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APPROXIMATION FOR GENERATING FUNCTIONAL OF τ -FUNCTIONS BASED ON LIMIT THEOREM FOR STOCHASTIC PSEUDOPROCESSES

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1. Introduction

A structural analogy between the classical statistical mechanics and the quantum mechanics as well as the quantum field theory [1] suggests that the probabilistic concepts on which the statistical methods are based, may be transferred to the quantum domain after some modifications [2]. These are drastic in fact since they include an analytic continuation of real parameters of the, e.g., Wiener processes occuring in statistical mechanics, to the purely imaginary ones relevant for quantum mechanics. For this reason we called those objects the quantum Wiener processes or, shortly, pseudoprocesses [3]. Many results of a theory of probabilities could formally be transposed on the quantum oase by the above prooedure. The results on the so-called Euolidean quantum field theory and their rigorous extension to the Minkowski quantum field theory [4] indicate that the idea of an analytic continuation is sound enough and deserves further exploration.

In this paper we shall tray to employ the idea of the central limit theorem of a theory of probabilities [5] in the context of the relativistic pseudoprocesses in terms of which the Minkowski quantum field theory may be written. It permits us to write down the main objects of a theory as the limits of some approximate expressions which, first, could be simpler for numerical calculations and, second, could possibly give a closer fits of the experimental facts than the limiting expressions. If the latter would be time one would get a very natural realization of the correspondence principle - new expressions coincide with the old ones after some limit has been taken. We shall consider it as a working hypothesis which should be checked. For the sake of simplicity, we consider here the simplest case of single, scalar interacting field. The considerations are mainly heuristic.

2. Basic formulae and definitions

Let us consider a scalar, neutral self-interacting field with a Lagrangian

$$L_{int}[\varphi] = \int d^4x\, L_{int}[\varphi(x)] = \mathcal{J}[\varphi^2] \qquad (2.1)$$

A generating functional for the τ- functions has the form

$$\mathcal{J}[p] = N^{-1}\exp\left(i L_{int}\left[-i\frac{\delta}{\delta p}\right]\right)\exp\left(-\frac{i}{2}\, p K^{-1} p\right) \qquad (2.2)$$

$$= N^{-1}\exp\left(i\mathcal{J}\left[-2i\frac{\delta}{\delta u}\right]\right)\exp\left[-\frac{i}{2}\, p(K+u)^{-1}p\right]\exp\left[\frac{1}{2}\mathrm{Tr}\ln(K+u)^{-1} - \frac{1}{2}\mathrm{Tr}\ln K^{-1}\right]_{|u=0} . \qquad (2.3)$$

Here the constant N is determined by the normalization condition

$$\mathcal{J}[0] = 1 \qquad (2.4)$$

and K^{-1} stands for a causal Green's function

$$K^{-1} = -\Delta^c , \quad K = \Box - \varkappa^2, \quad \varkappa = \frac{mc}{\hbar} . \qquad (2.5)$$

In a course of derivation of the formula (2.3) the following identities have been used [6]

$$\exp\left(i\mathcal{J}\left[-\left(\frac{\delta}{\delta p}\right)^2\right]\right) = \exp\left(i\mathcal{J}\left[-2i\frac{\delta}{\delta u}\right]\right)\exp\left[-\frac{i}{2}\,u\cdot\left(\frac{\delta}{\delta p}\right)^2\right]_{|u=0} \qquad (2.6)$$

and

$$\exp\left[-\frac{i}{2}\,u\cdot\left(\frac{\delta}{\delta p}\right)^2\right]\exp\left(-\frac{i}{2}\, pK^{-1}p\right) = \exp\left[-\frac{i}{2}\, p(K+u)^{-1}p\right]\exp\left[\frac{1}{2}\mathrm{Tr}\ln(K+u)^{-1} - \frac{1}{2}\mathrm{Tr}\ln K^{-1}\right] . \qquad (2.7)$$

Now, using the "proper time" representations [7-10]

$$(K+u)^{-1}_{y,x} = -\frac{\lambda}{2}\int_0^\infty dt\, \exp\left(-\frac{\lambda}{2}\varkappa^2 t\right)\left[\exp\frac{\lambda t}{2}(\Box + u)\right]_{y,x} \qquad (2.8)$$

and

$$\mathrm{Tr}\ln(K+u)^{-1} - \mathrm{Tr}\ln K^{-1} = \int_0^\infty \frac{dt}{t}\exp\left(-\frac{\lambda\varkappa^2}{2}t\right)\int d^4z\left[\exp\frac{\lambda t}{2}(\Box+u) - \exp\frac{\lambda t}{2}\Box\right]_{z,z} , \qquad (2.9)$$

