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ON LYAPUNOV STABILITY OF CHARGED SCALAR PARTICLE-LIKE SOLUTIONS

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Аджит Кумар **E2-81-85**

Об устойчивости по Ляпунову заряженных скалярных частицеподобных решений

Прямым методом Ляпунова исследована устойчивость заряженных солитонов /включая пульсоны/, описываемых комплексным скалярным полем. Показано, что прямым следствием теоремы Ляпунова об устойчивости является их /солитонов/ неустойчивость. Сформулированы необходимые и достаточные условия Q-устойчивости /устойчивость с дополнительным условием фиксации заряда/ безузловых заряженных скалярных солитонов.

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Ajit Kumar **E2-81-85**

On Lyapunov Stability of Charged Scalar Particle-Like Solutions

Lyapunov's direct method is applied to study the stability of charged solitons (including pulsons), described by the complex scalar field. It is shown that the direct consequence of Lyapunov's theorem of stability is their instability. Some necessary and sufficient conditions for the Q-stability (stability with the additional condition of charge fixation) of nonnodal charged scalar solitons are established.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

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INTRODUCTION

The construction of a self-consistent unified field theory, which could explain the existing mass spectrum of elementary particles and predict the new ones, is a long cherished desire of theorists. In this connection the idea, due to Einstein, of finding out the dynamical principle which could allow us to control the distribution of the excited states of strongly interacting matter, which we observe as elementary particles, is very attracting. The unified field theory, which should inevitably be nonlinear, is one of such dynamical principles. According to this concept all observable elementary particles and their interactions are manifestations of some unified, or as it is called fundamental nonlinear field. The superiority of such theory would have been not only in determining the mass spectrum of elementary particles but also in erradicating the divergencies that appear in the orthodox quantum field theory.

In nonlinear field theory elementary particles are described by regular solutions to the field equations. Nonlinear field equations may have regular solutions at rest or moving with constant velocity. Such solutions with field amplitude, considerably different from zero, are localized in a finite region of space. The energy of such solutions is finite and they are called "solitons" /1,17/, "lumps" /3,4/ or particle-like solutions. Regular localized solutions with distinct topological properties are often called "kinks" /5/. As a matter of fact, solitons, keeping in view their origin, are essentially one-dimensional (space) objects and, therefore, we shall use the name "particle-like solutions" (PLS) in our paper.

<u>Definition 1</u>. Three-dimensional regular solutions to nonlinear classical field equations are called particle-like solutions, if (a) they have finite energy and other physical characteristics and (b) they are localized in a small region of space at any instant of time.

Here under localization we mean the following: Let us introduce the average radius r^* of the regular solution by the formula

$$r^* = \int r \cdot \epsilon \, dV / \int \epsilon \, dV,$$

where ϵ stands for the energy density of the field.

Definition 2. Regular solutions to nonlinear field equations are called localized in space if $r^* < \delta$ for any moment of time. Here δ is an arbitrary but finite constant ($\delta > 0$).

Thus, both the requirements of the definition 1 can be fulfilled if

(a) the regular solutions are sufficiently smooth and
(b) they are finite at the origin and tend to zero at spatial infinity sufficiently fast.

Further, if we want to describe in a unified way stable as well as unstable particles in the framework of nonlinear field theory, we must require, in addition to the above-mentioned properties of PLS, fulfilment of another one, namely, their stability in Lyapunov sense /24/.

Note, that the requirement of stability is an additional one, whereas the finiteness of physical characteristics and localization of regular solutions constitute the necessary and sufficient conditions for them to be particle-like.

As is mentioned above, the investigation of the stability of PLS is very important. Firstly, the stability requirement of such solutions plays a vital role in nonlinear field theory (NFT), namely, when we look for NFT that allow the existence of stable PLS, which may be supposed to describe stable elementary particles. Secondly, the criterion of stability restricts to some extent the freedom in the choice of the basic field equations for the construction of an adequate self-consistent field theory of elementary particles.

In the present paper we shall investigate in detail the Lyapunov stability of charged scalar PLS. The purpose of this paper is to review some of our results on the Lyapunov stability of charged scalar PLS.

1. METHOD OF INVESTIGATION

First of all we must give the physical definition of the stability of the particle-like solutions. From this point of view one must take into account all possible perturbations that an elementary particle continuously experiences. Unfortunately, such a formulation of the problem of stability leads to tremendous mathematical difficulties if one wants to find its solution. Therefore, we confine ourself with the traditional treatment of this problem in Lyapunov sense when the perturbation (arbitrary) is switched on at the initial moment and then the time-evolution of the perturbed system is studied, i.e., the stability of the dynamical system with respect to initial perturbations is considered.

