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SUM RULES AND PION FORM FACTOR IN QCD

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1. INTRODUCTION

The discovery that perturbation theory can be applied to study exclusive high momentum transfer processes (see, e.g., refs. [1-3] and references therein) has been an important step in the development of perturbative QCD. In particular, asymptotic freedom enables one to easily reproduce in the asymptotic $Q^2 \to \infty$ region the well-known quark counting rules for electromagnetic form factors of hadrons [4]. However, for experimentally accessible momentum transfers Q the agreement between existing theory [1-3] and experimental data for pion form factor is very poor (see, e.g., ref. [5]).

This observation, nevertheless, should not be treated as an evidence against QCD itself because the perturbative QCD approach [1-3] is applicable only for asymptotically large Q^2, and the extrapolation of the asymptotic QCD formulas into the region of moderately large Q^2 is not justified. For pion, e.g., the main $1/Q^2$-contribution in the region $Q^2 \to \infty$ is due to the hard rescattering (one-gluon exchange) subprocess (fig. 1a). However, a straightforward use of the asymptotic formalism in the $Q^2 \lesssim 10$ GeV^2 region leads to the conclusion that the mean virtuality of the gluon (fig. 1a) is much smaller than 1 GeV^2 [6]. In such a situation it is, of course, misleading to rely on asymptotic freedom. According to the standard formalism [1-3], the gluon line corresponding to virtuality smaller than some $\lambda^2 \sim 1$ GeV^2 should be absorbed by the soft pion wave function. The resulting diagram looks like that shown in fig. 1b. In the asymptotic analysis [1-3] this diagram is simply ignored because the upper estimate for its large $-Q^2$ behaviour is only $1/Q^4$ [1]. However, a complete evaluation of its contribution within the perturbative QCD approach is impossible, since the region of small quark virtualities is dominated by nonperturbative effects. Earlier one of the authors (A.R.) attempted to take these effects into account within the framework of a QCD inspired model [7]. In the present letter we outline a new approach* to exlusive processes in QCD that has

*In the course of our investigations we have learned that a similar approach was independently proposed by B.L. Ioffe and A.V. Smilga.

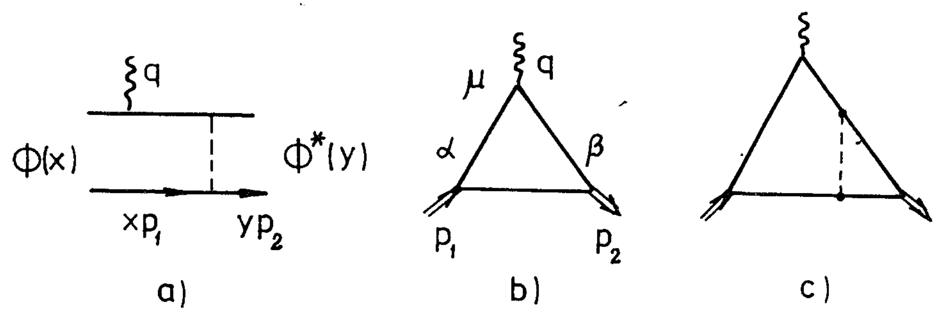

Fig. 1. Diagrams relevant to calculation of the pion form factor in QCD: a) Asymptotic perturbative QCD diagram; b) Lowest-order diagram of the QCD sum rule approach; c) One of 2-loop diagrams of the QCD sum rule approach.

much wider applicability than the asymptotic analysis [1-3]. The basic idea of the approach is the duality between quarks and hadrons: we show, in particular, that one can obtain the pion form factor by calculating the quark diagrams (the lowest-order ones are shown in figs. 1b,c) with local quark "pion" vertices and averaging the result over the appropriate duality interval.