where $\lambda = \frac{i\hbar}{m}$,

we see that a basic role is played here by the function

$$f(t;y,x) = \left[\exp\frac{\lambda t}{2}(\Box + u)\right]_{y,x} \qquad (2.10)$$

which in addition depends functionally on $u(x)$. It solves the following Cauchy's problem

$$\left\{\partial_t - \frac{\lambda}{2}\left[\Box_y + u(y)\right]\right\} f(t;y,x) = 0$$
$$\lim_{t\downarrow 0} f(t;y,x) = \delta(y-x) \qquad (2.11)$$

which is a special case of a general Cauchy's problem

$$\left\{ \partial_s + \alpha_\mu \partial^\mu - \frac{\lambda}{2} g_{\rho\rho} \beta_\mu^\rho \beta_\nu^\rho \partial^\mu \partial^\nu + c \right\} \varphi(s,y) = 0 \tag{2.12}$$

$$\lim_{s \uparrow t} \varphi(s,y) = f(y) .$$

Its solution may be written in a form analogous to the Feynman-Kac formula

$$\varphi(s,y) = Q\left\{ f \circ x(t;s,y) \exp \int_s^t c[\tau, x(\tau;s,y)] d\tau \right\} , \tag{2.13}$$

where the pseudoprocess $x(t;s,y)$ solves the following integral stochastic equations

$$x_\mu(t;s,y) = y_\mu + \int_s^t \alpha_\mu[\tau, x(\tau;s,y)] d\tau + \int_s^t \beta_\mu^\nu[\tau, x(\tau;s,y)] dz_\nu(\tau) \tag{2.14}$$

$\mu = 0,1,2,3$.

The basic pseudoprocesses $z_\mu(t)$ are independent Wiener pseudoprocesses characterized by the requirements

$$Q\{z_\mu(t)\} = 0 , \quad Q\{z_\mu(t) z_\nu(t)\} = -\lambda_\mu \delta_{\mu\nu} t \tag{2.15}$$

$$\lambda_\mu = \lambda g_{\mu\mu} , \quad g_{00} = -g_{11} = -g_{22} = -g_{33} = 1 .$$

We follow here closely the notation of our previous paper [3] to which we refer the reader for further details concerning the relativistic pseudoprocesses.

Putting in the equations (2.14) $\alpha_\mu = 0$, $\beta_{\mu\nu} = g_{\mu\nu}$ one obtains

$$x(t;s,y) = y + z(t) - z(s) = y + z'(t-s) . \tag{2.16}$$

Here $z'(t)$ stands for another Wiener pseudoprocess which for the sake of convenience will be labelled by $z(t)$ without prime. It gives for the function $\varphi(s,y)$ the result

$$\varphi(s,y) = v(t-s,y) , \tag{2.17}$$

where the function $v(t,y)$ is

$$v(t,y) = Q\left\{ f[y+z(t)] \exp \int_0^t c[y+z(\tau)] d\tau \right\} . \tag{2.18}$$

It satisfies the conditions following from (2.12)

$$\left\{ \partial_t - \frac{\lambda}{2}\left[\Box_y + u(y) \right] \right\} v(t,y) = 0 \tag{2.19}$$

$$v(0,y) = f(y) ,$$

where we put

$$u(y) = \frac{2}{\lambda} c(y) . \tag{2.20}$$

Comparing these conditions with those for $f(t;y,x)$ one infers that it may be written as follows

$$f(t;y,x) = Q\left\{ \delta[y - x + z(t)] \exp \frac{\lambda}{2} \int_0^t u[y+z(\tau)] d\tau \right\} . \tag{2.21}$$

This is a convenient formula which we are going to exploit further. First of all, we shall express the $J[p]$ generating functional in terms of this function. Using the formulae (2.3), (2.8) and (2.9) we may write

$$J[p] = \text{const.} \exp\left(i J\left[-2i \frac{\delta}{\delta u}\right] \right) \exp\left(-\frac{i}{2} p A[u] p \right) B[u] \Big|_{u=0} , \tag{2.22}$$

where we denoted

$$A[u;y,x] = (K+u)^{-1}_{y,x} = -\frac{\lambda}{2} \int_0^\infty dt \exp\left(-\frac{\lambda \varkappa^2}{2} t \right) f(t;y,x) , \tag{2.23}$$

$$B[u] = \exp\left\{ \frac{1}{2} \int_0^\infty \frac{dt}{t} \exp\left(-\frac{\lambda \varkappa^2}{2} t \right) \int d^4 z \, f(t;z,z) \right\} . \tag{2.24}$$

The new constant is again determined by the normalization condition (2.4).