Lyapunov considered dynamical systems in n-dimensional Euclidean space E_n, described by the equations

$$\frac{d\vec{x}}{dt} = X(\vec{x}, t); \quad \vec{x} = (x_1, x_2, \dots, x_n) \tag{1.1}$$

defined in some domain $D(\sum_{i=1}^{n} x_i^2 < H, \quad t \geq 0)$ and having the equilibrium solution $\vec{x} = 0$. As a measure of perturbation he considered the metric distance

$$\rho(\vec{x}, t) = \|\vec{x}\| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2} . \tag{1.2}$$

Definition 3. The equilibrium solution $\vec{x} = 0$ is called stable if for each $\epsilon > 0$ and given t_0 there exists a number $\delta > 0$ such that for the initial perturbation $\vec{x}_0$, satisfying the condition $\|\vec{x}_0\| < \delta$, the inequality $\|\vec{x}\| < \epsilon$ holds for any $t > t_0$. In the opposite case equilibrium solution $\vec{x} = 0$ is unstable. Moreover, if δ does not depend upon t_0, the stability is called to be uniform.

For the investigation of stability Lyapunov used continuous and unique valued function $V(\vec{x},t)$ with certain properties.

(a) The function $V(\vec{x}, t)$ is called to be positive definite in a certain domain D, if (i) $V(0,t) = 0$ and (ii) there exists a function $W(\vec{x})$ $(W(0) = 0$ and $W(\vec{x}) > 0$ for $\vec{x} \in D$ such that $V(\vec{x},t) > W$

(b) The function $V(\vec{x}, t)$ is said to have infinitesimally small higher limit, if for $\|\vec{x}\| \to 0$ $V(\vec{x}, t) \to 0$ uniformly in t.

In this case the following theorem (Lyapunov's theorem) give the answer to the question of stability of the equilibrium solution $\vec{x} = 0$.

Theorem. If the equations of motion are such that there exists a positive definite function $V(\vec{x}, t)$ such that $dV/dt \equiv \dot{V} < 0$, then the equilibrium solution $\vec{x} = 0$ is stable. If $V(\vec{x},t)$ allows infinitesimally small higher limit, then the stability will be uniform in t_0.

Lyapunov's method can easily be generalized to distributed systems because the latter can be considered to be the limiti case of a dynamical system in E_n for $n \to \infty$. Actually, the fie equations can always be written in the form of a system of first order differential equations for a multi-component fiel function $\psi = \begin{pmatrix}\psi_1 \\ \vdots \\ \psi_s\end{pmatrix}$. Let the region R of the three-dimensional space in which the field ψ is defined be finite. Then R can be divided into n small cells of volume ΔV. In every i-th cell one can define the average value of the field function

$\psi^{(1)}$. In this case the equations for $\psi^{(1)}$ will be of the form (1.1) and for them Lyapunov's method can directly be applied. Here the metric distance from the equilibrium $\psi^{(i)}=0$ can be defined as

$$(\sum_{j=1}^{s}\sum_{i=1}^{n}|\psi_j^{(i)}|^2\Delta V_i)^{1/2}, \qquad (1.3)$$

which in the limiting case when $n\to\infty$ gives

$$\rho=(\sum_{j=1}^{s}\int|\psi_j|^2 dV)^{1/2}.$$

It is clear that in our case of distributed systems we shall have to consider Lyapunov's functional instead of Lyapunov's function.

For our further investigation we shall use Lyapunov's method generalized to two metrics, ρ_0 and ρ, for the description of initial and current perturbations, respectively. This generalization is due to A.A.Movchan /6/.

2. INSTABILITY OF CHARGED SCALAR PARTICLE-LIKE SOLUTIONS IN LYAPUNOV SENSE

Let us consider the particle-like solution described by the complex scalar field

$$\phi(t,\vec{x}): R^1\times R^3\to \mathbb{C}^1,$$

satisfying the natural boundary condition

$$\lim_{|\vec{x}|\to\infty}\phi(x)=0;\quad x\equiv(t,\vec{x}).$$

Let the Lagrangian density of the theory have the Lorentz-invariant form:

$$\mathfrak{L}=-\frac{1}{2}F(s,p,q), \qquad (2.1)$$

F being arbitrary nonlinear function of the invariants $s=\phi^*\phi$, $p=-\partial_\mu\phi^*\partial^\mu\phi$, $q=J_\mu J^\mu$; $J_\mu=\frac{i}{2}[\phi^*\partial_\mu\phi-\partial_\mu\phi^*\phi]$. The field equations are written as

$$\partial_\mu(-iF_q J^\mu)\phi+\partial_\mu(-F_p\partial^\mu\phi)-2iF_q J^\mu\partial_\mu\phi-F_s\phi=0. \qquad (2.2)$$

