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Q-STABILITY OF PARTICLE-LIKE SOLUTIONS IN SCALAR ELECTRODYNAMICS

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In our previous papers[1-4] we studied the Lyapunov stability of particle-like solutions, described by the complex scalar field $\phi(x): R^1 \times R^3 \to \mathbb{C}^1$; $x \equiv (t, \vec{x})$ satisfying the natural boundary condition $\lim_{|\vec{x}| \to \infty} \phi(x) = 0$, and showed that the direct consequence of Lyapunov's theorem of stability is their instability and that they can have only conditional stability, for example, Q-stability. Besides we showed that nodal particle-like solutions (including pulsons) are even Q-unstable. In the present paper we shall establish the necessary and sufficient conditions for the Q-stability of particle-like solutions in scalar electrodynamics (see also ref.[9]).

Let the Lagrangian of the theory have the Lorentz-invariant form;

$$\mathcal{L} = -\frac{1}{2} G(s, p, q) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \tag{1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field tensor and G is an arbitrary nonlinear function of the invariants

$$s = \phi^*\phi, \quad p = -D^*_\mu \phi^* D^\mu \phi, \quad q = J_\mu J^\mu ;$$
$$J_\mu = \frac{i}{2}\{\phi^* D_\mu \phi - D^*_\mu \phi^* \phi\}, \qquad D_\mu = \partial_\mu - ieA_\mu , \tag{2}$$

where e is the interaction constant. Summation over repeated indices is understood (unless otherwise specified). Let the corresponding field equations have the stationary regular solutions

$$\phi_0(x) = u(\vec{x}) e^{-i\omega t}, \quad u^* = u, \quad \omega = \text{const};$$
$$A^\circ_\mu = (A^\circ_0, \vec{0}), \quad A^\circ_0 = A^\circ_0(\vec{x}). \tag{3}$$

We shall investigate the Lyapunov stability of the nonnodal regular solutions (3) with the additional condition of charge fixation (Q-stability[2,6-8]).

Let M denote the set of functions obtained from $\{\phi_0, A^\circ_\mu\}$ by means of the symmetry transformations of the theory. Then by definition $\{\phi(x), A_\mu\} \notin M$ denote the perturbed solutions.

Now let

$$\phi(x) = \Phi(x)\cdot e^{-i\omega t}, \quad \Phi(x) = u(\vec{x}) + \xi(x); \quad A_\mu = A_\mu^\circ + a_\mu,$$
$$\xi = \xi_1 + i\xi_2, \quad \xi_i^* = \xi_i, i = 1,2; \quad a_\mu = (a_0, \vec{a}). \tag{4}$$

We introduce the metrics

$$\rho_0 = [\int d^3x \{|\dot{\xi}|^2 + |\vec{\nabla}\xi|^2 + |\xi|^2 + \vec{e}_0^{\,2} + \vec{b}_0^{\,2} + u^2(a_0^2 + \vec{a}^{\,2})\}]^{1/2},$$

where

$$\vec{e}_0 = -\mathrm{grad}\, a_0 - \frac{1}{c}\cdot(\partial\vec{a}/\partial t), \quad \vec{b}_0 = \mathrm{rot}\,\vec{a}$$

for the characterization of the initial perturbations and

$$\rho = [\int d^3x \{|\dot{\xi}_1|^2 + |\xi_1|^2 + (\vec{e}')^2 + (\vec{b}')^2 + u^2(\vec{a}^{\,2} + \vec{a}'^{\,2})\}]^{1/2},$$

where

$$\vec{e}' = -\mathrm{grad}\, a_0' - \frac{1}{c}(\partial\vec{a}'/\partial t), \quad \vec{b}' = \mathrm{rot}\,\vec{a}',$$
$$a_0' = a_0 - (\dot{\xi}_2/eu), \quad \vec{a}' = \vec{a} - \mathrm{grad}(\xi_2/eu)$$

for the characterization of current perturbations.

Definition. The regular solutions $\{\phi_0, A_\mu^\circ\}$ are stable in the Lyapunov sense with respect to the metrics ρ_0, ρ if for each $\epsilon > 0$ there exists the number $\delta(\epsilon) > 0$ such that from $\rho_0 < \delta$ it follows that $\rho < \epsilon$ for any $t > 0$.

Let us now establish the necessary and sufficient conditions for the Q-stability of the nonnodal particle-like solutions (3) by choosing the Lyapunov's functional in the form

$$V = E - \omega Q, \tag{5}$$

where E is the field energy and Q is the total charge. We must find out in which case the functional (5) will satisfy all the conditions of the theorem of stability /5/, i.e., in which case the stationary solutions (3) will realize its minimum. For this we must investigate the sign of the second variation of the functional (5). The second variation of Lyapunov's functional can be written in the form

$$\delta^2 V = (\dot{\xi}_1, G_p\dot{\xi}_1) + e^2(u a_0', \kappa u a_0') + e^2(u\vec{a}', (G_p - sG_q)\cdot u\vec{a}') +$$
$$+ 2e^2(u\vec{a}, (G_p - sG_q)\cdot u\vec{a}) + (\xi_1, \hat{J}\xi_1) + \tag{6}$$
$$+ \frac{1}{4\pi}\{(\vec{e}', \vec{e}') + (\vec{b}', \vec{b}')\},$$

where $(\cdot,\cdot)$ denotes the scalar product in $L_2(R^3)$, the Hermitian operator $\hat{J}$ has the form

$$\hat{J} = G_s + 2sG_{ss} + \frac{q}{s^2}[-G_p + 6sG_q - 4sG_{ps} + 8s^2G_{qs}] - \mathrm{div}[G_p\vec{\nabla} + 2G_{pp}\vec{\nabla}u(\vec{\nabla}u\,\vec{\nabla})] + \frac{2q^2}{s^3}(G_{pp} - 4sG_{pq} + 4s^2G_{qq}) + \mathrm{div}[\frac{q}{s^2}(G_{pp} - 2sG_{qq})\cdot\vec{\nabla}s - G_{ps}\vec{\nabla}s] \quad (7)$$

and $\kappa = G_p - sG_q - \frac{2q}{s}(G_{pp} - 2sG_{pq} + s^2G_{qq})$.

