\boldsymbol{R}^{4,4}$ is embedded in $\mathbf{C}^{4,4}$ as a hypersurface specified by an axial ($H^m$) and a spinor ($H^\mu$ ,

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*It has already been used for both minimal /7,8/ and non-minimal /9,10/ $N=1$ supergravities.

**$\mathbf{C}^{n,k}$ means a complex superspace with n vector and k spinor coordinates.

$\bar{H}^{\mu}$ ) superfields. Einstein supergravity is described by a one-parameter (n) family of supergroups preserving a certain relation between the Berezinians (superdeterminants) of the $C^{4,4}$ and $C^{4,2}$ coordinate transformations. This relation becomes particularly simple for two values of n. For $n=-1/3$ the $C^{4,2}$ supervolume is preserved and this is the case of minimal supergravity. For $n=0$ the $C^{4,4}$ supervolume is preserved. This case exhibits a number of new features. First, in the Wess-Zumino gauge there is a local U(1) invariance. Second, a peculiar geometric invariant emerges. It is the Berezinian of the change of variables from left- to right-handed parametrization of $R^{4,4}$ which in this and only this case transforms as a (dimensionless) scalar superfield. It corresponds to an invariant subset of 8+8 fields. The latter can, and moreover, have to be constrained in order to write down an action. Third, unlike all other cases of N =1 supergravity here the action is not the invariant volume of $R^{4,4}$ (the latter just vanishes (cf. refs. [4, 11] ) when the whole 8+8 subset is eliminated). The action is now given by a new type of invariant [4] involving the U(1) part of the vielbeins. The constraint reducing the number of fields from 20+20 to 12+12 can be solved explicitly in terms of fields in the WZ gauge. The resulting theory is exactly the one of ref. [2]. A solution of this constaint in terms of superfields is presented in [6].

Unfortunately, it is not always so easy to solve explicitly the superfield constraints in a theory. In certain cases it might be even impossible, in particular in extended supergravity. Therefore an alternative approach seems to be of great importance. It consists in introducing the constraints into the action by means of Lagrange multipliers and then obtaining them as equations of motion. We do not know why this has not been attempted even in such simple cases, as N =1 minimal supergravity or super-Yang-Mills theory, etc. Probably, the greatest difficulty is to get rid of the Lagrange multipliers at the end, i.e., to eliminate them from the equations of motion and obtain equations involving only the initial dynamic variables. Here we show an example where this program can be successfully carried out. Hopefully, a similar approach would work in more complex cases, such as N=2 supergravity.

An analysis of U(1) supergravity has already been made in ref. [4] in the framework of real $R^{4,4}$ geometry supplemented by appropriate algebraic constraints. When translated into this language our results are consistent with those of the above authors.

In an Appendix we discuss a relaxed version of our constraint which leads to a model with 16+16 fields. 12+12 of them describe the U(1) minimal supergravity multiplet coupled to a 4+4 "notoph" multiplet (superspin 1/2 off-shell, 0 on-shell).

Parts of the results of this paper have been reported at the Second International Seminar on Quantum Gravity, Moscow, October 1981 [12].

## II. COMPLEX SUPERSPACE

Let us first recall the geometric framework for nonminimal supergravity developed in paper [10] in the spirit of refs. [7,9,13]. Consider a complex superspace

$$\mathbf{C}^{4,4} = \{z_L\} = \{x^m_L,\ \theta^\mu_L,\ \bar\phi^{\dot\mu}_L\}, \tag{1}$$

where $x^m_L$ are 4 complex vector coordinates and $\theta^\mu_L, \bar\phi^{\dot\mu}_L$ are 4 complex spinor ones. The conjugated coordinates will carry an index R:

$$\{z_R\} = \{x^m_R = (x^m_L)^+,\ \bar\theta^{\dot\mu}_R = (\theta^\mu_L)^+,\ \phi^\mu_R = (\bar\phi^{\dot\mu}_L)^+\}. \tag{2}$$

To distinguish these two parametrizations of $\mathbf{C}^{4,4}$ we call them left- and right-handed.

Now we introduce a gauge group in $\mathbf{C}^{4,4}$. We choose it to be the group of analytic transformations of the coordinates which leave the chiral subspace *

$$\mathbf{C}^{4,2} = \{\zeta_L\} = \{x^m_L,\ \theta^\mu_L\} \tag{3}$$

invariant. In other words, the group has a "triangular" structure

$$\begin{aligned} \delta x^m_L &= \lambda^m(x_L, \theta_L), \\ \delta\theta^\mu_L &= \lambda^\mu_L(x_L, \theta_L), \\ \delta\bar\phi^{\dot\mu}_L &= \bar\rho^{\dot\mu}(x_L, \theta_L, \bar\phi_L), \end{aligned} \tag{4}$$

where $\lambda^m$ and $\lambda^\mu$ are chiral superfunctions-parameters and $\bar\rho^{\dot\mu}$ is a general one.