2. DERIVATION OF SUM RULES

Technically, our analysis is based on the QCD sum rule approach [8] that has proved to be a very effective tool for studying the "static" properties of hadrons, such as masses, leptonic widths, etc. To analyse the pion form factor, we consider the three-point amplitude

$$T_{\mu\alpha\beta}(p_1, p_2) = i^2 \int e^{-ip_1x + ip_2y} \langle 0 | T\{ j^+_\alpha(x) J_\mu(0) j_\beta(y)\} | 0\rangle d^4x\, d^4y \quad (1)$$

(for notation see fig. 1b), where J_μ is the electromagnetic current and $j_\alpha = \bar{d}\gamma_5\gamma_\alpha u$ is the axial current. The latter satisfies the necessary condition that it should have nonzero projection onto the pion state $|P\rangle$

$$\langle 0 | j_\alpha(0) | P\rangle = i f_\pi P_\alpha , \quad (2)$$

where $f_\pi \simeq 133$ MeV is the pion decay constant. In principle, one may use also a pseudoscalar combination $\bar{d}\gamma_5 u$. In this case, however, there appear serious complications due to direct interactions of quarks with instantons [9,10], such interactions being absent for the axial current.

The amplitude $T_{\mu\alpha\beta}(p_1, p_2)$ is the sum of various structures, and the corresponding invariant amplitudes T_i depend on

3 variables: $p_1^2, p_2^2, q^2=(p_1-p_2)^2$. Owing to asymptotic freedom, one may calculate $T_i(p_1^2,p_2^2,q^2)$ in the deep Eucliden region p_1^2, p_2^2, $q^2 \lesssim -\lambda^2 \sim -1\ \mathrm{GeV}^2$. To extract the desired information about the form factors of physical states, we use the double dispersion relation

$$T_i(p_1^2,p_2^2,q^2) = \frac{1}{\pi^2}\int_0^\infty ds_1 \int_0^\infty ds_2 \frac{\rho_i(s_1,s_2,q^2)}{(s_1-p_1^2)(s_2-p_2^2)} + \dots \quad (3)$$

The terms not written explicitly in eq. (2) are polynomials in p_1^2 and/or p_2^2. They disappear after one applies to eq. (3) the Borel procedure B_{12}

$$\hat{B}_{12} = \lim_{n_1\to\infty} \lim_{n_2\to\infty} \left\{ \frac{(-p_1^2)^{n_1}(-p_2^2)^{n_2}}{(n_1-1)!\,(n_2-1)!} \left(\frac{d}{dp_1^2}\right)^{n_1}\left(\frac{d}{dp_2^2}\right)^{n_2} \right|_{\substack{p_1^2=n_1M_1^2,\\ p_2^2=n_2M_2^2}} \quad (4)$$

which is a straightforward generalization of that used in ref.[8] Applying B_{12} to eq. (3) gives

$$\Phi_i(M_1^2,M_2^2,Q^2) = \frac{1}{\pi^2}\int_0^\infty \frac{ds_1}{M_1^2}\int_0^\infty \frac{ds_2}{M_2^2}\,\rho_i(s_1,s_2,q^2)\exp\left\{-\frac{s_1}{M_1^2}-\frac{s_2}{M_2^2}\right\}, \quad (5)$$

where $Q^2=-q^2$ and $\Phi_i = \hat{B}_{12}T_i$.

The pion contribution into the spectral density is proportional to

$$\langle 0|j_\beta|p_2\rangle\langle p_2|J_\mu|p_1\rangle\langle p_1|j_\alpha^+|0\rangle \sim$$
$$\sim f_\pi^2 F_\pi(Q^2)\, p_1^\alpha p_2^\beta (p_1^\mu + p_2^\mu) \quad (6)$$

and the first idea is to extract from $T_{\mu\alpha\beta}$ the structure $p_1^\alpha p_2^\beta (p_1^\mu + p_2^\mu)$. However, there exist also other structures ($p_1^\beta p_2^\alpha$, $p_1^\beta p_1^\alpha$, $p_2^\beta p_2^\alpha$) that coincide with $p_1^\alpha p_2^\beta$ for $q=0$. This complication disappears if all the basic structures are expanded in $P=p_1+p_2$ and $q=p_2-p_1$. Then the relevant structure is $P_\alpha P_\beta P_\mu$ and the simplest way to extract the corresponding invariant amplitude (hereafter referred to as T) is to multiply $T_{\alpha\beta\mu}$ by $n^\alpha n^\beta n^\mu$, where n is a light-like vector orthogonal to q ($n^2=0$, $(nq)=0$, $(nP)\neq 0$).