Next our task consists of finding an approximate expression for the function $f(t;y,x)$ which in turn will yield an approximate value for $A[u;y,x]$, $B[u]$ and the $J[p]$ functional. To do this we shall utilize the known central limit theorems of a theory of probabilities, modified suitably to our case.

3. Limit theorem for pseudoprocesses

According to our principle stated in the Introduction, we modify the genuine limit theorems by an analytic continuation in the parameters of random variables involved. For instance, one of the limit theorems, suitable for our needs will be modified as follows:

Let $\xi_{\mu 1}, \xi_{\mu 2}, \dots$ be a sequence of the independent random variables, having the same distributions, such that

$$Q\{\xi_{\mu i}\} = 0 , \qquad \begin{array}{l} \mu = 0,1,2,3 \\ i = 1,2,\dots \end{array} \tag{3.1}$$

$$Q\{\xi_{\mu i}\xi_{\nu j}\} = -\lambda_\mu \delta_{\mu\nu}\delta_{ij}$$

$$\lambda_\mu = \lambda g_{\mu\mu} . \tag{3.2}$$

Let us denote a partial sums

$$S_{\mu k} = \xi_{\mu 1} + \dots + \xi_{\mu k} \tag{3.3}$$

and a pseudoprocess

$$\xi_n(\tau) = \sqrt{\tfrac{t}{n}} \sum_{k\frac{t}{n}<\tau} \xi_k \tag{3.4}$$

then the distributions of functionals depending on $\xi_n(\tau)$ will weakly converge to the distributions of the Wiener pseudoprocess $z(\tau)$ when $n \to \infty$,

$$F\Big[\sqrt{\tfrac{t}{n}} \sum_{k\frac{t}{n}<\tau} \xi_k\Big] \xrightarrow[n\to\infty]{} F[z(\tau)] . \tag{3.5}$$

It means that the average values of both the sides coincide in the limit.

In order to apply these considerations to our case we divide first a time interval $[0,t]$ into pieces using the points

$$\tau_k = k\frac{t}{n} \qquad k = 0,1,\dots,n \tag{3.6}$$

and write

$$f(t;y,x) = \lim_{n\to\infty} Q\Big\{\delta[y-x+z(t)] \exp\frac{\lambda}{2}\sum_{k=0}^{n-1} u\big[y+z(k\tfrac{t}{n})\big]\frac{t}{n}\Big\}. \tag{3.7}$$

Now, we make the replacements

$$z(k\tfrac{t}{n}) \longrightarrow \sqrt{\tfrac{t}{n}}\,(\xi_1 + \dots + \xi_k) \tag{3.8}$$

and get finally

$$f(t;y,x) = \lim_{n\to\infty} f_n(t;y,x) , \tag{3.9}$$

where

$$f_n(t;y,x) = Q\Big\{\delta\big[y-x+\sqrt{\tfrac{t}{n}}\,(\xi_1+\dots+\xi_n)\big] \cdot \exp\frac{\lambda}{2}\sum_{k=0}^{n-1} u\big[y+\sqrt{\tfrac{t}{n}}\,(\xi_1+\dots+\xi_k)\big]\frac{t}{n}\Big\} . \tag{3.10}$$

Therefore, we obtain for the generating functional

$$J[p] = \lim_{n\to\infty} J_n[p] \tag{3.11}$$

with approximate generating functional

$$J_n[p] = \text{const}\cdot \exp\left(i\,J\left[-2i\frac{\delta}{\delta u}\right]\right)\exp\left(-\frac{i}{2}pA_n[u]p\right)B_n[u]\Big|_{u=0}, \tag{3.12}$$

where

$$A_n[u;y,x] = -\frac{\lambda}{2}\int_0^\infty dt\, \exp\left(-\frac{\lambda\varkappa^2}{2}t\right) f_n(t;y,x) \tag{3.13}$$

and

$$B_n[u] = \exp\left\{\frac{1}{2}\int_0^\infty \frac{dt}{t}\exp\left(-\frac{\lambda\varkappa^2}{2}t\right)\int d^4z\, f_n(t;z,z)\right\}. \tag{3.14}$$

This is the result which was advocated in the Introduction. Clearly, there are, probably, many possibilities of constructing different approximation schemes for $J[p]$. We have just proposed a very natural one from the probabilistic standpoint. Another application of the limit theorem, concerning the structure of the transition amplitude of a pseudoprocess, may be found in [11].

