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A NEW INTEGRABLE MODEL OF QUANTUM FIELD THEORY IN THE STATE SPACE WITH INDEFINITE METRIC

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The nonlinear Schrödinger equation

$$i\,\Psi_t + \Delta\Psi + 2\kappa|\Psi|^2\,\Psi = 0 \tag{1}$$

is the most important one among equations of mathematical physics. It occures naturally in the weak coupling limit in the framework of every nonrelativistic polynomial nonlinear theory. The one-dimensional NLS displays have in addition the unique property - the property of complete integrability [1] that allows in principle to solve classical Cauchy problem. A complete integrable classical system may be as a rule put in the corresponding quantum one according to the quantum version of ST [2]. The factorization of scattering matrix in such systems simplifies processes allowed but makes them essentially poor. This is why the interest enlarges now to investigate integrable systems with internal symmetries. In fact such systems manifest more interesting dynamics including even processes of fission and fussion of solitons, polarization (colour) exchange and so on [3,4].

The two component NLS with $U(2)$ isosymmetry has been considered at the classical level in ref. [4] and its quantum version in ref. [5]. In the first case we have elliptically polarized electromagnetic wave in a medium with the dispersion $\omega = k^2 - 2\kappa|\vec{E}|^2$, in the latter it is a gas of "coloured" Bose-particles with point-like interaction of attraction or repulsion type.

But more interesting model a mixture of two "gravitating" and "antigravitating" gases may be treated only in the framework of non-compact pseudo-unitary group of internal symmetry $U(1,1)$ (see ref. [6]). Classical equations describing such system occure for a one-dimensional continual version of the Hubbard model [7]. Allowance for "colour" degrees of freedom for these two gases implies passing to non-compact isogroup $U(p,q)$ with p and q being the numbers of pure "colour" states of gravitating and antigravitating particles respectively. So we have two sets of pure colour states in the system.

1. The classical system of equations in our case is of the form

$$i\Psi_t + \Psi_{xx} + 2\kappa(\bar{\Psi}\Psi)\Psi = 0, \tag{2}$$

where $(\Psi)_a = \Psi^{(a)}(x,t)$ is the n-component column vector (n=p+q), $\bar{\Psi} = \Psi^{+}\gamma_0$ is the Dirac conjugate row vector and $\gamma = \mathrm{diag}(\underbrace{+1,\dots+1}_{p}, \underbrace{-1,\dots-1}_{q})$.

Inner product, invariant with respect to the isogroup U(p,q) transformations is as follows:

$$(\bar{\Psi}\Psi) = \sum_{a=1}^{p} |\Psi^{(a)}|^2 - \sum_{a=p+1}^{n} |\Psi^{(a)}|^2. \tag{3}$$

The Hamiltonian of the system may be expressed in terms of canonical conjugate variables $\Psi^{(a)}$ and $\bar{\Psi}^{(a)}$

$$\{\Psi^{(a)}(x), \bar{\Psi}^{(b)}(y)\} = i\delta^{ab}\delta(x-y)$$

in the following way

$$H = \int_{-\infty}^{\infty} dx\,[(\bar{\Psi}_x \Psi_x) - \kappa(\bar{\Psi}\Psi)^2]. \tag{4}$$

System (2) may be studied by means of the following auxiliary linear problem of $(n+1)\times(n+1)$ order [8]:

$$Xf = 0, \tag{5}$$

where $(f)_i = f_i(x,t)$, $(i=1,\dots,n+1)$,

$$X = \frac{d}{dx} + P(x,\lambda), \quad P(x,\lambda) = i\lambda\Sigma - Q(x),$$

$$\Sigma = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{n}I_n \end{pmatrix}, \quad Q(x) = \begin{pmatrix} 0 & i\bar{q}(x) \\ iq(x) & 0 \end{pmatrix},$$

$$q_a = \kappa^{1/2}\Psi^{(a)}(x,t).$$

The matrix Jost solutions $\phi(x,\lambda)$ and $\Psi(x,\lambda)$ to linear system (5) under vanishing boundary conditions $q|_{|x|\to\infty} \to 0$ are determined by their asymptotic values