The calculation of $\Phi \equiv B_{12}T$ is most simply performed by the Feynman parametrization or its exponential version ("α-representation", see e.g., ref.[11]). Then it is straightforward to obtain the contribution of the diagram 1b:

$$\Phi^{(1b)}(M_1^2,M_2^2,Q^2) = \frac{3}{2\pi^2(M_1^2+M_2^2)}\int_0^1 x(1-x)\exp\left\{-\frac{xQ^2}{(1-x)(M_1^2+M_2^2)}\right\}dx \times \quad (7)$$

$$\times\left[\frac{2}{3}\exp\left\{-\left(\frac{m_u^2}{1-x}+\frac{m_d^2}{x}\right)\left(\frac{1}{M_1^2}+\frac{1}{M_2^2}\right)\right\}+\frac{1}{3}\exp\left\{-\left(\frac{m_u^2}{x}+\frac{m_d^2}{1-x}\right)\left(\frac{1}{M_1^2}+\frac{1}{M_2^2}\right)\right\}\right].$$

The x-variable may be interpreted as the fraction of the total pion momentum carried by the passive quark in the infinite momentum frame. Note, that for massless quarks $\Phi^{(1b)} \sim 1/Q^4$ as $Q^2\to\infty$, in agreement with the general analysis described in refs.[1,12]. In what follows, we neglect $m_u, m_d \sim 10$ MeV $\ll\lambda$.

Notice further that eq. (5) is the (double) Laplace transformation in $1/M_i^2$. Applying the inverse transformation to eq. (7) gives the free-quark spectral function

$$\rho_0(s_1,s_2,q^2) = \frac{3(Q^2)^2}{2}\left\{\left(\frac{d}{dQ^2}\right)^2+\frac{Q^2}{3}\left(\frac{d}{dQ^2}\right)^3\right\}\frac{1}{\sqrt{(s_1+s_2+Q^2)^2-4s_1s_2}}. \quad (8)$$

In the real world $\rho(s_1,s_2,q^2)$ differs, of course, from $\rho_0(s_1,s_2,q^2)$. The difference is most pronounced for small s_1, s_2. In particular, $\rho(s_1,s_2,q^2)$ contains the pion term

$$\rho_{\pi\pi}(s_1,s_2,q^2) = \pi^2 f_\pi^2 F_\pi(Q^2)\,\delta(s_1-m_\pi^2)\,\delta(s_2-m_\pi^2). \quad (9)$$

Furthermore, $\rho(s_1,s_2,q^2)$ vanishes below the 3π-threshold (i.e., in the region $m_\pi^2 < (s_1,s_2) < 9m_\pi^2$), and only in the region, where both s_1 and s_2 are sufficiently large, ρ is close to ρ_0. This means that $\Phi(M_1^2,M_2^2,Q^2)$ also differs from the free-quark value (7). As it was argued in ref.[8], nonperturbative corrections $(1/M^2)^N$ are much more important than the perturbative ones*. Taking into account the contributions proportional to $(\alpha_s/\pi)\langle G^a_{\mu\nu}G^a_{\mu\nu}\rangle$ and $\alpha_s\langle\bar{q}q\rangle^2$ (cf. [8]) and using eqs. (5), (9), we arrive at the following representation for the pion form factor

$$\begin{aligned} f_\pi^2 F_\pi(Q^2) &= \frac{1}{\pi^2}\int_0^{s_0} ds_1\int_0^{s_0} ds_2\,\rho_0(s_1,s_2,q^2)\exp\left\{-\frac{s_1+s_2}{M^2}\right\} + \\ &+ \frac{\alpha_s\langle G^a_{\mu\nu}G^a_{\mu\nu}\rangle}{12\pi M^2} + \frac{176\pi\alpha_s\langle\bar{q}q\rangle^2}{81M^4}\left(1-\frac{2}{11}\frac{Q^2}{M^2}\right) - \\ &- \Delta(s_0,M^2,Q^2) + 0(1/M^6) + 0(\alpha_s(s_0)/\pi), \end{aligned} \quad (10)$$