Let the field equations (2.2) have the stationary regular solution

$$\phi_0(x)=\psi(\vec{x})e^{-i\omega t};\qquad \psi^*=\psi,\quad \omega=\text{const} \qquad (2.3)$$

describing the charged PLS at rest. Let M denote the set of functions obtained from ϕ_0 by means of symmetry transformations of the theory, i.e.,

$$M=\{\phi_0'=T_g\phi_0\},$$

T_g being the representation of the continuous group of symmetry of the theory (excluding time transformation), then by definition, the function $\phi(x)\notin M$ describes the perturbed solution. Let

$$\phi(x)=\Phi(x)e^{-i\omega t}.$$

Following Movchan /6/ and Slobodkin /7/ we introduce the metrics ρ_0 and ρ for the characterization of the initial perturbation $\xi^\circ=\Phi(0,\vec{x})-\psi(\vec{x})$ and the current perturbation $\xi=\Phi(x)-\psi(x)$, respectibely. Putting

$$\xi=\xi_1+i\xi_2,\quad \xi_i^*=\xi_i,\quad i=1,2,$$

we choose the metrics in the form

$$\rho_0=\sum_{i=1}^{2}\{\|\dot{\xi}_i^\circ\|+\|\xi_i^\circ\|^s\};\qquad \rho=\inf_{\psi\in M}\sum_{i=1}^{2}\|\xi_i\|,$$

where $\|\cdot\|$ and $\|\cdot\|^s$ designate the norms in $L_2(R^3)$ and Sobolev space $W_2^1(R^3)$, respectively, and $\dot{\xi}_i^\circ=\partial_t\xi_i|_{t=0}$.

<u>Definition 4.</u> The regular solution ϕ_0 is stable in Lyapunov sense with respect to the metrics ρ_0, ρ, if for each $\epsilon>0$ there exists a number $\delta(\epsilon)>0$ such, that from $\rho[\xi^\circ]<\delta$ it follows that $\rho[\xi]<\epsilon$ for any $t>0$.

Let us now consider a lemma /16/ of variational calculus that will be useful for our further investigations. Let the functional

$$V[\phi]=\int v(\phi,\partial_\mu\phi)d^3x$$

be defined in the class of sectionally smooth functions $\phi(t,\vec{x})$: $R^1\times R^3\to R^n$, $\phi(t,\infty)=0$ and let it be invariant with respect to the continuous symmetry group of the theory given by the parameters $\alpha=\{\alpha_i\}$. Let $u(\alpha;t,\vec{x})$ be the family of its extremal fields in which $u(0;t,\vec{x})$ is the one under our consideration.

<u>Lemma</u>. If there exist the constants c_i, not all equal to zero, such that the linear combination $\sum_i c_i(\frac{\partial u}{\partial\alpha_i})_{\alpha=0}=0$ on some boundaryless surface S separating in R^3 a domain Ω of nonzero measure, then the extremal $u(0;t,\vec{x})$ does not realize even weak minimum of the functional V, i.e., $\delta^2 V$ is sign changing in the neighbourhood of $u(0,x)$.

Proof. Let the extremal $u(0;x) \equiv u_0$ give weak minimum to the functional V. Consider the ϵ-neighbourhood of u_0:

$u = u_0 + \epsilon\eta$.

As u_0 gives weak minimum to the functional V, the adjoint functional

$$\delta^2 V = \int (A\eta^2 + 2B\eta\eta' + C\eta'^2)\, d^3x, \qquad (2.4)$$

where prime designates differentiation with respect to the argument and A, B and C stand for the second derivatives of v, calculated at the point u_0, with respect to its arguments, must be either positive or equal to zero on all alowable curves η [8]. Hence any sectionally smooth function η, which is equal to zero on S and for which $\delta^2 V[\eta] = 0$, must consist of the parts of the extremals of this functional. Besides that on the surface S it must satisfy the Weierstrass-Erdmann matching condition. Take the function

$$\eta_0 = \begin{cases} 0 & \text{for } \vec{x} \in \Omega, \\ \beta & \text{for } \vec{x} \notin \Omega, \quad \beta \equiv \sum_i C_i \left(\frac{\partial u}{\partial \alpha_i}\right)_{\alpha=0} \end{cases}.$$