The next step is to introduce the real superspace

$$\mathbf{R}^{4,4} = \{z\} = \{x^m,\ \theta^\mu,\ \bar\theta^{\dot\mu}\} \tag{5}$$

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* We thank Prof. Yu.I.Manin for pointing out that the term "quotient superspace" would be more correct for $\mathbf{C}^{4,2}$, i.e., $\mathbf{C}^{4,2} = \mathbf{C}^{4,4}/\mathbf{C}^{0,2}$. This is suggested by the form of the transformations (4).

as a hypersurface in $\mathbf{C}^{4,4}$, e.g.,

$$x^m = \mathrm{Re}\, x_L^m, \quad \theta^\mu = \theta_L^\mu, \quad \bar\theta^{\dot\mu} = \bar\theta_R^{\dot\mu},$$
$$H^m(x, \theta, \bar\theta) = \mathrm{Im}\, x_L^m,$$
$$H^\mu(x, \theta, \bar\theta) = \phi_R^\mu - \theta_L^\mu, \quad \bar H^{\dot\mu}(x, \theta, \bar\theta) = \bar\phi_L^{\dot\mu} - \bar\theta_R^{\dot\mu}. \tag{6}$$

Here the coordinates of $\mathbf{C}^{4,4}/\mathbf{R}^{4,4}$ are made arbitrary functions of the coordinates of $\mathbf{R}^{4,4}$. The superfunctions $H^m, H^\mu, \bar H^{\dot\mu}$ define the hypersurface and simultaneously determine the (curved) geometry of $\mathbf{R}^{4,4}$. The group (4) induces the following transformations

$$x'^m = x^m + \frac{1}{2}[\lambda^m(x_L, \theta_L) + \bar\lambda^m(x_R, \bar\theta_R)],$$
$$\theta'^\mu = \theta^\mu + \lambda^\mu(x_L, \theta_L), \tag{7.a}$$
$$\bar\theta'^{\dot\mu} = \bar\theta^{\dot\mu} + \bar\lambda^{\dot\mu}(x_R, \bar\theta_R),$$

$$\delta H^m = H'^m(x', \theta', \bar\theta') - H^m(x, \theta, \bar\theta) = \frac{1}{2i}[\lambda^m(x_L, \theta_L) - \bar\lambda^m(x_R, \bar\theta_R)],$$
$$\delta H^\mu = H'^\mu(x', \theta', \bar\theta') - H^\mu(x, \theta, \bar\theta) = \rho^\mu(x_R, \bar\theta_R, \phi_R) - \lambda^\mu(x_L, \theta_L), \tag{7.b}$$
$$\delta \bar H^{\dot\mu} = \bar H'^{\dot\mu}(x', \theta', \bar\theta') - \bar H^{\dot\mu}(x, \theta, \bar\theta) = \bar\rho^{\dot\mu}(x_L, \theta_L, \bar\phi_L) - \bar\lambda^{\dot\mu}(x_R, \bar\theta_R).$$

Here $z_L$:

$$x_L^m = x^m + iH^m(x, \theta, \bar\theta), \quad \theta_L^\mu = \theta^\mu$$
$$\bar\phi_L^{\dot\mu} = \bar\theta^{\dot\mu} + \bar H^{\dot\mu}(x, \theta, \bar\theta) \tag{7.c}$$

and their conjugates $z_R$ are now functions of $x, \theta, \bar\theta$ rather than independent coordinates. In what follows we shall refer to $z_L(z_R)$ of equation (7.c) as left- (right)-handed parametrization of $\mathbf{R}^{4,4}$.

The transformations (7) correspond to conformal supergravity. Restricting them appropriately one can obtain the transformation group of Einstein supergravity. Owing to the triangular structure of the group (4) the Berezinians of both the $\mathbf{C}^{4,4}$ and $\mathbf{C}^{4,2}$ transformations have multiplicative property. So we can single out subgroups by imposing a natural restriction

$$[\mathrm{Ber}(\frac{\partial z'_L}{\partial z_L})]^{3n+1} = [\mathrm{Ber}(\frac{\partial \zeta'_L}{\partial \zeta_L})]^{2n} \quad \text{and h.c.} \tag{8}$$

or, infinitesimally,

$$(3n+1)\frac{\partial}{\partial \bar\phi^{\dot\mu}_L}\bar\rho^{\dot\mu} = (n+1)\left(\frac{\partial\lambda^m}{\partial x^m_L} - \frac{\partial\lambda^\mu}{\partial\theta^\mu_L}\right). \tag{9}$$

Each value of n corresponds to a nonminimal formulation of supergravity with 20+20 fields [9]. There are only two exceptions.

At $n=-1/3$ equation (8) takes the form

$$\mathrm{Ber}(\frac{\partial \zeta'_L}{\partial \zeta_L}) = 1, \tag{10}$$

i.e., the transformations preserve the supervolume of $\mathbf{C}^{4,2}$. In this case the parameters $\rho^\mu, \bar\rho^{\dot\mu}$ are not restricted and with their help the spinor superfields $H^\mu, \bar H^{\dot\mu}$ can be gauged away (just as in conformal supergravity). Thus one recovers the minimal formulation with 12+12 fields. It has been described in detail earlier [7,8] and we are not going to discuss it here.

The second exceptional value, $n=0$, corresponds to the preservation of the total supervolume of $\mathbf{C}^{4,4}$. At $n=0$ equation (8) reduces to

$$\mathrm{Ber}(\frac{\partial z'_L}{\partial z_L}) = 1. \tag{11}$$

Respectively, the supervolume element $d^4x_L d^2\theta_L d^2\bar\phi_L$ is invariant. This value of n is connected with the new minimal version of supergravity as will be explained below.

## III. FIELD CONTENT AND TRANSFORMATIONS

The field content of each of the above described formulations and the meaning of the field transformations are revealed in the Wess-Zumino gauge. We shall do it here with the intention to show how the local U(1) group emerges in the case $n=0$ *.