*To treat initial and final states on equal footing and to simplify the calculation, we take henceforth $M_1=M_2=M$.

where $\Delta(s_0, M^2, Q^2) \equiv B(M^2,Q^2) - B_0(s_0, M^2, Q^2)$ is the difference between the "true" background contribution $B(M^2,Q^2)$

$$B(M^2,Q^2) = \frac{1}{\pi^2} \int_{\max\{s_1,s_2\}>9m_\pi^2}^{\infty} ds_1 \int^{\infty} ds_2\, \rho(s_1,s_2,q^2) \exp\{-\frac{s_1+s_2}{M^2}\} \tag{11}$$

and its free-quark analogue $B_0(s_0,M^2,Q^2)$

$$B_0(s_0,M^2,Q^2) = \frac{1}{\pi^2} \int_{\max\{s_1,s_2\}>s_0}^{\infty} ds_1 \int^{\infty} ds_2\, \rho_0(s_1,s_2,q^2) \exp\{-\frac{s_1+s_2}{M^2}\}. \tag{12}$$

3. QUARK-HADRON DUALITY

To get rid of power corrections in eq. (10), one should take $M^2 \to \infty$. Furthermore, for any fixed $Q^2 = Q_0^2$ the Δ-term can be also eliminated by an appropriate choice of the s_0-parameter. As a result, one obtains the following representation for $F_\pi(Q_0^2)$:

$$f_\pi^2 F_\pi(Q_0^2) = \frac{1}{\pi^2} \int_0^{s_0} ds_1 \int_0^{s_0} ds_2\, \rho_0(s_1,s_2, q^2 = -Q_0^2) \tag{13}$$

that is nothing else but a (generalized) finite-energy sum rule (FESR, cf.[13-15]) or a duality relation between the resonance (pion) and free-quark contributions. The functions $\rho_{\pi\pi}(s_1, s_2, q^2)$ and $\rho_0(s_1,s_2,q^2)$ are quite different but if one averages them over the relevant duality interval, the result is the same in both cases. The parameter s_0 can be interpreted then as the boundary between the pion duality interval and the next one related to the A_1-resonance. In such an interpretation s_0 is the same number (for all Q^2 values) at the midpoint of the interval between $m_\pi^2 \simeq 0$ and $m_{A_1}^2 \simeq 1.2$ GeV^2. This gives the estimate $s_0 \simeq 0.6$ GeV^2 [15].

Within the QCD sum rule approach[8] s_0 is not a free parameter. Rather, it is determined by the magnitude of the quark and gluon condensates $\langle\bar{q}q\rangle$ and $\langle GG\rangle$*. To extract s_0 from the sum rule (10), we analysed the M^2-dependence of its r.h.s. for a chosen Q_0^2 value and various s_0-values (taking $\Delta(s_0,M^2,Q^2) = 0$, as argued above). It is easy to establish (see fig.2) that for sufficiently large M^2 our theoretical "prediction" for $F_\pi(Q^2)$ has a very weak dependence on the auxiliary

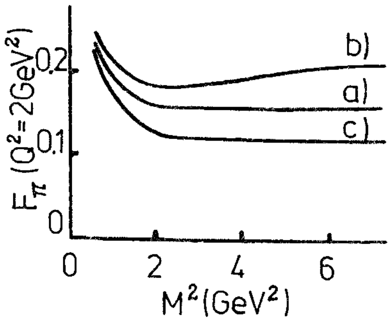

Fig.2. Typical dependence of $F_\pi(Q_0^2=2$ GeV$^2)$ on the auxiliary parameter M^2: a) $s_0=0.7$ GeV^2; b) $s_0=0.8$ GeV^2; c) $s_0=0.6$ GeV^2.