Now let us take into account the condition of charge fixation. In the linear approximation with respect to ξ we get

$$e(ua_0', \kappa u a_0') = (g, \xi_1), \quad (8)$$

where

$$g = -2\,\mathrm{div}[s(\omega + eA_0^\circ)(G_{pp} - sG_{pq})\cdot\vec{\nabla}u] + 2u(\omega + eA_0^\circ)\{G_p - 2sG_q + s(G_{ps} - sG_{qs}) - \frac{q}{s}(G_{pp} - 3sG_{pq} + 2s^2G_{qq})\}. \quad (9)$$

From (8) using Schwartz's inequality we get

$$e^2(ua_0', \kappa u a_0') \geq (g, \xi_1)^2\cdot(u, \kappa u)^{-1}. \quad (10)$$

Therefore, from (6) and (10) we have the estimate

$$\delta^2 V \geq (\dot{\xi}_1, G_p\dot{\xi}_1) + e^2(u\vec{a}', (G_p - sG_q)u\vec{a}') + 2e^2(u\vec{a}, (G_p - sG_q)u\vec{a}) + (\xi_1, \hat{K}\xi_1) + \frac{1}{4\pi}\{(\vec{e}', \vec{e}') + (\vec{b}', \vec{b}')\}; \quad (11)$$

$$\hat{K}\xi_1 = \hat{J}\xi_1 + g(g, \xi_1)\cdot(u, \kappa u)^{-1}.$$

For the solutions (3) to be Q-stable it is sufficient that $\delta^2 V \geq 0$. From (11) it is clear that if $G_p > 0$, $\kappa > 0$ $(G_p - sG_q) > 0$, then $\delta^2 V$ will be positive definite if the operator $\hat{K}$ has positive spectrum. Now for the spectrum of $\hat{K}$

to be positive it is necessary that the operator $\hat{J}$ has not more than one negative eigenvalue, because in the opposite case $(g, \xi_1) = 0$ can always be attained for $(\xi_1, \hat{J}\xi_1) < 0$.

Let $\lambda(\omega)$ be the first eigenvalue of $\hat{K}$. Then according to the theorem I of our papers /2,4/ $\lambda(0)$ is always negative. Now let us find the critical frequency ω_0 for which $\lambda(\omega_0) = 0$ and which determines the boundary of the domain of Q-stability $\omega > \omega_0$ if $(d\lambda/d\omega) \geq 0$ (owing to the symmetry $\omega \to -\omega$ it is sufficient to consider $\omega > 0$). As

$$\mathrm{Sgn} \min_{\rho=\epsilon} \delta^2 V = \mathrm{Sgn}(\omega - \omega_0) \qquad (12)$$

$u(\omega_0)$ is the saddle point of V with the curve of descent $u(\omega)$. Thus for $\omega = \omega_0$

$$\min_{\rho=\epsilon} \delta^2 V = \min_{\rho=\epsilon} (\xi_1, \hat{K}\xi_1) = 0 \qquad (13)$$

and is achieved when $\xi_1 = \frac{du}{d\omega} \equiv u_\omega$. Hence u_ω is the eigen-function of $\hat{K}$ corresponding to the eigenvalue equal to zero. This fact leads us to the following equation for the determination of ω_0

$$\frac{dQ}{d\omega} \equiv Q_{0\omega} = 0. \qquad (14)$$

Thus for $\omega > \omega_0$ $(u_\omega, \hat{K} u_\omega) > 0$ or $Q_{0\omega}(Q_{0\omega} - (u, \kappa u)) > 0$, whence with the help of (14) we get the following inequality for the determination of the domain of Q-stability

$$Q_{0\omega} < 0. \qquad (15)$$

Note that the zero-modes are excluded here according to the definition of the metric ρ.

Thus we formulate the following theorem.

Theorem. Nonnodal regular solutions (3) in the model (1) are Q-stable and the domain of Q-stability is determined by the inequality (15) if the following conditions hold:

(a) $G_p > 0$, $\kappa > 0$, $(G_p - sG_q) > 0$,

(b) $(d\lambda/d\omega) \geq 0$.

(c) the operator $\hat{J}$ has only one negative eigenvalue.

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REFERENCES

1. Ajit Kumar, Izvestiya Vuzov "Fizika", 1979, No. 9, p. 29.
2. Ajit Kumar, Nisichenko V.P., Rybakov Yu.P. Int. Jour. Theor. Phys., 1979, v. 18, No. 6.
3. Ajit Kumar. Physica Scripta. 1980, v. 22, No. 6. Ph. D.Thesis. M., 1980.
4. Ajit Kumar. JINR, E2-81-85, Dubna, 1981.
5. Movchan A.A. Appl.Math.Mech., 1960, v. 24, p. 988.
6. Makhankov V.G. Phys.Rep., 1978, v. 35, No. 1; JINR, P2-10362, Dubna, 1977.
7. Friedberg R., Lee T.D., Sirlin A. Phys.Rev., 1976, D13, p. 2739.
8. Katyshev Yu.V., Makhankov V.G. Phys.Letts., 1976, 57A, p. 10.
9. Morris T.F. Can. J. Phys., 1979, v.57, p.2171. Katyshev Yu.V. et al. JINR, P2-12941, Dubna, 1979.

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