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# BLOCK DISCRETE ZAKHAROV-SHABAT SYSTEM.

## II. Hamiltonian Structures

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§1. A number of nonlinear evolution equations, exactly soluble through the inverse scattering method (ISM)[1,2] are known to describe infinite-dimensional completely integrable Hamiltonian systems (see the review paper[3']), allowing a hierarchy of Hamiltonian structures[4,5]. Important examples of such systems, applicable in physics, are the multicomponent nonlinear Schrödinger (NLS) equations[6,7]. A number of completely integrable difference evolution equations (DEE) have been considered in refs[8-13] and the classes of DEE related to a number of auxiliary discrete linear problems have been described, see the review paper[2] and the references in it.

In the present paper we consider the discrete analogs of the NLS eqs., related to the block discrete Zalharov-Shabat system:

$$\psi(n+1,z)=(Z+Q(n))\psi(n,z),\quad Z=\begin{pmatrix} z\,1_s, & 0 \\ 0, & \frac{1}{z}1_p \end{pmatrix},\quad Q(n)=\begin{pmatrix} 0 & q(n) \\ r(n) & 0 \end{pmatrix}, \tag{1.1}$$

and to its equivalent eigenvalue problem

$$\left[\begin{pmatrix} D_+ & 0 \\ 0 & D_- \end{pmatrix}+\bar{Q}(n)-z\right]\bar{\psi}(n,z)=0,\qquad \bar{Q}(n)=\begin{pmatrix} 0, & -\bar{q}(n) \\ \bar{r}(n), & 0 \end{pmatrix}. \tag{1.2}$$

In (1.1) and (1.2) $q$ , $\bar{q}$ , $r^T$ , $\bar{r}^T$ are rectangular $s\times p$ matrices. The interrelations between (1.1) and (1.2) are given by:

$$\bar{q}(n)=q(n)\hat{v}_2(n),\quad \bar{r}(n)=v_2(n)r(n-1),\quad \bar{\psi}(n,z)=\begin{pmatrix} 1 & 0 \\ 0 & v_2(n) \end{pmatrix}\psi(n,z)$$

$$v_i(n)=\prod_{k=\infty}^{n} h_i(k),\quad \hat{v}_i\equiv v_i^{-1},\quad h_1(k)=1-q(k)r(k),\quad h_2(k)=1-r(k)q(k), \tag{1.3}$$

$$q(n)=\bar{q}(n)\hat{\bar{v}}_2(n),\quad r(n)=\bar{v}_2(n+1)\bar{r}(n+1),\quad \bar{v}_i(n)=\prod_{k=\infty}^{n}\bar{h}_i(k),$$

$$\bar{h}_1(k)=1+\bar{q}(k)\bar{r}(k+1),\quad \bar{h}_2(k)=1+\bar{r}(k+1)\bar{q}(k),\quad \bar{v}_i(n)=\hat{v}_i(n). \tag{1.4}$$

The present paper is a direct continuation of ref /14/, where the inverse scattering problem for the systems (1.1) and (1.2) is solved, and the expansions over the "squared" solutions of (1.1) and (1.2) are derived.

In §2 we list the necessary results from /14/. Using them in §3 we describe the classes of DEE related to (1.1) and (1.2), resp. Let us note that those classes contain the systems

$$i\frac{dq(n)}{dt}=q(n+1)h_2(n)+h_1(n)q(n-1)-2q(n),$$
$$-i\frac{dr(n)}{dt}=h_2(n)r(n-1)+r(n+1)h_1(n)-2r(n), \tag{1.5}$$

and the equivalent to:

$$i\frac{d\bar{q}(n)}{dt}=\bar{q}(n+1)+\hat{\bar{h}}_1(n-1)\bar{q}(n-1)-\bar{q}(n)\bar{r}(n)\bar{q}(n)-2\bar{q}(n),$$
$$-i\frac{d\bar{r}(n)}{dt}=\bar{r}(n-1)+\bar{r}(n+1)\hat{\bar{h}}_1(n)-\bar{r}(n)\bar{q}(n)\bar{r}(n)-2\bar{r}(n), \tag{1.6}$$

which in the continuous limit and after the reduction $q=\pm r^+$, $\bar{q}=\pm\bar{r}^+$ go into the multicomponent NLS eqs. In §4 we use the results in /14/ to derive the hierarchy of Hamiltonians structures for the discrete NLS equations. The Hamiltonians here appear naturally as linear combinations of the conserved quantities, so that the question solved is how to define the simplectic forms. For the case $s=p=1$ considered in /12,14/ the simplest simplectic form related to (1.1) is non-canonical: $\Omega_0\sim\sim\sum_{n=-\infty}^{\infty}\delta q(n)\wedge\delta r(n)(1-q(n)r(n))^{-1}$. The generalization to $s+p>2$ leads to great complications in $\Omega_0$, which now contains under the summation sign highly nonlocal and nonlinear expressions of q and r, see formula (3.8). From that point of view the linear problem (1.2) is preferable, since the necessary simplectic form $\bar{\Omega}_0$ is canonical.