("unphysical", cf.[16]) parameter M^2, but the onset of the asymptotic regime strongly depends on s_0. The "true" value of M^2 is evidently that for which the region of weak sensitivity to variations of M^2 is the largest one. For $Q_0^2=1,2$ and 3 GeV^2 this criterion gives the same value $s_0 = 0.7$ GeV^2. Furthermore, to check the self-consistency of the whole approach, we estimated the s_0-parameter also from the analysis of the two-point amplitude[4] related to $\langle T(j_\beta j_\alpha^+)\rangle$ and obtained the same result $s_0 = 0.7$ GeV^2. The accuracy of the method, however, should not be overestimated: it is not better than 10-20%.

4. DISCUSSION OF RESULTS

Using the explicit form of eq. (8), one can reduce eq. (13) to

$$F_\pi(Q^2) = \frac{s_0}{4\pi^2 f_\pi^2}\{1 - \frac{1+6s_0/Q^2}{(1+4s_0/Q^2)^{3/2}}\}. \tag{14}$$

Another formula can be obtained, if one substitutes the integration over the square $(0 \le s_1 \le s_0;\, 0 \le s_2 \le s_0)$ in eq. (13) by integration over the triangle $(0 \le s_1+s_2 \le S_0 = \sqrt{2}\, s_0)$ of equivalent area. This gives

$$F_\pi^{(TR)}(Q^2) = \frac{S_0}{8\pi^2 f_\pi^2 (1+Q^2/2S_0)^2}. \tag{15}$$

However arbitrary this substitution may seem, for $Q^2 \ge 0.2$ GeV^2 eq. (15) reproduces eq. (14) with better than 10% accuracy. So, eq. (15) is very convenient for a quick estimation of eq. (14) predictions.

Before comparing eq. (14) with experimental data, we want to stress that our analysis is reliable only in the $Q^2 \ge 1$ GeV^2

*In numerical estimates we use the values $(\alpha_s/\pi)\langle G_{\mu\nu}^a G_{\mu\nu}^a\rangle = 0.012$ GeV^4 and $\alpha_s\langle\bar{q}q\rangle^2 = 1.83\cdot 10^{-4}$ GeV^6 taken from ref.[8].

*This amplitude was analysed first by SVZ in ref.[8] using a slightly different method. In their fits SVZ used $s_0 = 0.75$ GeV^2.

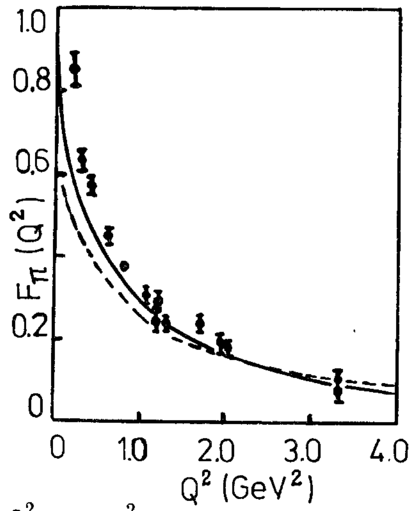

Fig.3. Comparison between experimental data (taken from ref.[17]) and our theoretical predictions based on eq. (10): $M^2 = \infty$ (solid line); $M^2 = 1.8$ GeV^2 (broken line).

region where the asymptotic freedom guarantees the absence of large corrections (of, say, $1/Q^2$ type). One should also remember that there exists some freedom in choosing s_0: $s_0 = 0.7 \pm 0.1$ GeV^2. Our curve corresponding to eq. (14) for $s_0 = 0.7$ GeV^2 is shown in fig.3 (solid line). In the region $Q^2 > 1$ GeV^2 it is in excellent agreement with the existing data[17].

