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PROPERTIES OF THE ALGEBRAS $L^{+}(D)$

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PROPERTIES OF THE ALGEBRAS $L^+(D)$

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Свойства алгебр $\mathfrak{L}^{+}(\mathfrak{D})$

Рассматриваются свойства алгебры всех операторов, которые вместе со своими сопряженными операторами отображают в себе данное линейное подмножество гильбертового пространства. Каждый автоморфизм и каждая производная этой алгебры являются внутренними. Их можно определить алгебраическим образом.

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Properties of the Algebras $\mathfrak{L}^{+}(\mathfrak{D})$

We consider properties of the algebra of all operators which together with its abjoints transform a given dense linear manifold of an Hilbert space into itself. This algebra admits inner *-automorphisms and derivations only and there is an algebraic characterisation of this algebra.

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1. Definitions, results.

Let $\mathcal{D}$ be a dense linear submanifold of the Hilbert space $\mathcal{H}$. With $\mathcal{L}^+(\mathcal{D})$ we denote the set of all such linear operators a from $\mathcal{D}$ into $\mathcal{D}$, $a\mathcal{D} \subseteq \mathcal{D}$, for which $\mathcal{D}$ is in the domain of definition of a^* and $a^*\mathcal{D} \subseteq \mathcal{D}$. $\mathcal{L}^+(\mathcal{D})$ is an algebra with respect of the ordinary addition and multiplication of operators. $\mathcal{L}^+(\mathcal{D})$ becomes a *-algebra by the involution $a \to a^+$, where a^+ is defined to be the restriction of a^* onto $\mathcal{D}$.

We shall prove the following theorems:

Theorem 1: Let τ be a *-isomorphism from $\mathcal{L}^+(\mathcal{D}_1)$ onto $\mathcal{L}^+(\mathcal{D}_2)$. Then there exists a unitary map u from $\mathcal{D}_1$ onto $\mathcal{D}_2$

(1) $$u\mathcal{D}_1 = \mathcal{D}_2$$

with

(2) $$\tau(a) = u a u^{-1} \quad \text{for all } a \in \mathcal{L}^+(\mathcal{D}).$$

Theorem 2: Every *-automorphism τ of $\mathcal{L}^+(\mathcal{D})$ is an inner one, i.e., there is a unitary element $u \in \mathcal{L}^+(\mathcal{D})$ with $\tau(a) = u a u^{-1}$ for all $a \in \mathcal{L}^+(\mathcal{D})$.

Theorem 2 is an obvious consequence of theorem 1. Note that these theorems suggest the existence of a "space-free" definition of $\mathcal{L}^+(\mathcal{D})$ (theorems 4 - 6).

Let us now remind that a derivation of $\mathcal{L}^+(\mathcal{D})$ is a linear map of $\mathcal{L}^+(\mathcal{D})$ into itself satisfying

(3) $$\varphi(ab) = \varphi(a)\cdot b + a\cdot\varphi(b).$$

Theorem 3 (P.Krüger): Is φ a derivation of $\mathcal{L}^+(\mathcal{D})$, then there exists an element $x \in \mathcal{L}^+(\mathcal{D})$ with

(4) $$\varphi(a) = xa - ax.$$

Hence every derivation is an inner one. [1]

One knows [2] that $\mathcal{L}^+(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space, is the von Neumann algebra of all bounded operators. Von Neumann has proved that every left ideal of this algebra is generated by a projection, i.e., an operator p with $p = p^* = p^2$ (see for instance [3]). The technique of this proof also works in the more general case of the $\mathcal{L}^+(\mathcal{D})$ algebras. We now explain shortly, how one can use these techniques to characterise the algebras $\mathcal{L}^+(\mathcal{D})$ abstractly.

<u>Definition 1:</u> Let $\mathcal{A}$ be a $*$-algebra. $\mathcal{A}$ is called an algebra with "property I" if and only if

(i) every proper left ideal contains a minimal left ideal,

(ii) every minimal left ideal is generated by a minimal projection, and

(iii) every element of every subalgebra $\mathcal{A}_o$, which contains an identity e_o, has a non-empty spectrum.