The parameters $\lambda^m, \lambda^\mu, \bar\rho^{\dot\mu}$ have the following decomposition consistent with equation (9):

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*As we learn from ref. [8] a similar analysis has been carried out in paper [14].

$$\lambda^m(x_L,\theta_L) = a^m + ib^m + \theta_L^\mu \chi_\mu^m + \theta_L\theta_L(c^m + id^m),$$

$$\lambda^\mu(x_L,\theta_L) = \epsilon^\mu + \theta_L^\nu[\delta_\nu^\mu(a+ib) + \omega_{(\nu}{}^{\mu)}] + \theta_L\theta_L\eta^\mu ,$$

$$\bar\rho^{\dot\mu}(x_L,\theta_L,\bar\phi_L) = \bar E^{\dot\mu} + \bar\phi_L^{\dot\mu}\frac{n+1}{3n+1}(-a - ib + \frac{1}{2}\partial_m a^m + \frac{i}{2}\partial_m b^m) +$$

$$+ \bar\phi_L^{\dot\nu}\bar\Omega_{(\dot\nu}{}^{\dot\mu)} + \theta_L^\nu c_\nu{}^{\dot\mu} + \theta_L\theta_L\bar D^{\dot\mu} + \theta_L^\nu\bar\phi_L^{\dot\mu}\frac{n+1}{3n+1}(-\eta_\nu + \frac{1}{2}\partial_m\chi_\nu^m) +$$

$$+ \theta_L^\nu\bar\phi_L^{\dot\nu}G_{\nu(\dot\nu}{}^{\dot\mu)} + \theta_L\theta_L\bar\phi^{\dot\mu}\frac{n+1}{2(3n+1)}\partial_m(c^m + id^m) +$$

$$+ \theta_L\theta_L\bar\phi_L^{\dot\nu}P_{(\dot\nu}{}^{\dot\mu)} . \tag{12}$$

All parameters in the r.h.s. of equation (12) are functions of $x_L$.

From equations (7), (12) one finds that $H^m$ can be gauged into

$$H^m(x,\theta,\bar\theta) = \theta^\mu\bar\theta^{\dot\mu}e^m_{\mu\dot\mu} + \bar\theta^2\theta^\mu\psi_\mu^m + \theta^2\bar\theta_{\dot\mu}\bar\psi^{m\dot\mu} + \theta^2\bar\theta^2A^m \tag{13}$$

by means of fixing the parameters $b^m, \chi_\mu^m, c^m, d^m$ in equation (12). Note that $a^m$ remains unrestricted and it serves as the parameter of general coordinate transformations. Further, $H^\mu$ transforms as follows

$$\delta H^\mu = E^\mu - \epsilon^\mu + \theta^\mu[-\frac{4n+2}{3n+1}a - \frac{2n}{3n+1}ib + \frac{n+1}{2(3n+1)}\partial_m a^m] +$$

$$+ \theta^\nu[\Omega_{(\nu}{}^{\mu)} - \omega_{(\nu}{}^{\mu)}] + \bar\theta_{\dot\nu}c^{\dot\nu\mu} - \theta\theta\eta^\mu + \bar\theta\bar\theta D^\mu + \tag{14}$$

$$+ \theta^\mu\bar\theta_{\dot\nu}\frac{n+1}{3n+1}(-\bar\eta^{\dot\nu} - i\partial_m(\tilde\sigma^m\epsilon)^{\dot\nu}] +$$

$$+ \theta^\nu\bar\theta_{\dot\nu}\bar G_{\dot\nu(\nu}{}^{\mu)} + \bar\theta^2\theta_\nu\bar P^{(\nu\mu)} + \dots ,$$

where the dots denote field-dependent terms. Now one sees that for $n \neq 0, -1/2, -1/3$ one can gauge $H^\mu$ into /9/

$$H^\mu(x,\theta,\bar\theta) = \theta^2\xi^\mu + \bar\theta^2\theta^\mu B + \theta^2\bar\theta_{\dot\mu}(v + iw)^{\mu\dot\mu} + \theta^2\bar\theta^2\beta^\mu \tag{15}$$

by means of fixing all parameters except $\epsilon^{\mu}(x)$ (local supersymmetry) and $\omega^{(\nu\mu)}(x)$ (local Lorentz). The components in equations (13), (15) correspond to the non-minimal set of fields.

## IV. PECULIARITIES OF THE n=0 CASE: U(1) LOCAL GROUP AND EXISTENCE OF INVARIANT

It is remarkable that for $n=0$ the parameter $ib(x)$ (of local $\gamma_5$, or U(1) transformations) drops out of equation (14), so it cannot be fixed and $H^{\mu}$ becomes

$$H^{\mu} = \theta^{\mu} iA + \theta^2 \xi^{\mu} + \bar{\theta}^2 \theta^{\mu} B + \theta^2 \bar{\theta}_{\dot{\mu}} (v + iw)^{\mu\dot{\mu}} + \theta^2 \bar{\theta}^2 \beta^{\mu} . \quad (16)$$

In comparison with equation (15) an additional real pseudoscalar field $A(x)$ appears. At the same time, however, $v^{\mu\dot{\mu}}$ undergoes gradient transformations with parameter $b(x)$, so the total number of components is again 20+20 *.