$$\phi(x,\lambda) \to \exp(-i\lambda\Sigma x) \qquad x \to -\infty,$$

$$\psi(x,\lambda) \to \exp(-i\lambda\Sigma x) \qquad x \to +\infty,$$

and coupled with the transition matrix

$$\phi(x,\lambda) = \Psi(x,\lambda)S(\lambda).$$

This matrix satisfies to unimodularity condition det $S(\lambda) = 1$ and that of pseudounitarity

$$\bar{S}(\lambda)S(\lambda) = I \tag{6}$$

(here $\bar{S} = \Gamma_0 S^{+}\Gamma_0$, $\Gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & \gamma_0 \end{pmatrix}$). As it was shown in ref. [8] n+1 elements of the transition matrix S namely $S_{11}(\lambda)$, $S_{\alpha\beta}(\lambda)$ (α, β =2, ..., n+1) are independent of time. Their simple analytical properties allow one to construct a complete system of the inverse transform [8].

Conserving element $S_{11}(\lambda)$ being invariant under isogroup transformations generates an infinite series of local polynomial conservation laws. Denumerable series of such integrals of motion $I_{11}^{(k)}$ may be obtained via the asymptotic expansion

$$\ln S_{11}(\lambda) \sim \sum_{k=1}^{\infty} \frac{I_{11}^{(k)}}{(\frac{n+1}{n} i\lambda)^k}, \quad \lambda \to \infty .$$

They are generalization of the well-known integrals for U(1) NLS and one can derive them simply by means of passing from $|\Psi|^2$ to $U(p,q)$ invariant inner product (3). The block $S_{\alpha\beta}(\lambda)$ generates n series of non-local integrals of motions barring n^2 ones corresponding to $U(p,q)$ group generators. The $U(p,q)$ invariant integrals of motions $I_{11}^{(k)}$ $(k=1,2,\dots)$ are in involution with one another and with noninvariant integrals generated by $S_{\alpha\beta}(\lambda)$:

$$\{I_{11}^{(k)}, I_{11}^{(m)}\}=0, \quad \{I_{11}^{(k)}, I_{\alpha\beta}^{(m)}\}=0.$$

The integrals $I_{\alpha\beta}^{(k)}$ are noninvolutive and generate an infinite parameter algebra.

Generalizing the procedure of ref. [1] one may calculate the Poisson brackets between the transition matrix elements

$$\{S_{k\ell}(\lambda), S_{ps}(\mu)\} = \frac{1}{\frac{n+1}{n}(\lambda-\mu)} \lim_{x\to\infty} (S_{p\ell}(\lambda) S_{ks}(\mu) e^{-i(\lambda-\mu)(\Sigma_{pp}-\Sigma_{kk})x} - S_{p\ell}(\mu) S_{ks}(\lambda) e^{-i(\lambda-\mu)(\Sigma_{ss}-\Sigma_{\ell\ell})x}). \tag{7}$$

Here the Poisson brackets are

$$\{A, B\} = i \sum_{a=1}^{n} \int_{-\infty}^{\infty} dx \left(\frac{\delta A}{\delta q_a(x)} \frac{\delta B}{\delta \bar{q}_a(x)} - \frac{\delta B}{\delta q_a(x)} \frac{\delta A}{\delta \bar{q}_a(x)} \right).$$

All above statements may be proved using (7). In particular

$$\{S_{11}(\lambda), S_{1\alpha}(\mu)\} = \frac{1}{\frac{n+1}{n}(\lambda-\mu)} S_{11}(\lambda) S_{1\alpha}(\mu), \quad (\lambda \neq \mu) \tag{8}$$

whence

$$\{\ln S_{11}(\lambda), S_{1\alpha}(\mu)\} = \frac{1}{\frac{n+1}{n}(\lambda-\mu)} S_{1\alpha}(\mu).$$

It gives when expanded in powers of λ^{-1}

$$\{H, S_{1\alpha}(\mu)\} = \frac{dS_{1\alpha}(\mu)}{dt} = i\left(\frac{n+1}{n}\right)^2 \mu^2 S_{1\alpha}(\mu)$$

or (9)

$$S_{1\alpha}(\mu, t) = e^{i\left(\frac{n+1}{n}\right)^2 \mu^2 t} S_{1\alpha}(\mu, 0)$$

which implies the elements $S_{1\alpha}(\mu)$ to be angle type variables and simply depend on time.