Let us first add some remarks. A projector p is minimal in $\mathcal{A}$ iff $p \neq 0$ and $pq = qp$ implies $pq = p$ for every projector q of $\mathcal{A}$. If $\mathcal{A}_o$ is an algebra with identity e_o, then the spectrum of one of its elements a is the set of all complex numbers λ such, that $(a - \lambda e_o)^{-1}$ does not exist in $\mathcal{A}_o$.

We now construct an example of a $*$-algebra with property I. Let T be an index set (an abstract set) and assume to be associated to every $t \in T$ an algebra $\mathcal{L}^+(\mathcal{D}_t)$. Then the $*$-algebra

$$\prod_{t \in T} \mathcal{L}^+(\mathcal{D}_t) \equiv \mathcal{L}^+(\mathcal{D}_t, t \in T) \tag{5}$$

consists of all functions $t \to x(t)$ defined on T with $x(t) \in \mathcal{L}^+(\mathcal{D}_t)$ together with the composition laws

$$(x_1 + x_2)(t) = x_1(t) + x_2(t)\ , \quad (x_1 x_2)(t) = x_1(t)\, x_2(t)\ ,$$
$$(x^+)(t) = x(t)^+ \ , \quad (\lambda x)(t) = \lambda x(t)$$

This construction provides us with a $*$-algebra.

<u>Theorem 4:</u> $\mathcal{L}^+(\mathcal{D}_t, t \in T)$ satisfies property I.

<u>Theorem 5:</u> Let $\mathcal{A}$ be a $*$-algebra with property I. Then there exists up to $*$-isomorphisms one and only one algebra $\mathcal{L}^+(\mathcal{D}_t, t \in T)$ and a $*$-isomorphism τ of $\mathcal{A}$ into $\mathcal{L}^+(\mathcal{D}_t, t \in T)$ which maps the set of all minimal projectors of $\mathcal{A}$ <u>onto</u> the set of all minimal projectors of $\mathcal{L}^+(\mathcal{D}_t, t \in T)$.

<u>Definition 2:</u> A $*$-algebra is called a "type I_d algebra" if the following two conditions are fullfilled:

1) $\mathcal{A}$ has property I

2) Let τ be a $*$-isomorphism from $\mathcal{A}$ into a $*$-algebra $\mathcal{B}$ with property I. If τ maps the set of all minimal projectors of $\mathcal{A}$ onto the set of all minimal projectors of $\mathcal{B}$, then τ maps $\mathcal{A}$ <u>onto</u> $\mathcal{B}$.

<u>Theorem 6:</u> A $*$-algebra is a type I_d algebra if and only if it is $*$-isomorph to a certain algebra $\mathcal{L}^+(\mathcal{D}_t, t \in T)$.

According to theorem 6 the centre of a type I_d algebra is a discrete one, i.e., it is generated by its own minimal projectors. Especially, a type I_d algebra, which is to an algebra of bounded operators isomorphic, is a W^*-algebra with discrete centre.

2. Algebras with property I.

To prove the theorems we need some further insight in the considered class of algebras.

Theorem 7: For every *-algebra with property I the following statements are true:

1) If p is a minimal projector, then there exists a positive linear form f with

$$(6) \qquad p a p = f(a)\cdot p \qquad \text{for all } a \in \mathcal{A}$$

2) If $\mathcal{A}$ contains only one minimal projector p_0, then p_0 is the identity element of $\mathcal{A}$ and $\mathcal{A}$ is isomorphic to the algebra of complex numbers.

We beginn with the second assertion. For every non-zero $a \in \mathcal{A}$ the left ideal $\mathcal{A}a$ contains a minimal projector p_0. The case $\mathcal{A}a = 0$ can be excluded, because in this situation a and the zero form a left ideal, that has to contain a minimal projector and this is impossible. Now there is an element a' with $a = a'p_0$ and thus $(a-a')p_0 = 0$. By the same reasoning $a - a' = b p_0$ and from $p_0^2 = p_0$ it follows $a = a'$. So we see $a p_0 = a$, $p_0 a^* = a^*$ for all $a \in \mathcal{A}$ and p_0 is the identity of $\mathcal{A}$. For every $a \in \mathcal{A}$ there should be a complex number λ such that $a - \lambda p_0$ is not inversible. It follows $a = \lambda p_0$ because otherwise $\mathcal{A}(a - \lambda p_0) \ni p_0$ wich contradicts the assumption that λ belongs to the spectrum of a. The second assertion of the theorem is now available and the first assertion becomes obvious: The subalgebra $p\mathcal{A}p = \mathcal{A}_0$, where p is a minimal projector of $\mathcal{A}$, has to satisfy property I too. In virtue of the minimality of p, no projector different from p is in $\mathcal{A}_0$. Therefore, $\mathcal{A}_0$ is isomorphic to the algebra of complex numbers and $p a p = f(a) p$ with some number $f(a)$. Clearly, f depends linearly on a and

$p a^* a p = f \cdot p$ has to be a positive element of $\mathcal{A}$. Hence f is a positive linear form.