So, in the family of nonminimal sets of fields there is one and only one allowing for local U(1) transformations. This is not yet the set for the new minimal version of $N=1$ supergravity as we still have 20+20 fields instead of 12+12. However, it turns out that 8+8 fields of this set form a subset closed under supersymmetry transformations. This can be shown by the following clear geometrical reasoning. As was stressed above, for $n=0$ the $\mathbf{C}^{4,4}$ supervolume is preserved. Consequently, both $d^8 z_L$ and $d^8 z_R = (d^8 z_L)^+$ are invariant. On the real hypersurface (6) $d^8 z_L$ and $d^8 z_R$ are connected by the change of variables (see equation (7.c)) $z_L \to z \to z_R$:

$$d^8 z_L = \mathrm{Ber}\left(\frac{\partial z_L}{\partial z}\right) d^8 z = \mathrm{Ber}\left(\frac{\partial z_L}{\partial z}\right) \cdot \mathrm{Ber}\left(\frac{\partial z}{\partial z_R}\right) \cdot d^8 z_R . \quad (17)$$

Therefore the quantity

$$U(x, \theta, \bar{\theta}) = \mathrm{Ber}\left(\frac{\partial z_L}{\partial z_R}\right) = \mathrm{Ber}\left(\frac{\partial z_L}{\partial z}\right) \cdot \mathrm{Ber}^{-1}\left(\frac{\partial z_R}{\partial z}\right) \quad (18)$$

is invariant under the transformations (4), (8) for and only for $n=0$. The explicit form of $U(x, \theta, \bar{\theta})$ can be easily calculated.

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*Note that for $n=-1/2$ the parameter $a(x)$ drops out but $\partial_m a^m(x)$ remains and the gauge can still be fixed as in equation (15) although thus restricting the general coordinate transformations /9/.

$$\mathrm{Ber}\left(\frac{\partial z_L}{\partial z}\right) = \mathrm{Ber}\begin{pmatrix} \delta_n{}^m + i\partial_n H^m & 0 & \partial_n \bar{H}^{\dot\mu} \\ i\partial_\nu H^m & \delta_\nu^\mu & \partial_\nu \bar{H}^{\dot\mu} \\ i\bar\partial_{\dot\nu} H^m & 0 & \delta_{\dot\nu}{}^{\dot\mu} + \bar\partial_{\dot\nu} H^{\dot\mu} \end{pmatrix} = \tag{19}$$

$$= \frac{\det(\delta_n^m + i\partial_n H^m)}{\det(\delta_{\dot\nu}{}^{\dot\mu} + \bar\Delta^{\dot\mu}\bar H_{\dot\nu})},$$

where [8]

$$\bar\Delta_{\dot\mu} = -\bar\partial_{\dot\mu} - i\bar\partial_{\dot\mu} H^m \cdot (1 - i\partial H)^{-1}_m{}^n \partial_m$$

so

$$U(x,\theta,\bar\theta) = \frac{\det(\delta_n{}^m + i\partial_n H^m)\cdot \det(\delta_\nu{}^\mu + \Delta_\nu H^\mu)}{\det(\delta_{\dot\nu}{}^{\dot\mu} + \bar\Delta^{\dot\mu}\bar H_{\dot\nu})\cdot \det(\delta_n{}^m - i\partial_n H^m)}. \tag{20}$$

Clearly, $UU^+ = 1$, therefore $U = \exp(iu)$. The real superfield $u(x,\theta,\bar\theta)$ is the carrier of the invariant 8+8 subset. It is a new quantity not yet encountered either in minimal or nonminimal supergravity. Its origin is essentially in the complex structure of $\mathbb{C}^{4,4}$ and it cannot be explained in the framework of real superspace geometry. It is neither a torsion nor a curvature component, nor anything else known in real supergeometry.

## V. CONSTRAINTS ON THE PREPOTENTIALS

Since U is an invariant object it can be used to write down constraints. In fact, one must do that if one wishes to construct an action. Indeed, as was mentioned above, the field $v_a = \frac{1}{2}(\sigma_a)_{\mu\dot\mu} v^{\mu\dot\mu}$ in equation (16) (as well as in equation (13)) transforms as a gauge field for U(1). However, its dimension is $cm^{-2}$ *, so it cannot have a normal kinetic term of the type $F_{ab}F^{ab}$. The only way it can enter a Lagrangian is to be coupled to a divergenceless (i.e., constrained) axial vector field. This is, indeed, the case realized in the new minimal version by Sohnius and West [2]

* $[H^\mu] = [\theta^\mu] = cm^{1/2}$ but all components of $H^\mu$ have to include a factor $\kappa$, $[\kappa] = cm$, since they vanish in the flat limit.

and Akulov et al. [1]. The corresponding constraint is

$$U = 1. \tag{21}$$

The solution to it is easily found in terms of components in the WZ gauge (13), (16):

$$A = 0, \quad \xi^{\mu} = 0, \quad B = 0, \quad w_a = -\frac{1}{2}\partial_m e_a^{\ m}, \quad \beta^{\mu} = i\partial_m \psi^{m\mu}, \tag{22.a}$$

$$\partial_m (A^m - e_a^m v^a) = 0 . \tag{22.b}$$

Equation (22.b) means that

$$A^m - e_a^m v^a = \epsilon^{mnk\ell} \partial_n a_{k\ell}, \quad a_{k\ell} = -a_{\ell k} = a_{k\ell}^{+} \tag{23}$$

so the "notoph" [15] $a_{k\ell}$ of Sohnius and West [2] and Akulov et al. [1] (together with its additional invariance $\delta a_{k\ell} = \partial_k b_\ell - \partial_\ell b_k$) appears as a solution to the constraint.

A weaker constraint will be discussed in the Appendix.