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§2. Let us consider the linear problem (1.2) (or (1.1) with a potential $\bar{Q}(n)$ (or $Q(n)$) falling off fast enough* for $n\to\pm\infty$, and such, that $\det h_1(n)=\det h_2(n)\neq 0$ for all n. Let us also assume, that the discrete spectrum of (1.2) (or (1.1))

*It is enough if $\bar{Q}_{ij}(n)\sim c^{|n|}$ for $n\to\pm\infty$, where $c=\mathrm{const}$, $|c|<1$.

$\Delta = \Delta^+ \cup \Delta^-$, $\Delta^\pm \equiv \{ z_{\alpha\pm} - z_{\alpha\pm}, |z_{\alpha\pm}| \gtrless 1, \alpha = 1,\dots,N \}$ is finite and simple. Under these conditions one can show that both systems (1.1) and (1.2) have the same minimal set of scattering data

$$\mathfrak{T} \equiv \{ \rho^\pm(z),\ |z| = 1,\ \rho^\pm_\alpha,\ z_{\alpha\pm},\ \alpha = 1,\dots,N \}, \tag{2.1}$$

which uniquely determines both the potentials $Q(n)$, $\bar{Q}(n)$ and the Jost solutions of (1.1) and (1.2). Besides one can derive the following expansions for the potential $\bar{w}(n) = \begin{pmatrix} \bar{q}^T(n) \\ \bar{r}(n) \end{pmatrix}$ and its variation $\sigma_3 \delta \bar{w}(n)$ over the "squared" solutions $\{ V^\pm_{ij} \}$ of the system (1.2):

$$\begin{aligned} \bar{w}_A(n) = & -\frac{i}{2\pi} \oint_{S^1} dz \sum_{i<j} [ V^+_{ji}(n,z)(\rho^+_A)_{ji} - V^-_{ij}(n,z)(\rho^-_A)_{ij} ] + \\ & + 2 \sum_{\alpha=1}^{N} \sum_{i<j} [ V^{(\alpha)+}_{ji}(n)(\rho^+_{\alpha,A})_{ji} + V^{(\alpha)-}_{ij}(n)(\rho^-_{\alpha,A})_{ij} ] \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} \sigma_3 \delta \bar{w}(n) = & \frac{i}{2\pi} \int_{S^1} dz \sum_{i<j} [ V^+_{ji}(n,z)(\delta\rho^+)_{ji} - V^-_{ij}(n,z)(\delta\rho^-)_{ij} ] - \\ & -2 \sum_{\alpha=1}^{N} \sum_{i<j} \{ V^{(\alpha)+}_{ji}(n)(\delta\rho^+_\alpha)_{ji} + \dot{V}^{(\alpha)+}_{ji}(n)(\rho^+_\alpha)_{ji}\, \delta z_{\alpha+} + V^{(\alpha)-}_{ij}(n)(\delta\rho^-_\alpha)_{ij} + \dot{V}^{(\alpha)-}_{ij}(n)(\rho^-_\alpha)_{ij}\, \delta z_{\alpha-} \}. \end{aligned} \tag{2.3}$$

In (2.2) $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ is a constant quasi-diagonal matrix, $S^1$ is the positively oriented unit circle, and

$$\bar{w}_A(n) = \begin{pmatrix} (A_1 \bar{q}(n) - \bar{q}(n) A_2)^T \\ \bar{r}(n) A_1 - A_2 \bar{r}(n) \end{pmatrix}, \quad \sigma_3 \delta \bar{w}(n) = \begin{pmatrix} \delta \bar{q}^T(n) \\ -\delta \bar{r}(n) \end{pmatrix},$$

$$\rho^+_A = \rho^+ A_1 - A_2 \rho^+, \quad \rho^-_A = \rho^- A_2 - A_1 \rho^-, \tag{2.4}$$

$$\rho^+_{\alpha,A} = \rho^+_\alpha A_1 - A_2 \rho^+_\alpha, \quad \rho^-_{\alpha,A} = \rho^-_\alpha A_2 - A_1 \rho^-_\alpha,$$

$$V^\pm_{ij}(n,z) = V^{(\alpha)\pm}_{ij}(n) + (z - z_{\alpha\pm}) \dot{V}^{(\alpha)\pm}_{ij}(n) + O((z - z_{\alpha\pm})^2).$$

From the results in ref. /14/ it is possible to obtain analogical expansions for $w(n)$ and $\sigma_3 \delta w(n)$ over the system $\{ Y^\pm_{ij} \}$ of "squared" solutions of (1.1); the correspinding expansion coefficients will be simply related to the ones in (2.1),(2.2).