The property (6) is an essential characteristicum of minimal projectors for property I algebras. This shows

Theorem 8: Let $\mathcal{A}$ be a *-algebra. Denote by $\mathfrak{M}(\mathcal{A})$ the set of all such projectors p of $\mathcal{A}$ for which (6) is fulfilled with a certain linear form f.

$\mathcal{A}$ has property I if and only if

$$p a p = 0 \qquad \text{for all } p \in \mathfrak{M}(\mathcal{A})$$

implies $a = 0$ in $\mathcal{A}$.

The proof proceeds in two steps. Firstly we need

Lemma 1: $\mathfrak{M}(\mathcal{A})$ consists of minimal projectors of $\mathcal{A}$.

From $p a p = f(a) p$ for all $a \in \mathcal{A}$ and $f(b^* b) \neq 0$ we have

$$(7) \qquad q = b p b^* / f(b^* b) \in \mathfrak{M}(\mathcal{A})$$

and

$$(8) \qquad q a q = \frac{f(b^* a b)}{f(b^* b)} \cdot q .$$

We see this in the following way: $p \in \mathfrak{M}(\mathcal{A})$ and $p\tilde{q} = \tilde{q}$ implies $f(\tilde{q}) p = p\tilde{q}p = \tilde{q}p = \tilde{q}$ for projectors $\tilde{q}$ and thus $p = \tilde{q}$. Therefore $\mathfrak{M}(\mathcal{A})$ consists of minimal projectors only. The other part of the lemma is a straight-forward application of equ. (6).

We can now be sure that $\mathfrak{M}(\mathcal{A})$ consists of all minimal projectors if $\mathcal{A}$ has property I. In this case $\mathcal{A}a \ni \mathcal{A}p$ with a certain $p \in \mathfrak{M}(\mathcal{A})$ for a given $a \neq 0$ and we get $ba = p$. Now $f(pba) \neq 0$ implies by positivity $f(b^* p b) \neq 0$ and we obtain $b^* p b a b^* p b = b^* a b \neq 0$. According to lemma 1 it is $q = \lambda b^* p b \in \mathfrak{M}$ with some λ and $q a q \neq 0$. To prove the other part of the theorem 8 we choose an element $a \neq 0$ out of a given left ideal $\mathfrak{J}$. According to the assumption we can find $p \in \mathfrak{M}$ with $p a p \neq 0$. By (6) one shows $f(a) \neq 0$ and the positivity of f implies $\lambda^{-1} = f(aa^*) \neq 0$. Now $q = \lambda a^* p a \in \mathcal{J} \cap \mathcal{M}$ shows that $\mathcal{J}$ contains the minimal subideal $\mathcal{A}q$ and theorem 8 is proved.

As a consequence of theorem 8, every *-algebra with property I is a reduced one [3].

Theorem 8 implies theorem 4 in virtue of

<u>Lemma 2:</u> Let $\mathcal{A} = \mathcal{L}^+(\mathcal{D}_t, t \in T)$. For every $\xi_t \in \mathcal{D}_t$, $\langle \xi_t, \xi_t \rangle = 1$ the element

$$(px)(t') = 0 \;, \quad t \neq t'$$
$$(px)(t)\eta_t = \langle \xi_t, \eta_t \rangle \xi_t \;, \quad \eta_t \in \mathcal{D}_t$$

is a minimal projector and there are no other minimal projectors in $\mathcal{A}$.

Indeed, every projector q of $\mathcal{A}$ defines new projectors by $q(t) = q_t(t)$, $q_t(t') = 0$ for $t \neq t'$. q_t is smaller than q and if q was minimal and $q_t \neq 0$ then $q = q_t$. One sees that q_t projects $\mathcal{D}_t$ onto a one-dimensional subspace of $\mathcal{D}_t$ provided q_t is a minimal projector. On the other hand, every one-dimensional subspace of $\mathcal{D}_t$ defines its projector and this projector is a **minimal** one.