2. It is natural to try putting above classical problem into the corresponding quantum one. For the sake of simplicity we consider the $U(1,1)$ model. In the Hubbard model for example the Schrödinger wave functions $u(x,t) = \phi_\uparrow + \phi_\downarrow$, $v(x,t) = \phi_\uparrow - \phi_\downarrow$ may be expressed in terms of Ψ_1 and Ψ_2 (see eq. (2)) as follows: $u^* = \Psi_1$, $v = \Psi_2$.

Suppose them to be subject to the following standard commutation relations

$$[u(x), u^+(y)] = \frac{2}{\kappa}\delta(x-y), \quad [u(x), y(y)] = 0 = [u^+(x), u^+(y)],$$

$$[v(x), v^+(y)] = \frac{2}{\kappa}\delta(x-y), \quad [v(x), v(y)] = 0 = [v^+(x), v^+(y)].$$

The state space becomes Hilbert Space but eigenvalues of the free field Hamiltonian may be negative*.

There is another possibility which we prefer here: it is associated with indefinite metrics. We pass for that to the variables Ψ_1, Ψ_2 or just the same to q_1 and q_2 with the commutation relations

$$[q_a(x), q_b^+(y)] = (\gamma_0)_{ab}\delta(x-y), \quad [q_a(x), q_b(y)] = 0 = [q_a^+(x), q_b^+(y)] \qquad (10)$$

being valid in the state space with indefinite metrics. All the eigenvalues of the free field Hamiltonian are now positive. The fact that the metrics are indefinite should be not surprising since the transition matrix is pseudo-unitary already in the classical limit. There arise difficulties in the quantum field theory connected to the interpretation of results for in such theory negative probabilities appear according to the orthodox sense. Note that similar difficulties occur in the Yang-Mills theory with non-compact gauge group $SL(2,C)$ important for gauge theory of gravity /9/. The group

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* Following Haag (see /10/), one might say that the kinetic aspect does, but dynamical aspect does not allow an underlying Hilbert Space.

non-compactability makes the system energy to be no positively definite. Passing to an indefinite metrics allows one to make the energy to be positive definite but the unitarity of the scattering matrix is now inconsistent with gauge symmetry.

In this connection we would like to quote Nagy /10/ :"... that arguments based on the usual Lagrangian formulation are sometimes superficial; actually the decision on the interpretability requires a knowledge of the exact solution - needless to say, this is in most cases an unsolvable task".

As it will be however seen the inverse quantum technique enables us to advance far in getting an exact solution in our case whereby it will throw light upon the possibilities of non-compact inner symmetry groups in the Yang-Mills theory as well*.

In what follows we apply the usual technique of the quantum inverse transform. The infinitesimal evolution operator $L_n(\lambda)$ when written for a chain spaced in the range $(-L,L)$ assumes at the nth-interval $(x_n-\Delta, x_n)$ the form

$$L_n(\lambda)=\begin{pmatrix} 1_n-i\lambda\Delta & i\bar{q}_{1n} & i\bar{q}_{2n} \\ iq_{1n} & 1_n+i\frac{\lambda}{2}\Delta & 0 \\ iq_{2n} & 0 & 1_n+i\frac{\lambda}{2}\Delta \end{pmatrix} \tag{11}$$

where $\Delta=L/N$, $q_{a_n}=\int_{x_n-\Delta}^{x_n} q_a(x)dx$ and

$[q_{a_n}, \bar{q}_{b_m}]=\Delta\delta_{ab}\delta_{nm}, (\bar{q}=q^+\gamma_0)$.