According to the theorem of variational calculus, stating that the difference of two infinitesimally close extremals sets the second variation of the functional equal to zero, we get

$$\delta^2 V[\eta_0] = 0. \qquad (2.5)$$

From (2,4) and (2.5), after integrations by parts, requiring the fulfilment of Weierstrass-Erdmann condition we get that the following condition must hold

$$(\eta'_0)_{S-0} = (\eta')_{S+0} \quad . \qquad (2.6)$$

But the condition (2.6) cannot be fulfilled because

$$(\eta'_0)_{S-0} = 0 \quad \text{and} \quad (\eta'_0)_{S+0} = \vec{\nabla}\beta \neq 0.$$

Hence $\delta^2 V$ is sign changing in the neighbourhood of u_0.

Sometimes the class of regular solutions

$$\phi_0(x) = \psi(t, \vec{x}) \cdot e^{-i\theta(t)}\ ; \quad \psi^* = \psi\ , \qquad (2.7)$$

which includes, in particular, the so-called pulsons [1], is considered.

Now, using the above lemma, we shall prove a couple of theorems regarding the instability of the regular solutions (2.7) in Lyapunov sense.

Theorem 1. The regular solutions (2.7) are unstable in Lyapunov sense in any model (2.1).

Proof. As the field equations are invariant under time-translation, the dynamical system under consideration is autonomous. According to the general theorem of stability [6] the motion of the autonomous dynamical system (2.1) is stable with respect to the metrics ρ_0, ρ, if and only if in some neighbourhood of ϕ_0 there exists a Lyapunov's functional $V[\phi]$, such that

(a) it does not increase along the trajectories of the system;

(b) it is continuous with respect to the metric ρ_0 and

(c) it is positive definite with respect to the metric ρ.

Thus, if we suppose ϕ_0 to be stable, then there must exist a positive definite functional $V[\phi]$ for which ϕ_0 is the extremal field. Since the model (2.1) is invariant under 3-translations $\vec{x} \to \vec{x} + \vec{a}$, $\vec{a} = \text{const}$, then $\phi(t, \vec{x} + \vec{a})$ is also the extremal field for the functional V. Hence according to the above lemma

$$\beta = \vec{C} \cdot \vec{\nabla}\psi$$

must not be equal to zero on any surface. However, the equation $\partial_x \psi = 0$ $(\vec{C} = \{1, 0, 0\})$ can be satisfied on some surface S because ψ is regular and $\psi(t, \infty) = 0$. Hence according to the lemma $\delta^2 V$ is sign changing, which contradicts the stability of ϕ_0 and proves the theorem.

From this theorem it follows that only conditional stability of the regular solutions (2.3) and (2.7) can be achieved In general, from the physical point of view, several conditions, such as conservations of charge, momentum, angular momentum, leptonic charge, baryonic charge, etc., can be imposed on the initial perturbations. We choose the condition o charge fixation [1, 13-16, 23]

$$Q[\phi] = \frac{i}{2}\int d^3x (F_p - sF_q)(\phi^*\partial_t\phi - \partial_t\phi^*\phi) = Q[\phi_0] \equiv Q_0 . \qquad (2.8$$

Stability under the condition (2.8) will be called Q-stabili

Theorem 2. Regular nodal solutions (2.7) are Q-unstable i any model (2.1).

Proof. As $\psi(t, \vec{x}) = 0$ on the nodal surface, for the extrema fields $\phi_0 \cdot e^{i\alpha}$, allowed by the model (2.1), all the condition of the lemma are fulfilled. So the proof of theorem 1 can automatically be extended to this case too; the condition (2. being satisfied by choosing

$$\dot{\xi}_1 = \dot{\xi}_2 = 0, \qquad ||\xi_1|| \ll ||\xi_2|| . \tag{2.9}$$

The condition (2.9) can readily be observed if we expand the functional of charge Q up to the second order in ξ.

3. SUFFICIENT CONDITIONS FOR THE Q-STABILITY OF NONNODAL PARTICLE-LIKE SOLUTIONS

Now we shall investigate in detail the Q-stability of nonnodal regular solutions (2.3). For this purpose we need the explicit form of the Lyapunov's functional V. Therefore, let us begin with the study of some of the fundamental properties of Lyapunov's functional.