Let us mention two further properties of $\mathcal{L}^+(\mathcal{D}_t, t \in T)$. For every pair of projectors $p, q \in \mathcal{M}$ we distinguish two possibilities: Either they project into the same or in different $\mathcal{D}_t$. Let us denote by $\mathcal{M}_t$ the set of all minimal projectors that are defined according to lemma 2 by the subspaces of $\mathcal{D}_t$. Then $\mathcal{M}$ is the union of the $\mathcal{M}_t$, $t \in T$ and $\mathcal{M}_t \cap \mathcal{M}_{t'}$ is empty for $t \neq t'$. **One immediately sees that two projectors** belong to the same $\mathcal{M}_t$ if and only if there is an a

with $paq \neq 0$. Of course, the later condition can be extended to an arbitrary property I algebra, the proof of this fact is evident.

<u>Lemma 3:</u> Let $\mathcal{A}$ be a *-algebra with property I. There is an index set T and a decomposition of $\mathcal{M}(\mathcal{A})$ in disjunct sets $\mathcal{M}_t(\mathcal{A})$, $t \in T$ such, that $q, p \in \mathcal{M}(\mathcal{A})$ belong to the same t if and only if there is an $a \in \mathcal{A}$ with $paq \neq 0$.

Now suppose $qbp \neq 0$ for $q, p \in \mathcal{M}_t(\mathcal{A})$. The element $d = qb$ satisfies $dpd^* = qbpb^*q = \lambda q$ and $\lambda \neq 0$, for $\mathcal{A}$ is reduced and $\lambda q = (qbp)(qbp)^*$. This gives

<u>Lemma 4:</u> $p, q \in \mathcal{M}_t(\mathcal{A})$ if and only if there is a positive linear form f and an element $b \in \mathcal{A}$ such, that equ. (7) and (8) are valid.

3. Representations.

Let

(9) $\quad \tau : \quad a \to \tau(a) \;, \quad a \in \mathcal{A}$

be a *-representation of the *-algebra $\mathcal{A}$ with domain of definition $\mathcal{D}_\tau$. If $q \in \mathcal{M}(\mathcal{A})$ and $\tau(q) \neq 0$, then the functional g defined by $qaq = g(a)q$ is a vector state of τ. Indeed, for $\Phi \in \mathcal{D}_\tau$ and $\Psi = \tau(q)\Phi \neq 0$ we have $\langle \Psi, \tau(a)\Psi \rangle = g(a)\langle \Psi, \Psi \rangle$. If now (7) and (8) is valid for the projector $p \in \mathcal{M}(\mathcal{A})$, we conclude $\tau(p) \neq 0$ and with f as defined by (6) we have $\langle \Psi', \tau(a)\Psi' \rangle = f(a)\langle \Psi', \Psi' \rangle$ with a vector $\Psi' = \tau(p)\Psi'$. Now $\tau(p)$ is a projector and hence

$$g(p)\langle \Psi, \Psi \rangle = \langle \Psi, \tau(p)\Psi \rangle \geq \frac{|\langle \Psi, \tau(p)\Phi_0 \rangle|^2}{\langle \Phi_0, \Phi_0 \rangle}$$

for all Φ. Setting $\Phi = \Psi'$ we get

$$g(p) \geq |\langle \Psi, \Psi' \rangle|^2 / \langle \Psi, \Psi \rangle \langle \Psi', \Psi' \rangle$$

and the equality sign holds for $\Psi' = \tau(p)\Psi$.