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APPENDIX

In order to illustrate how the modified limit theorem works we shall apply it to the simplest case of the Feynman propagator. We hava a formulae

$$\Delta^c(x-y;\varkappa^2) = \left(\frac{1}{2\pi}\right)^4\int d^4p\,\frac{\exp[ip(x-y)]}{\varkappa^2-p^2-i\epsilon} = \tag{A.1}$$

$$= \frac{\lambda}{2}\int_0^\infty dt\,(t;y,x)\exp\left[t\left(\frac{c^2}{2\lambda}-\epsilon\right)\right] \tag{A.2}$$

where the transition amplitude $(t;y,x)$ is given by

$$(t;y,x) = -i\,(2\pi\lambda t)^{-2}\exp\left\{\frac{(x-y)^2}{2\lambda t}\right\} \tag{A.3}$$

$$= \left(\frac{1}{2\pi\hbar}\right)^4\int d^4p\,\exp\left\{\frac{i}{\hbar}\left[\frac{p^2}{2m}t + p(x-y)\right]\right\} \tag{A.4}$$

$$= Q\{\delta[y-x+z(t)]\}\,. \tag{A.5}$$

According to the prescription (3.8) we may write

$$Q\{\delta[y-x+z(t)]\} = \lim_{n\to\infty} Q\left\{\delta\left[y-x+\sqrt{\tfrac{t}{n}}\left(\xi_1+\dots+\xi_n\right)\right]\right\} \tag{A.6}$$

and therefore

$$\Delta^c(x-y;\varkappa^2) = \lim_{n\to\infty}\Delta^c_n(x-y;\varkappa^2), \tag{A.7}$$

where we denoted

$$\Delta^c_m(x-y;\varkappa^2)=$$

$$=\frac{\lambda}{2}\int_0^\infty dt\, Q\left\{\delta\left[y-x+\sqrt{\tfrac{t}{n}}(\xi_1+\cdots+\xi_n)\right]\right\}\exp\left[t\left(\frac{c^2}{2\lambda}-\epsilon\right)\right]. \tag{A.8}$$

In order to perform the average operation Q we use the Fourier representation of the $\delta-$ function, independence of the variables ξ_k and equality of their distributions we have

$$Q\left\{\delta\left[y-x+\sqrt{\tfrac{t}{n}}(\xi_1+\cdots+\xi_n)\right]\right\}= \tag{A.9}$$

$$=\left(\frac{1}{2\pi}\right)^4\int d^4z\,\exp[iz(y-x)]\left(Q\left\{\exp\left[-i\sqrt{\tfrac{t}{n}}(z\cdot\xi)\right]\right\}\right)^n.$$

Furthermore, because of the independence of ξ_μ components we have

$$Q\left\{\exp\left[-i\sqrt{\tfrac{t}{n}}(z\cdot\xi)\right]\right\}=\prod_{\mu=0}^{3}Q\left\{\exp\left[-i\sqrt{\tfrac{t}{n}}\,g_{\mu\mu}z_\mu\xi_\mu\right]\right\}. \tag{A.10}$$

Hence we arrived at a product of the characteristic functions of the variables ξ_μ. They may be found if we recall the distribution amplitudes of ξ_μ

$$M\{\xi_\mu\in A\}=(-2\pi\lambda_\mu)^{-1/2}\int_A dx\,\exp\left(\frac{x^2}{2\lambda_\mu}\right). \tag{A.11}$$

From this we easily compute that

$$Q\{\exp(-ia\xi_\mu)\}=(-2\pi\lambda_\mu)^{-1/2}\int_{-\infty}^{\infty}dx\,\exp\left(-iax+\frac{x^2}{2\lambda_\mu}\right)=\exp\left(\frac{a^2}{2}\lambda_\mu\right) \tag{A.12}$$

and finally

$$Q\left\{\exp\left[-i\sqrt{\tfrac{t}{n}}(z\cdot\xi)\right]\right\}=\exp\left(\frac{\lambda}{2}\frac{t}{n}z^2\right). \tag{A.13}$$

Therefore, we obtain the result

$$Q\left\{\delta\left[y-x+\sqrt{\tfrac{t}{n}}(\xi_1+\cdots+\xi_n)\right]\right\}=\left(\frac{1}{2\pi}\right)^4\int d^4z\,\exp\left[iz(y-x)+\frac{\lambda t}{2}z^2\right] \tag{A.14}$$

which does not show any dependence on n. Performing the integration over t as it is indicated in the formula (A.8) we get

$$\Delta^c_n(x-y;\varkappa^2)=\Delta^c(x-y;\varkappa^2). \tag{A.15}$$

Hence, the convergence is immediate in this case.

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