Our field equations are invariant under time inversion (which corresponds to the requirement of inversibility of microprocesses). Hence our Lyapunov's functional, which, irrespective of the perturbed motion, must be positive definite in the neighbourhood of the stable solution ϕ_0, must also be invariant under time inversion, i.e.,

$$V[\phi(t,\vec{x})] = V[\phi(-t,\vec{x})] . \tag{3.1}$$

However, the time-derivative of V changes its sign along the trajectory of the system under the transformation $t \to -t$:

$$\frac{d}{dt} V[\phi(t,\vec{x})] \to -\frac{d}{dt} V[\phi(-t,\vec{x})] = -\frac{d}{dt} V[\phi(t,\vec{x})] . \tag{3.2}$$

Therefore, if for the direct motion we have $\frac{dV}{dt} \equiv \dot{V} \leq 0$, for the reversed motion we shall have $\dot{V} \geq 0$. Hence, we won't contradict Lyapunov's theorem of stability if and only if

$$\dot{V} \equiv 0 . \tag{3.3}$$

The condition (3.3) requires the Lyapunov's functional V to be the integral of motion in our case. So let us now write the variational principle for the stationary regular solutions (2.3). We get

$$\delta S = \delta \int_{t_1}^{t_2} dt \int d^3x \{ \{ \frac{\partial \mathfrak{L}}{\partial \dot{\phi}_0} \dot{\phi}_0 + \frac{\partial \mathfrak{L}}{\partial \dot{\phi}_0^*} \dot{\phi}_0^* \} - H \}$$

$$= \delta \int_{t_1}^{t_2} dt \int d^3x [-i\omega \frac{\partial \mathfrak{L}}{\partial \dot{\phi}_0} \phi_0 + i\omega \frac{\partial \mathfrak{L}}{\partial \dot{\phi}^*} \phi_0^* - H]$$

$$= \delta \int_{t_1}^{t_2} dt (\omega Q_0 - E) = (t_1 - t_2)\, \delta(E - \omega Q_0) , \tag{3.4}$$

where E stands for energy, S designates the action function and Q_0 is the total charge. Thus, we choose the Lyapunov's functional V in the form

$$V = E - \omega Q . \tag{3.5}$$

Note, that this particular choice of Lyapunov's functional can also be justified from the point of view of the method of the chains of integrals of motion, due to Chetaev. As it will be clear afterwards, in our case, the combination $(E - \omega Q)$ does not contain any sign changing term, linear in ξ, explicitly.

Further we must find out in which case the functional (3.5) will satisfy all the conditions of the theorem of stability, i.e., in which case the stationary state ϕ_0 will realize its minimum. This problem is solved in variational calculus and leads to the investigation of the sign of the second variation of Lyapunov's functional.

The second variation of the functional (3.5) can be written as

$$\delta^2 V = \frac{1}{2} \int d^3x [F_p \dot{\xi}_1^2 + (F_p - 2\omega^2 s F_{pp} - sF_q -$$

$$- 2\omega^2 s^3 F_{qq} - 2\omega^2 s F_{pq}) \cdot \dot{\xi}_2^2 + F_p (\vec{\nabla} \xi_1)^2 +$$

$$+ 2F_{pp} (\vec{\nabla} \xi_1 \cdot \vec{\nabla} \psi)^2 + (F_p - sF_q)(\vec{\nabla} \xi_2)^2 + \{ F_s +$$

$$+ 2sF_{ss} + \omega^2 (-F_p + 6sF_q - 4sF_{sp} + 8s^2 F_{sq}) + \tag{3.6}$$

$$+ 2\omega^4 s (F_{pp} + 4s^2 F_{qq}) - 8\omega^4 s^2 F_{pq} +$$

$$+ \mathrm{div}[(\omega^2 F_{pp} - 2\omega^2 s F_{pq} - F_{ps}) \vec{\nabla} s] \} \xi_1^2 +$$

$$+ \{ F_s - \omega^2 F_p - (\vec{\nabla}\psi)^2 F_q + 2\omega^2 s F_q + \mathrm{div}[\frac{1}{2} \vec{\nabla} s F_q] \} \xi_2^2 \cdot$$