Theorem 9: For any $p, q \in \mathfrak{M}(\mathcal{A})$ and

(10) $\quad p a p = f(a) p \quad , \quad q a q = g(a) q \quad , \quad a \in \mathcal{A}$

every *-representation τ of $\mathcal{A}$ with $\tau(p) \neq 0$ satisfies

(11) $\quad g(p) = f(q) = \sup \dfrac{|\langle \Psi, \Psi' \rangle|^2}{\langle \Psi, \Psi \rangle \langle \Psi', \Psi' \rangle}$

where the supremum runs over all $\Psi, \Psi' \in \mathcal{D}_\tau$ with the restriction

(12) $\quad \tau(p)\Psi = \Psi \quad , \quad \tau(q)\Psi' = \Psi'$

We are now in the position to show theorem 5. Let $\mathcal{A}$ be a *-algebra with property I. With T we denote the index set given by lemma 3. For every $t \in T$ we choose $p_t \in \mathfrak{M}_t(\mathcal{A})$ and define f_t by $p_t a p_t = f_t(a) p_t$. Let us now perform the GNS-representation τ_t of $\mathcal{A}$ determined by f_t with domain of definition $\mathcal{D}_t$ and cyclic vector $\Phi_t \in \mathcal{D}_t$, $f_t(a) = \langle \Phi_t, \tau_t(a) \Phi_t \rangle$. It is $\tau_t(p_t)\Phi_t = \Phi_t$. If for some $\Phi \in \mathcal{D}_t$ we have $\tau_t(p_t)\Phi = \Phi$, then $\tau_t(p_t a)\tau_t(p_t)\Phi = \tau_t(p a)\Phi$ and with the help of (6) we find Φ depending linearly on Φ_t. This shows that $\tau_t(p_t)$ is a one-dimensional projector. The same conclusion can be drawn for every $\tau_t(q)$ with $q \in \mathfrak{M}_t(\mathcal{A})$ by similar arguments. Lemmata 1 and 4 now indicate a one-to-one correspondence between $\mathfrak{M}_t(\mathcal{A})$ and the set of all one-dimensional subspaces of $\mathcal{D}_t$.

Hence the vectors (12) form one-dimensional spaces and equ. (12) is valid without performing the operation "sup" ! We construct the direct sum τ of the representations τ_t, $t \in T$, and the result is a *-isomorphism of $\mathcal{A}$ into $\mathcal{L}^+(\mathcal{D}_t, t \in T)$ with properties required by theorem 5.

We consider now a second *-representation $\tilde{\tau}$ into $\mathcal{L}^+(\tilde{\mathcal{D}}_t, t \in T)$ with the same properties. Then the one-dimensional subspaces of $\mathcal{D}_t$ and $\tilde{\mathcal{D}}_t$ are given by $\tau_t(p_t)\mathcal{D}_t$ and $\tilde{\tau}(p_t)\tilde{\mathcal{D}}_t$ and there is a one-to-one correspondence

(13) $\quad \tilde{\tau}(p_t)\tilde{\mathcal{D}}_t \quad \leftrightarrow \quad \tau(p_t)\mathcal{D}_t$

As proved above, the transition probabilities between one-dimensional subspaces remain unchanged by the mapping (13). Applying a theorem of Wigner [4] there is a unitary or anti-unitary one-to-one mapping u_t from $\mathcal{D}_t$ onto $\tilde{\mathcal{D}}_t$ with

(14) $\quad \tilde{\tau}(p_t) u_t = u_t \tau(p_t)$

Considering now with the help of (14)the validity of

$$\tilde{\tau}(q)\{\tilde{\tau}(a)u_t - u_t\tau(a)\}\tau(q) = \{\tilde{\tau}(q a q)u_t - u_t\tau(q a q)\} = 0$$

for every minimal projector q we get

(15) $\quad u^{-1}\tilde{\tau}(a)u = \tau(a) \quad , \quad u = \sum u_t$

Applying this to ia too, one proves linearity of u.

By this way we have not only proved theorem 5 but also a generalisation of theorem 1. Indeed, let $\mathcal{A} = \mathcal{L}^+(\mathcal{D}_t, t \in T)$, τ the identic automorphism and $\tilde{\tau}$ a *-isomorphism onto $\mathcal{L}^+(\tilde{\mathcal{D}}_t, t \in T)$. There is a unitary map u of the direct sum of all $\mathcal{D}_t$ onto the direct sum of all $\tilde{\mathcal{D}}_t$ which impliments $\tilde{\tau}$.