It is also interesting to analyse the predictions for $F_\pi(Q^2)$ resulting from the basic sum rule (10) for finite M^2. As argued in ref.[8], one should choose M^2 so as to reduce to 30% both the power corrections (that blow up as $M^2 \to 0$) and the background contribution $B(M^2, Q^2)$) (that grows as $M^2 \to \infty$). For two-point functions considered by SVZ[8] such a compromise is really possible for $M^2 \simeq s_0$. Alas, this strategy fails for eq. (10): to reduce the power corrections to 30% at the reference momentum $Q_0^2 = 2$ GeV^2, one should choose M^2 in the region $M^2 \gtrsim 1.8$ GeV^2, where $B(M^2, Q^2)$ yields more than 70% compared to the total sum. Moreover, since the first term in the r.h.s. of eq. (10) rapidly diminishes with growing Q^2, the power corrections (that are almost constant for $Q^2 \lesssim 6$ GeV^2) exceed the 30% boundary just above the reference point. If one takes $M^2 = 1.8$ GeV^2, the power corrections reach 100% for $Q^2 = 6$ GeV^2. This means that for sufficiently large Q^2 one looses the control over the $1/M^2$-expansion. There exists a simple explanation of this phenomenon. Note, that the main contribution into $\Phi^{(1b)}$ (eq. (7)) for large Q^2 gives the region $x \sim M^2/Q^2$, where the passive quark has the virtuality $k^2 \sim M^4/Q^2$. Thus, the region $Q^2 \gtrsim M^4/m_\rho^2$ is in fact beyond the scope of the asymptotic freedom. Of course, taking M^2 sufficiently large, one can control the power corrections for arbitrarily high Q^2 values. The most radical way out is to take $M^2 = \infty$. Then the power corrections are absent altogether. However, this is achieved at the price of the (infinite) growth of the $B(M^2, Q^2)$-term. Still, it is the difference $\Delta = B - B_0$ that matters. As argued

above, the choice $M^2 = \infty$ reduces the whole M^2-busyness to finding the appropriate "effective threshold" s_0. It is worth noting also that the finite-M^2 results for $F_\pi(Q^2)$ in the Q^2 region where power corrections are under control, do agree with the $M^2 = \infty$ result, eq. (14) (see fig.3).

5. CONCLUDING REMARKS

Thus, the one-gluon-exchange contribution for available Q^2 is of litte importance. However, the situation changes drastically in the asymptotic $Q^2 \to \infty$ region, because fig. 1c gives in this region $1/Q^2$-contribution (corresponding to quark counting rules[4] and asymptotic QCD analysis[1-3]) whereas eq. (14) behaves asymptotically as $1/Q^4$. Thus, extrapolating the asymptotic QCD formulas[1-3] into the region of moderately large Q^2 one should not expect for a good description of experimental data, and vice versa: good fits of existing data extrapolated into the large-Q^2 region may be irrelevant to the "true" asymptotic behaviour of $F_\pi(Q^2)$. In particular, as highly encouraging we consider the result obtained by Dubnicka, Dubnickova and Meshcheryakov[18]. Analysing all data on $F_\pi(Q^2)$ (both in spacelike and timelike regions) and taking properly into account the analytic properties of $F_\pi(Q^2)$, they obtained as a best fit the curve that behaves just like $1/Q^4$ in the asymptotic region.

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Нестеренко В.А., Радюшкин А.В. Правила сумм и формфактор пиона в квантовой хромодинамике E2-82-204

Излагается подход к исследованию электромагнитного формфактора пиона в квантовой хромодинамике, основанный на использовании правил сумм в КХД. Полученная теоретическая кривая для $F_\pi(Q^2)$ находится в хорошем согласии с имеющимися экспериментальными данными.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1982

Nesterenko V.A., Radyushkin A.V. Sum Rules and Pion Form Factor in QCD E2-82-204

We propose an approach to the investigation of the pion electromagnetic form factor in QCD based on the systematic use of the QCD sum rule technique. The theoretical curve obtained for $F_\pi(Q^2)$ is in good agreement with existing experimental data.

The investigation has been performed at the Laboratory of the Theoretical Physics, JINR.

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