It can also be written as

$$\delta^2 V = (\dot{\xi}_1, F_p \dot{\xi}_1) + (\dot{\xi}_2, h\dot{\xi}_2) + \sum_{i=1}^{2} (\xi_i, \hat{J}_i \xi_i), \tag{3.7}$$

where (.,.) designates scalar product in $L_2(R^3)$,

$$h = F_p - sF_q - 2\omega^2 s(F_{pp} - 2sF_{pq} + s^2 F_{qq}) \tag{3.8}$$

and the Hermitian operators $\hat{J}_i$ (i = 1,2) have the form

$$\begin{aligned}\hat{J}_1 = &- \mathrm{div}[F_p \vec{\nabla} + 2F_{pp} \vec{\nabla}\psi(\vec{\nabla}\psi\, \vec{\nabla})] + \\ &+ \mathrm{div}[(\omega^2 F_{pp} - 2\omega^2 sF_{pq} - F_{ps}) \cdot \vec{\nabla} s] + F_s + 2sF_{ss} + \\ &+ \omega^2(-F_p + 6sF_q - 4sF_{ps} + 8s^2 F_{qs}) + 2\omega^4 s(F_{pp} + 4s^2 F_{qq} - 4sF_{pq}),\end{aligned} \tag{3.9}$$

$$\begin{aligned}\hat{J}_2 = &- \mathrm{div}[(F_p - sF_q)\vec{\nabla}] - \mathrm{div}[\tfrac{1}{2}\vec{\nabla} s \cdot F_q] + \\ &+ F_s - \omega^2 F_p + F_q(\omega^2 s - p).\end{aligned} \tag{3.10}$$

So for $\delta^2 V$ to be positive it is necessary that $F_p > 0$ and $h > 0$. Using the condition of charge fixation in linear approximation with respect to ξ we get

$$(\dot{\xi}_2, hu) = (g, \xi_1), \tag{3.11}$$

where

$$\begin{aligned}g = &- \mathrm{div}[(F_{pp} - sF_{pq}) 2\omega s \vec{\nabla}\psi] + 2\omega u\{F_p - \\ &- (2F_q + sF_{sq} - F_{ps}) s - \omega^2 s(F_{pp} - 3sF_{pq} + 2s^2 F_{qq})\}.\end{aligned} \tag{3.12}$$

From (3.11) using Schwartz's inequality we get

$$(\dot{\xi}_2, h\dot{\xi}_2) \geq (g, \xi_1)^2 \cdot (\psi, h\psi)^{-1}. \tag{3.13}$$

As a result we can write

$$\delta^2 V \geq (\dot{\xi}_1, F_p \dot{\xi}_1) + (\xi_2, \hat{J}_2 \xi_2) + (\xi_1, \hat{K}\xi_1) \equiv W[\dot{\xi}_1, \xi], \tag{3.14}$$

where

$$\hat{K}\xi_1 = \hat{J}_1 \xi_1 + g(g, \xi_1) \cdot (\psi, h\psi)^{-1}. \tag{3.15}$$

Let us study the functional W and find out the conditions under which it will be positive definite with respect to the metric ρ which can be enlarged by including $\|\dot{\xi}_1\|$. Note that due to the field equations $\hat{J}_2 \psi = 0$, i.e., ψ is the eigenfunction of the operator $\hat{J}_2$ with eigenvalue equal to zero. Therefore, according to Courant's theorem about the positivity of the first eigenfunction of the Hermitian differential operator of second order /9/ the spectrum of $\hat{J}_2$ will not be negative because $\psi > 0$. The zero mode is excluded here according to the definition of ρ because for $\xi_2 = \psi \in M$ $\rho[\psi] \equiv 0$.

Now let us investigate the spectrum of $\hat{K}$. In this case the situation is a bit complicated because $\hat{K}$ is an integro-differential operator and, therefore, Courant's theorem cannot be used for it as a whole. However, Courant's theorem is applicable to the operator $\hat{J}_1$. We shall use this fact and the geometrical properties of Lyapunov's functional in our further investigations.

It is clear from (3.15) that for the spectrum of $\hat{K}$ to be positive it is necessary that $\hat{J}_1$ does not have more than one negative eigenvalue, because in the opposite case it is always possible to make the scalar product (g, ξ_1) equal to zero for $(\xi_1, \hat{J}_1 \xi_1) < 0$. Now let $\lambda(\omega)$ be the first eigenvalue of $\hat{K}$. Then, according to theorem 1 $\lambda(0)$ (or $\lambda(0+\delta)$, δ being the infinitesimally small constant) is always negative. Now, let us find the critical frequency ω_0 for which $\lambda(\omega_0) = 0$ and which determines the domain of Q-stability $\omega > \omega_0$ if $(d\lambda/d\omega) \geq 0$ (due to the symmetry $\omega \to -\omega$ it is sufficient t consider $\omega > 0$). As