Matrix $S_L(\lambda)$ - the transition operator from $-L$ to L (monodromy matrix) is

$$S_L(\lambda)=\lim_{\Delta\to 0}\prod_{-N+1\leq n\leq N} L_n(\lambda).$$

Commutation relations of matrix elements $L_n(\lambda)$ and $L_n(\mu)$ are realized via the c-number (9x9) matrix $R(\lambda-\mu)$:

$$R(\lambda-\mu)(L_n(\lambda)\otimes L_n(\mu))=(L_n(\mu)\otimes L_n(\lambda))R(\lambda-\mu). \tag{12}$$

Tensor product matrices $L_n(\lambda)\otimes L_n(\mu)$ operate in the space $\mathfrak{L}_1(\boldsymbol{R}^1, \mathbf{C}^3\otimes\mathbf{C}^3)$. Following Kulish /5/ we take vectors in this space to be

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* The system of self-dual equations for Yang-Mills field has been shown to be also an integrable one /11/, but the Bäcklund transformations for it are closed only in the framework of non-compact gauge group SL(N,C) rather than in its compact subgroup SU(N) /12/.

$$(f \otimes f')^T = (f_\alpha \otimes f'_\beta \,,\, f_\alpha \otimes f'_3 ,\quad f_3 \otimes f'_\beta \,,\, f_3 \otimes f'_3),$$

where

$$f \equiv f(x,\lambda)\,,\; f' \equiv f(x,\mu),\quad (\alpha,\beta = 1,2).$$

Then matrix R is the following

$$R(\lambda) = \begin{pmatrix} r(\lambda) & 0 & 0 & 0 \\ 0 & b(\lambda)e & c(\lambda)e & 0 \\ 0 & c(\lambda)e & b(\lambda)e & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\quad r(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(\lambda) & c(\lambda) & 0 \\ 0 & c(\lambda) & b(\lambda) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{13}$$

$$b(\lambda) = \frac{1}{1 - \frac{3}{2} i\lambda},\quad c(\lambda) = \frac{-\frac{3}{2} i\lambda}{1 - \frac{3}{2} i\lambda},\quad b(\lambda) + c(\lambda) = 1,$$

and e is the unit (2x2) matrix, $r(\lambda)$ is the R-matrix of the one-component U(1) symmetric NLS. Local commutation relation (12) implies similar one for overall transition matrix, which becomes in continuum limit

$$R_1(\lambda-\mu)(S(\lambda) \otimes S(\mu)) = (S(\mu) \otimes S(\lambda))R_2(\lambda-\mu) \tag{14}$$

with

$$R_1(\lambda-\mu) = R_2(\lambda-\mu) = R_\infty(\lambda-\mu),\qquad (\lambda \neq \mu),$$

and

$$R_\infty(\lambda) = \begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 0 & c & & & & \\ & & 1/c' & 0 & & & & \\ & & & & \ddots & & & \\ & & & & 1 & & & \\ & & & & & \ddots & & \\ & & & & & & \begin{matrix} 0 & 0 & c & 0 \\ 0 & b & 0 & c \\ 1/c' & 0 & 0 & 0 \\ 0 & c & 0 & b \end{matrix} & \\ & & & & & & & \ddots \\ & & & & & & & & 1 \end{pmatrix},\quad c'(\lambda,\lambda) = c(\mu,\lambda) = \frac{\frac{3}{2} i(\lambda-\mu)}{1 + \frac{3}{2} i(\lambda-\mu)},$$

$$b(\mu,\lambda) = 1 - c(\mu,\lambda).$$

By using (14) we are now able to get commutation relations for monodromy matrix, in particular

$$[S_{11}(\lambda),\ S_{12}(\mu)] = \frac{1}{\frac{3}{2} i(\lambda-\mu)} S_{12}(\mu) S_{11}(\lambda),$$

$$(\lambda \neq \mu)$$

$$[S_{11}(\lambda),\ S_{13}(\mu)] = \frac{1}{\frac{3}{2} i(\lambda-\mu)} S_{13}(\mu) S_{11}(\lambda),$$

that is completely in accordance with the classical result (8).

Note that our R-matrix (13) and that of the vector NLS /5/ are similar despite the transition matrices are essentially different in these two models. In fact our TM operates in the state space with indefinite metrics and have a different commutation relations for some complex conjugate and nonconjugate elements.

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