## VI. INVARIANT INTEGRALS AND ACTION PRINCIPLE

The constraint (21) enables us to write down an action. To this end we first need an invariant integral for $R^{4,4}$. Let $R^{4,4}$ be parametrized by $z_L^M$ (or their cojugates $z_R^M$) defined in equation (7.c) instead of $z^M$. Then, according to the geometric meaning of our gauge group (4), (11) the following integrals

$$\begin{aligned} I_L &= \int d^8 z_L \Phi_L (z_L) = \int d^8 z \; \mathrm{Ber}\left(\frac{\partial z_L}{\partial z}\right) \Phi(z) , \\ I_R &= \int d^8 z_R \Phi_R (z_R) = \int d^8 z \; \mathrm{Ber}\left(\frac{\partial z_R}{\partial z}\right) \Phi(z) \end{aligned} \tag{24}$$

are invariant, Here $\Phi(z)$ is a real scalar superfield, and $\Phi_L(z_L) = \Phi_R(z_R)^+ = \Phi(z)$. Further, as a consequence of the constraint (21)

$$\begin{aligned} &\mathrm{Ber}\left(\frac{\partial z_L}{\partial z}\right) = \mathrm{Ber}\left(\frac{\partial z_R}{\partial z}\right) = \\ &= \left[ \frac{\det(\delta_n^{\ m} + i\partial_n H^m) \cdot \det(\delta_n^{\ m} - i\partial_n H^m)}{\det(\delta_{\dot\nu}^{\ \dot\mu} + \bar\Delta^{\dot\mu} \bar H_{\dot\nu}) \cdot \det(\delta_\nu^{\ \mu} + \Delta_\nu H^\mu)} \right]^{1/2} \equiv E , \end{aligned} \tag{25}$$

therefore

$$I_L = I_R = \int d^8z\, .E.\Phi(z). \qquad (26)$$

Note that the density $E$ is in fact the Berezinian of vielbeins for the curved $R^{4,4}$ with local $U(1)$ in the tangent space (see Sect. VIII). If we choose $\Phi(z)=1$ in equation (24) the integrals will vanish and so will the integral in equation (26), i.e., the invariant volume of $R^{4,4}$ (the same phenomenon was observed by Howe et al. /4/ (see also ref. /11/ )). So, the supervolume of $R^{4,4}$ is not an adequate action for $n=0$ unlike all cases with $n \neq 0$. If we had some nontrivial dimensionless scalar superfield $\Phi$ constructed out of the prepotentials we could put it in equation (26) and try this as an action; however, the only such object is U (20) and it is 1 in our case.

Fortunately, the unique properties of the superspace in this case provide another way of constructing an action. Suppose that $\Phi$ in equation (26) is not a scalar but transforms as follows:

$$\delta\Phi(x, \theta, \bar\theta) = L + R\, , \qquad (27)$$

where

$$L = L(x_L\, , \theta_L\, ), \quad R = R(x_R\, , \bar\theta_R\,) = L^+$$

are some left- and right-handed $C^{4,2}$ (chiral) parameters. Then

$$\delta \int d^8 z E\Phi = \int d^4 x_L d^2\theta_L d^2\bar\phi_L \cdot L(x_L\, , \theta_L\,) + \text{h.c.} = 0$$

because $L(R)$ is independent of $\bar\phi_L$ ($\phi_R$ ). Such type of invariant was proposed in ref. /4/ . In our approach the superfield $\Phi$ can be constructed in terms of prepotentials

$$\Phi = \ln F\, ,$$

$$F = \det{}^{-1/4}\left(\frac{1}{4}[\Delta\, , \sigma_a\ \Delta]H^m\right)\cdot \det{}^{-1/8}(\delta_n^m + \partial_n H^k \partial_k H^m\,)\times \qquad (28)$$

$$\times\, [\det(\delta_\nu{}^\mu + \Delta_\nu H^\mu\,)\cdot \det(\delta_{\dot\nu}^{\dot\mu} + \bar\Delta^{\dot\mu} \bar H_{\dot\nu}\,)]^{3/8}.$$

It transforms according to equation (27):

$$\delta \ln F = L + R\, ,$$

$$L(x_L\, , \theta_L\,) = \frac{\partial\lambda^m}{\partial x_L^m} - \frac{\partial\lambda^\mu}{\partial\theta_L^\mu}\, , \quad R = L^+\, , \qquad (27')$$

where $L(R)$ is the variation of the $C^{4,2}$ volume element. In fact, $F$ is a part of the vielbeins $E_a^M, E_{\dot a}^M$ (see below).

Now we are prepared to write down the action for the new minimal version. Putting equations (25), (28) into the invariant integral one finds

$$S = \frac{1}{\kappa^2} \int d^4x\, d^2\theta\, d^2\bar{\theta}\, E \ln F\,, \tag{29}$$

which should be considered together with the constraint (21). Inserting the component field solution (22), (23) to this constraint into the action (29) one obtains exactly the action of Sohnius and West /2/.

## VII. IMPLEMENTING THE CONSTRAINT IN THE ACTION BY MEANS OF LAGRANGE MULTIPLIER

An action with the dynamic variables restricted by a constraint poses serious problems. For instance, obtaining equations of motion by variation is a nontrivial task. Further, the quantization is rather difficult, etc.

As pointed out in the Introduction, a possible way out (besides the explicit solution of the constraint which is not always so easy) is to introduce Lagrange multipliers in the action and obtain the constraints as equations of motion. In the case of $U(1)$ supergravity this can be done as follows.