The last part of the proof of theorem 5 contains the following statement:

<u>Theorem 10:</u> Let τ be a *-isomorphism of $\mathcal{L}^+(\mathcal{D}_t, t\in T)$ onto $\mathcal{L}^+(\tilde{\mathcal{D}}_{t'}, t'\in\tilde{T})$. Then there exists a unitary map u from $\sum \mathcal{D}_t, t\in T$ onto $\sum \tilde{\mathcal{D}}_{t'}, t'\in\tilde{T}$ and a map j from T onto $\tilde{T}$ with

$$u\,\mathcal{D}_t = \tilde{\mathcal{D}}_{j(t)}$$

and

$$\tau(a) = u\,a\,u^{-1}, \quad a\in\mathcal{L}^+(\mathcal{D}_t, t\in T).$$

Theorem 10 implies the theorems 1 and 2 and shows how to prove theorem 6: We have to consider an imbedding

$$\mathcal{A} = \mathcal{L}^+(\mathcal{D}_t, t\in T) \leq \mathcal{B} \qquad \text{with } \mathfrak{M}(\mathcal{A}) = \mathfrak{M}(\mathcal{B}).$$

Theorem 5 tells us, that we need to consider the case

$$\mathcal{A} = \mathcal{L}^+(\mathcal{D}_t, t\in T) \leq \mathcal{L}^+(\tilde{\mathcal{D}}_t, t\in T) = \mathcal{B}, \quad \mathfrak{M}(\mathcal{A}) = \mathfrak{M}(\mathcal{B})$$

only. Further, $\mathcal{A}$ and $\mathcal{B}$ have to be *-isomorph (theorem 5) and hence there is a *-isomorphism from $\mathcal{B}$ onto $\mathcal{A}$, i.e., into $\mathcal{B}$ that leaves stable the set of all minimal projectors as a whole. This *-isomorphism has therefore to be an *-automorphism and it follows $\mathcal{B} = \mathcal{A}$.

<u>4. Proof of theorem 3.</u>

Let φ be a derivation of $\mathcal{L}^+(\mathcal{D})$. Using an idea of P.Krüger we construct the element x of eq. (4) explicitely. For any two vectors ξ, η of $\mathcal{D}$ we define $P_{\xi,\eta}$ by

$$(P_{\xi,\eta})\eta = \xi, \quad (P_{\xi,\eta})\eta' = 0 \quad \text{for all } \eta'\perp\eta\,!$$

Now $\xi \to P_{\xi,\eta}$ is a linear map of $\mathcal{D}$ into $\mathcal{L}^+(\mathcal{D})$ and we have $a P_{\xi,\eta} = P_{a\xi,\eta}$ for all $a\in\mathcal{L}^+(\mathcal{D})$.

Now we define

$$x\eta = \varphi(P_{\eta,\xi})\,\xi$$

and get a linear map $\eta\to x\eta$ from $\mathcal{D}$ into $\mathcal{D}$. Now

$$\varphi_1(a) = xa - ax, \quad a\in\mathcal{L}^+(\mathcal{D})$$

is a map of $\mathcal{D}$ into $\mathcal{D}$ for every $a\in\mathcal{L}^+(\mathcal{D})$ and

$$\varphi_1(a)\eta = \varphi(P_{a\eta,\xi})\xi - a\varphi(P_{\eta,\xi})\xi = \{\varphi(aP_{\eta,\xi}) - a\varphi(P_{\eta,\xi})\}\,\xi$$

shows that

$$\varphi_1(a)\eta = \varphi(a)P_{\eta,\xi}\,\xi = \varphi(a)\eta.$$

Hence $\varphi_1 = \varphi$, Substituting $a = P_{\eta,\xi}$ we get $\langle\xi, x\xi\rangle = 0$.

Next we consider $\psi(a) = \varphi(a^*)^*$. ψ is again a derivation and we construct as above $y\eta = \psi(P^*_{\eta,\xi})^*\xi$ so that $\psi(a) = [y, a]$ and

$$\langle [y,a]\eta_1, \eta_2\rangle = \langle \eta_1, [x, a^*]\eta_2\rangle.$$

Choosing $\eta_1 = \xi$, $a = P_{\bar{\eta},\xi}$ we obtain with $\langle\xi, x\xi\rangle = \langle\xi, y\xi\rangle = 0$

$$\langle y\bar{\eta}, \eta_2\rangle = -\langle\bar{\eta}, x\eta_2\rangle.$$

Now y maps $\mathcal{D}$ into $\mathcal{D}$ and $x^+ = -y$ so that $x, y\in\mathcal{L}^+(\mathcal{D})$ and the theorem is proved.

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