$$\mathrm{Sgn} \min_{\rho=\epsilon} \delta^2 V = \mathrm{Sgn}(\omega - \omega_0), \tag{3.16}$$

therefore, $\psi(\omega_0)$ is the saddle point of the functional V with the curve of descent $\psi(\omega)$. If we move along the curve we shall reach the point ω_0 at which $\delta^2 V = 0$. Therefore, for $\omega = \omega_0$

$$\min_{\rho=\epsilon} \delta^2 V = \min_{\rho=\epsilon} (\xi_1, \hat{K}\xi_1) = 0 \tag{3.17}$$

and is attained for $\xi = (d\psi/d\omega) \equiv \psi_\omega$. So $\hat{K}\psi_\omega = 0$, i.e., ψ_ω is the eigenfunction of $\hat{K}$ corresponding to the eigenvalue equal to zero. The relation $\hat{K}\psi = 0$ along with the equa tion $\hat{J}_2\psi = 0$ and a bit of algebra leads to the following equation for the determination of the critical frequency ω_0 [10,11]

$$\frac{d}{d\omega}[\omega\int(F_p - sF_q) s\, d^3x] \equiv Q_{0\omega} = 0. \qquad (3.18)$$

If $\omega > \omega_0$ then

$$(\psi_\omega, \hat{K}\psi_\omega) > 0$$

or

$$Q_{0\omega}(Q_{0\omega} - (\psi, h\psi)) > 0. \qquad (3.19)$$

From (3.18) and (3.19) we get the inequality for the determination of the domain of Q-stability [12,13,15,22]

$$Q_{0\omega} < 0. \qquad (3.20)$$

Note, once again, that the zero modes of the type $\xi_1 = C_i\, \partial_i \psi$ are excluded according to the definition of the metric ρ. Thus we come to the following conclusion.

Theorem 3. Nonnodal regular solutions (2.3) in the model (2.1) are Q-stable and the domain of Q-stability is determined by the inequality (3.20) if the following conditions hold:

(a) $(d\lambda/d\omega) \geq 0$;

(b) the operator $\hat{J}_1$ has only one negative eigenvalue.

4. ONE EXAMPLE

Let us see the following model (other examples can be found in [14-16])

$$F = p + s - \frac{s^n}{n} - 2aq^{1/2}; \quad a = \text{const} > 0. \qquad (4.1)$$

In this case for $1 < n < 3$ and $\beta^2 = (1-\omega^2-2a|\omega|) > 0$ there exist regular spherically symmetric solutions (2.3) to the corresponding field equations. Changing the variables

$$x = r\beta, \quad v(x) = \psi \cdot \beta^{-(1/(n-1))} \qquad (4.2)$$

we can easily establish that for $\omega > 0$

$$Q_0(\omega) = \text{const}\cdot(\omega + a)\cdot\beta^{-3+\frac{2}{(n-1)}}.$$

So the condition (3.20) is fulfilled only for $n < 5/3$. Let us take $n = 3/2$. This case was studied in [2]. For this model the operator $\hat{K}$ has the form

$$\hat{K} = -\Delta - 2\psi + (1 - \omega^2 - 6a\omega) + \frac{4(\omega + 2a)^2}{1 + a/\omega} \cdot \hat{P}_\psi, \qquad (4.3)$$

where $\hat{P}\psi$ is the projection operator on $\psi/\|\psi\|$. With the help of (4.2) the equation $\hat{K}u = \lambda u$ can be written in the following form:

$$(-\Delta - 2v(x) + 1 - \mu)\cdot u + \nu\hat{P}_v u = \tilde{\lambda} u, \qquad (4.4)$$

where

$$\mu = 4a\omega\beta^{-2}, \quad \nu = (\omega + 2a)^2/\beta^2(1 + a/\omega), \quad \tilde{\lambda} = \lambda\beta^{-2}. \qquad (4.5)$$

Differentiating (4.4) with respect to ω we get

$$\frac{d\tilde{\lambda}}{d\omega} = (v, u)^2\frac{d\nu}{d\omega} - \frac{d\mu}{d\omega}. \qquad (4.6)$$

As $\frac{d\nu}{d\omega} > 0$ and $\frac{d\mu}{d\omega} > 0$ for small a $(\frac{d\tilde{\lambda}}{d\omega}) > 0$. The last relation replaces the first condition of theorem 3 because $\text{sgn}\,\lambda = \text{sgn}\,\tilde{\lambda}$.