The first problem encountered when trying to implement the constraint (21) into the action (29) is that the latter ceases to be invariant. Indeed, the transformation (27') of $\ln F$ in equation (29) leaves $S$ invariant only if equation (21) holds. Therefore, one has to compensate for this transformation. To this end one introduces a real pseudoscalar superfield $\phi(x, \theta, \bar{\theta})$ transforming as follows

$$e^{i\phi'} = e^{i\phi}\,.\,\mathrm{Ber}\left(\frac{\partial\zeta'_L}{\partial\zeta_L}\right).\,\mathrm{Ber}^{-1}\left(\frac{\partial\zeta'_R}{\partial\zeta_R}\right),$$

or infinitesimally

$$\delta\phi = -i\,(L - R)\,, \tag{30}$$

where $L(R)$ is given in equation (27'). This is obviously a group covariant law. Further, consider the integral

$$\begin{aligned} I &= \frac{1}{\kappa^2} \int d^8z_L (\ln F + i\phi) = \\ &= \frac{1}{\kappa^2} \int d^8z\, E U^{1/2} (\ln F + i\phi) \end{aligned} \tag{31}$$

(see equations (19), (20), (25)). Evidently,

$$\delta I = \frac{1}{\kappa^2}\int d^8 z_L(\delta \ln F + i\delta\phi) = \frac{1}{\kappa^2}\int d^8 z_L 2L(x_L, \theta_L) = 0,$$

so I in equation (31) is invariant. Its real part

$$\tilde{S} = \frac{1}{2}(I + I^+) =$$
$$= \frac{1}{2\kappa^2}\int d^8 z E[U^{1/2}(\ln F + i\phi) + U^{-1/2}(\ln F - i\phi)] \tag{32}$$

is a generalization of the action (29)* (when $\phi=0$ and the constraint (21) holds, equation (32) reduces to equation (29)). It is invariant without the help of the constraint. Furthermore, the variation of $\phi$ in equation (32) produces just the constraint (21) as an equation of motion. So, the superfield $\phi$ plays a dual role: it is a compensating superfield for the main term in the action and a Lagrange multiplier for the constraint. It remains to see how one can eliminate the Lagrange multiplier from the rest of the equations.

The variation of $H^a, \bar{H}^{\dot a}$ in equation (32) gives the following equation (up to terms vanishing owing to the already obtained equation (21)):

$$\bar{\Delta}_{\dot a}(\ln F + i\phi) = 0 \qquad \text{and h.c.} \tag{33}$$

It is indeed a covariant equation since

$$\delta(\ln F + i\phi) = 2L(x_L, \theta_L) \quad \text{and} \quad \bar{\Delta}_{\dot a} L = 0.$$

Equation (33) has the following general solution

$$\phi = -\frac{i}{2}(\ell - r), \tag{34.a}$$

$$\ln F = \frac{1}{2}(\ell + r), \tag{34.b}$$

where

$$\ell = \ell(x_L, \theta_L) \qquad \text{and} \qquad r = r(x_R, \bar{\theta}_R) = (\ell)^+$$

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*We are grateful to Dr. B.M.Zupnik for an important improvement of the form of the action (32).

are arbitrary chiral superfields. So, equation (34.a) fixes the Lagrange multiplier (up to gauge freedom, see equation (30)'). Equation (34.b) means that $\ln F$ vanishes up to an arbitrary chiral part. Since the gauge freedom (27') in $\ln F$ is just of the type (34.b) one can conclude that the gauge invariant part of $\ln F$ is zero. In Section VIII we shall see that $\ln F$ is the prepotential for the local U(1) invariance in the tangent (super) space, and the corresponding gauge invariants are the U(1) curvatures $F_{AB}$. So, our second equation of motion (33) is equivalent to

$$F_{AB} = 0 \qquad \text{on-shell.} \tag{35}$$

We shall come back to this below.

Finally, the variation of $H^m$. It is not hard to see (using the already derived equation (21)) that in the corresponding equation of motion only the derivatives of $\phi$ appear. They can be replaced by derivatives of $\ln F$ according to equation (33). Thus, the Lagrange multiplier can be eliminated completely. The resulting equation involves only $H^m$, $H^\mu$ and $\bar{H}^\mu$ and has the form

$$G_a = 0, \tag{36}$$

where $G_a$ is a certain torsion component (see Section VIII). This is not surpising since $G_a$ is the only covariant vector of the right dimension in the theory. It is indeed the equation of motion found in ref. /4/.

The last question is whether our second equation (33) in the form (35) is compatible with equation (36). In the differential geometry formalism (Section VIII) one can show that all the non-vanishing components of $F_{AB}$ are covariant derivatives of $G_a$, so equation (35) is itself a corollary of equation (36).

Ending this section we would like to point out two similarities between the case discussed here and the $N=2$ supergravity theory /10,11/. First, in both cases the invariant volume of the real superspace vanishes as a consequence of the constraints. Second, both actions are not invariant unless the constraints are imposed. On these grounds one may hope that the method of Lagrange multipliers can be applied to the $N=2$ case successfully.

## VIII. DIFFERENTIAL GEOMETRY IN $R^{4,4}$

The geometry of the real superspace $R^{4,4}$ is determined by the fact that it is embedded as a hypersurface in a complex superspace. This means that such invariant characteristics of $\mathbf{R}^{4,4}$ as torsion and curvature can be simply calculated instead of being postulated.

The development of the differential geometry formalism for $\mathbf{R}^{4,4}$ is a straightforward procedure (see ref. [8]). Notice that it can be done before imposing the constraint (21) (the latter is needed only for the action). Here we shall recall just the main steps.