Now let us show that the operator $\hat{J}_1$ has only one negative eigenvalue for $\omega = 0$. Consider the eigenvalue problem

$$(-\Delta + 1 - 2\psi)\chi = \epsilon\chi. \qquad (4.7)$$

Separating in (4.7) angular variables

$$\chi_{\ell k} = R_\ell(r)\, Y_{\ell k}(\theta, a)$$

one can verify that for $\ell = 1$ $R_1 = d\psi/dr$ and $\epsilon = 0$. Further, as $\psi(r)$ is monotonic [18] $R_1(r)$ does not have internal zeroes and hence, according to Courant's theorem $\epsilon = 0$ is the minimal eigenvalue for $\ell = 1$. Therefore $\epsilon > 0$ only for S-state. According to Sturm's theorem [19] the number of S states with $\epsilon < 0$ is equal to the internal zeroes of the solution to the equation

$$(-d^2/dr^2 + 1 - 2\psi)\, y(r) = 0 \qquad (4.8)$$

with the boundary conditions $y(0) = 0$ and $y'(0) = 1$. Numerical calculations show that $y(r)$ has only one internal zero for $r \approx 1.32$. Hence for small a all the conditions of theorem 3 are fulfilled. The domain of Q-stability is given by

$$\left\{\left[\frac{1+a^2}{2}\right]^{1/2}-a\right\}<\omega<\left\{(1+a^2)^{1/2}-a\right\}. \qquad (4.9)$$

CONCLUSION

From our investigation of the Lyapunov-stability of the scalar particle-like solutions it follows that classical field theoretic models of elementary particles with quantized charge (topological charge) are perspective. In such models the condition of charge fixation $\Delta Q = 0$ is automatically fulfilled and the stability of extended particles has absolute character. It is due to the fact that the Hamiltonian H in such models can be estimated from below through the topological charge

$$H \geq \text{Const}\cdot|Q|.$$

Therefore, in the cases when the lower bound of the Hamiltonian is attained the corresponding PLS of the model are absolutely stable because the Lyapunov's functional $V = \int H d^3x$ automatically satisfies all the requirements of Lyapunov's theorem of stability.

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REFERENCES

1. Makhankov V.G. Phys.Rep., 1978, 35, p.1.
2. Synge J.L. Proc.Roy.Irish Acad. of Sci., 1961, 62A, p.17.
3. Coleman S. Lecture Notes, Erice Summer School, 1975.
4. Weder R. Commun.Math.Phys., 1977, 57, p.161; 1978, 63, p.1.
5. Finkelstein D. Journ.Math.Phys., 1966, 7, p.1216.
6. Movchan A.A. Appl.Math.and Mech., 1960, 24, p.988.
7. Slobodkin A.M. Proc. of the USSR Academy of Sciences, 1964, 157, p.63.
8. Akhiezer N.I. Lectures in Variational Calculus, Leningrad, 1955.

9. Courant R., Hilbert D. Methods of Mathematical Physics, Interscience Publishers, 1953, vol.1, chap.VI.
10. Zastavenko L.G. Appl.Math.and Mech., 1965, 29, p.430.
11. Rybakov Yu.P. In: Problems of Gravitation and Elementary Particle Theory", K.P.Staniukovich, ed. Atomizdat, Moscow 1966, p.68.
12. Vakhitov N.G., Kolokolov A.A. Izvestiya Vuzov, Radiophysic 1973, 16, p.1020.
13. Friedberg R., Lee T.D., Sirlin A. Phys.Rev., 1976, D13, p.2739.
14. Ajit Kumar, Izvestya Vuzov, Fizika, 1979, 9, p.29.
15. Ajit Kumar, Nisichenko V.P., Rybakov Yu.P. Int.J.Theor. Phys., 1979, 18, p.6.
16. Ajit Kumar. Phys.Scr., 1980, vol.22, No.6.
17. Ajit Kumar. In: Problems of Statistical and Quantum Physics, p.102. P.Lumumba University, Moscow, 1979 (In Russian).
18. Coleman S., Glaser V., Martin A. Comm.Math.Phys., 1978, 58, p.211.
19. Tricomi F.G. Differential Equations, ChapIII, Blakie and Son Ltd., London, 1961.
20. Skyrme T.H.R. Nucl.Phys., 1962, 31, p.556.
21. 't Hooft G. Nucl.Phys., 1974, B79, p.276.
22. Makhankov V.G. JINR, P2-10362, Dubna, 1977.
23. Katyshev Yu.V., Makhankov V.G. Phys.Lett., 1976, 57A, p.10.
24. Ajit Kumar. Ph.D.Thesis, M., 1980.

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