The derivative

$$\nabla_\alpha \Phi \equiv (1+\Delta H)^{-1}{}_\alpha{}^\beta \Delta_\beta \Phi = \frac{\partial}{\partial \phi_R^\alpha} \Phi(x_R, \phi_R, \bar{\theta}_R) \tag{37}$$

of a scalar superfield transforms homogeneously under the group (4), (8) (infinitesimally):

$$\delta(\nabla_\alpha \Phi) = -(\nabla_\alpha \rho^\beta)\nabla_\beta \Phi = \frac{1}{2}(\nabla^\beta \rho_\beta)\nabla_\alpha \Phi - (\nabla_{(\alpha} \rho^{\beta)})\nabla_\beta \Phi. \tag{38}$$

The second term in equation (38) is an induced Lorentz transformation in the tangent space, while the first one is an induced Weyl one. In fact, the component field analysis of Section III shows that in $x$-space there is only a U(1) tangent group. So, one should expect to have only it induced in the tangent superspace. Therefore one should compensate for the dilatation part in equation (38) by introducing a factor F into the definition of the spinor covariant derivative of a scalar weightless superfield

$$D_\alpha \Phi = F \nabla_\alpha \Phi = E_\alpha{}^M \partial_M \Phi . \tag{39}$$

This factor must transform as follows (see equations (9), (27'))

$$\delta F = -\frac{1}{4}(\nabla \rho + \bar{\nabla}\bar{\rho})F = \frac{1}{4}(L+R)F . \tag{40}$$

A full covariant derivative requires a connection

$$D_\alpha = E_\alpha{}^M \partial_M + \omega_{\alpha B}{}^C . \tag{41}$$

The latter has both Lorentz and U(1) parts. Further, the vector covariant derivative can be defined as

$$D_a \equiv E_a{}^M \partial_M = \frac{i}{4}\tilde{\sigma}_a^{\alpha\dot{\alpha}}\{D_\alpha, \bar{D}_{\dot{\alpha}}\} \tag{42}$$

thus automatically choosing the torsion components

$$T_{\alpha\dot{\beta}}{}^c = -2i(\sigma^c)_{\alpha\dot{\beta}}, \quad T_{\alpha\beta}{}^\gamma = 0, \quad T_{\alpha\dot{\beta}}{}^{\dot{\gamma}} = 0 . \tag{43}$$

The last of the equations (43) allows us to express the connection in terms of $H^m, H^\mu, \bar{H}^\mu$ and $F, \bar{F}$. In particular, for its U(1) part one finds

$$\omega_{aB}{}^{B} \sim F\, \nabla_a \ln F. \tag{44}$$

Having obtained the vielbeins $E_A^M$ we can define left-or right-handed vielbeins $\ell_A^M$ ($r_A^M$) by changing variables from $z^M$ to $z_L^M$ or $z_R^M$:

$$E_A^M \frac{\partial}{\partial z^M} = \ell_A^M \frac{\partial}{\partial z_L^M} = r_A^M \frac{\partial}{\partial z_R^M}. \tag{45}$$

According to equation (11) the Berezinians of $\ell_A^M$ and $r_A^M$ transform as scalars, so they can be put equal to some function of the scalar U (20) thus obtaining equations for the factors $F, \bar{F}$ (39). The particular choice

$$\mathrm{Ber}(\ell_A^M) = U^{-1/2} \tag{46}$$

leads to the form of $F = \bar{F}$ given in equation (28). Further, $\mathrm{Ber}(E_A^M)$ calculated with the above value of F is indeed equal to $E^{-1}$(25).

The last step is to calculate the invariant tensors (torsion components) using the covariant derivatives already defined. Our results agree with those of ref.[4] but we ought to point out the following. The quantity U (20) is an invariant of the group although there is no room for it among the torsion components. However, its covariant derivatives do appear as torsion components, e.g., $T_{ab}{}^{b}$ is expressed in terms of $D_\alpha U$; $T_{ab}{}^{\gamma}$, in terms of DDU, etc. So, the constraint (21) yields the vanishing of all those torsion components. In the framework of real superspace geometry U is not present. There, however, there is the constraint

$$T_{ab}{}^{b} = T_{\dot{a}b}{}^{b} = 0,$$

which is equivalent to

$$D_\alpha U = \bar{D}_{\dot{a}} U = 0 \tag{47}$$

in our language. Equation (47) implies U=const which is essentially the same as equation (21). This explains the agreement between the two approaches.

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APPENDIX: A WEAKER CONSTRAINT

Here we would like to discuss briefly a weaker constraint on the superfield U. In this case we get 4+4 additional degrees of freedom. They are a superanalogue of the "notoph" [16] (superspin 0 on-shell and 1/2 off-shell) which interacts with U(1) supergravity.

Consider the integral

$$I_1 = \frac{1}{\kappa^2} \int d^8 z_L \ln F = \frac{1}{\kappa^2} \int d^8 z \, \mathrm{Ber}\left(\frac{\partial z_L}{\partial z}\right) \ln F \qquad (A.1)$$

taken over $R^{4,4}$ in the left-handed parametrization. According to equation (27')

$$\delta I_1 = \frac{1}{\kappa^2} \int d^8 z_L (L + R) = \frac{1}{\kappa^2} \int d^8 z_L R \qquad (A.2)$$

because L does not depend on $\bar{\phi}_L$. Further, going to the right-handed parametrization we find

$$\delta I_1 = \frac{1}{\kappa^2} \int d^8 z_R \cdot U \cdot R = \frac{1}{\kappa^2} \int d^4 x_R d^2 \bar{\theta}_R \left(\frac{\partial^2}{\partial \phi_R^2} U\right) R \,, \qquad (A.3)$$

because now R does not depend on $\phi_R$. So, $I_1$ will be invariant if

$$\frac{\partial^2}{\partial \phi_R^2} U = 0. \qquad (A.4)$$

This is a covariant (the l.h.s. of equation (A.4) transforms as a scalar with a chiral weight) constraint weaker than equation (21). Notice that the quantity $I_1$ is not real since U is not 1 now. Furthermore, we can write down another non-trivial complex invariant

$$I_2 = \frac{1}{\kappa^2} \int d^8 z_L f(U) \,, \qquad (A.5)$$

where f(U) is any function of the scalar U.

The constraint (A.4) can be solved in terms of component fields. The pseudoscalar field A and the spinor $\xi^{\mu}$ in equation (16) remain unrestricted; B = 0 and $\beta^{\mu}$ is expressed in terms of $\xi^{\mu}$ and $\psi^{m\mu}$ from equation (13); finally, the vector $v^{\mu\dot{\mu}}$ and axial vector $w^{\mu\dot{\mu}}$ are constrained as follows

$$-2 \frac{A^m}{(1+iA)^2} + \frac{e^m_{\alpha\dot{\alpha}}}{1+iA} \left[ \frac{(v+iw)^{\alpha\dot{\alpha}}}{(1+iA)^2} + \frac{i}{2} \partial_n \left( \frac{e^{n\alpha\dot{\alpha}}}{1+iA} \right) \right] =$$
$$= \epsilon^{mnpq} \partial_n a_{pq} \equiv a^m \,, \quad \partial_m a^m = 0 \,. \qquad (A.6)$$

Here one has a complex, i.e., two real antisymmetric tensors $a_{pq} \neq a^*_{pq}$. The fields $A, \xi^{\mu}, \bar{\xi}^{\mu}, i(a_{pq} - a^*_{pq})$ form a superspin 1/2 multiplet (off-shell).

The invariant integral $I_1$ (A.1) gives rise to the following action (we consider the bosonic sector only):

$$S_1 = \frac{1}{\kappa^2} \int d^4x \det{}^{-1} \tilde{e}_a{}^m [ - \frac{1}{2} \frac{1}{(1-iA)^2} R(\tilde{e}) -$$

$$- \frac{3}{(1-iA)^4(1+iA)^2} \tilde{e}^m_{a\dot{a}} \tilde{e}^{na\dot{a}} \partial_m A \partial_n A \cdot (A^2 + 1 - 2iA) + \tag{A.7}$$

$$+ 8\tilde{a}^*_{a\dot{a}} \tilde{A}^{a\dot{a}} - \frac{3}{2} \frac{(1+iA)^4}{(1-iA)} \tilde{a}_{a\dot{a}} \tilde{a}^{a\dot{a}} +$$

$$+ \frac{3}{2}(1-iA)^2 \tilde{a}^*_{a\dot{a}} \tilde{a}^{*a\dot{a}} + 3(1+iA)^2 \tilde{a}_{a\dot{a}} \tilde{a}^{*a\dot{a}} ],$$

where

$$\tilde{e}_a{}^m = \frac{1}{2}(\tilde{\sigma}_a)^{a\dot{a}} (\det e)^{-1/2} e^m_{a\dot{a}}, \qquad a^{a\dot{a}} = \tilde{e}_m^{a\dot{a}} a^m,$$

$$\tilde{A}^{a\dot{a}} = \tilde{e}_m^{a\dot{a}} A^m - \frac{i}{8} \tilde{e}^n_{\beta\dot{\beta}} \partial_n \tilde{e}^{ma\dot{\beta}} \cdot \tilde{e}_m^{\beta\dot{a}} +$$
$$+ \frac{i}{8} \tilde{e}^n_{\beta\dot{\beta}} \partial_n \tilde{e}^{m\beta\dot{a}} \cdot \tilde{e}_m^{a\dot{\beta}} .$$

Here $R(\tilde{e})$ is the usual gravitational Lagrangian, the term with $\partial_m A \partial_n A$ is the kinetic term for the pseudoscalar $A$ and $a^*A$ is the U(1) covariant coupling of the notoph to the U(1) gauge field.

The second integral $I_2$ (A.5) for the particular choice $f(U) = U^p$ produces the action

$$S_2 = \frac{1}{\kappa^2} \int d^4x \det{}^{-1} \tilde{e} \cdot \frac{1}{2} p(p+1) \frac{(1+iA)^{2p-2}}{(1-iA)^{2p+2}} \times \tag{A.8}$$

$$\times [\frac{4}{(1-iA)^2} \tilde{e}^m_{a\dot{a}} \tilde{e}^{na\dot{a}} \partial_m A \partial_n A + (1+iA)^2 (a-a^*)_{a\dot{a}} (a-a^*)^{a\dot{a}} ].$$

This is an action for the notoph multiplet alone. Combining the real parts of equations (A.7) and (A.8) one can find actions with correct relative sign of the gravitational and pseudoscalar kinetic terms.

The fermionic part is just a combination of the Rarita-Schwinger action for $\psi^{m\mu}$ and the Dirac action for $\xi^{\mu}$. Owing to the dual nature of the notoph $a^m$ (spin 1 off-shell, 0 on-shell) the notoph multiplet describes superspin 0 on-shell.

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