In /14/ we introduced the operators $\Lambda_\pm$, $\bar{\Lambda}_\pm$

$$\Lambda_\pm = \bar{\Lambda}^\pm_2 \bar{\Lambda}^\pm_1, \quad \bar{\Lambda}_\pm = \Lambda^\pm_1 \Lambda^\pm_2 = \bar{\Lambda}^\pm_1 \bar{\Lambda}^\pm_2 \tag{2.5}$$

through the relations

$$(\Lambda_{\pm} - z^2) Y^{\pm}_{ji} = (\Lambda_{\pm} - z^2) Y_{ij} = 0,$$
$$(\bar{\Lambda}_{\pm} - z^2) V^{\pm}_{ji} = (\bar{\Lambda}_{\pm} - z^2) V_{ij} = 0, \qquad i<j. \tag{2.6}$$

The explicit forms of $\Lambda^{\pm}_i$ , $\bar{\Lambda}^{\pm}_i$ are given in the appendix in ref./14/.

Theorem 1. Let us be given the block-diagonal matrix function $F(z^2) = \sum_p F^{(p)} z^{2p}$ and let us denote by $w^{(p)}$ and $\bar{w}^{(p)}$ the quantities $w_{F(p)}$ and $\bar{w}_{F(p)}$ , see (2.4). Then in order that $w$ and $\bar{w}$ satisfy the DEE

$$i\sigma_3 \frac{d\bar{w}}{dt} + \sum_p \bar{\Lambda}^p_+ \bar{w}^{(p)} = 0, \quad i\sigma_3 \frac{dw}{dt} + \sum_p \Lambda^p_+ w^{(p)} = 0, \tag{2.7a,b}$$

it is necessary and sufficient, that the scattering data $\mathcal{T}$ (2.7) satisfy the linear equations:

$$i\frac{d\rho^{\pm}}{dt} - \rho^{\pm}_F(z,t) = 0, \qquad i\frac{d\rho^{\pm}_{\alpha}}{dt} - \rho^{\pm}_{\alpha,F}(t) = 0, \qquad \frac{dz_{\alpha\pm}}{dt} = 0,$$
$$\rho^{\pm}_F = \rho^{\pm}_A \big|_{A=F(z^2)}, \; \rho^{\pm}_{\alpha,F} = \rho^{\pm}_{\alpha,A} \big|_{A=F^{\pm}_{\alpha}}, \quad F^{\pm}_{\alpha} = F(z^2_{\alpha\pm}). \tag{2.8}$$

The proof is analogical to the ones in/15,13/. Let us insert the expansions* for $\bar{w}^{(p)}$ and $\sigma_3 \frac{d\bar{w}}{dt}$* in the l.h.s. of (2.7a) and make use of (2.6) to obtain

$$i\sigma_3 \frac{d\bar{w}}{dt} + \sum_p \bar{\Lambda}^p_+ \bar{w}^{(p)} = \frac{i}{2\pi} \int_{S1} dz \sum_{i<j} \{ [i\frac{d\rho^+_{ji}}{dt} - \sum_p z^{2p} \rho^{(p)+}(z)_{ji}] V^+_{ji}(n,z) -$$
$$-[i\frac{d\rho^-_{ij}}{dt} - \sum_p z^{2p} \rho^{(p)-}(z)_{ij}] V^-_{ij}(n,z) \} - \tag{2.9}$$
$$-2\sum_{\alpha=1}^{N} \sum_{i<j} \{ [i(\frac{d\rho^+_{\alpha}}{dt})_{ji} - \sum_p z^{2p}_{\alpha+} (\rho^{(p)+}_{\alpha})_{ji}] V^{(\alpha)+}_{ji}(n) +$$

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* The expansion for $\sigma_3 \frac{d\bar{w}}{dt}$ can be obtained from (2.2) by considering variations of the form $\delta\bar{w} = \bar{w}(n,t+\delta t) - \bar{w}(n,t)$ and retaining only the terms, proportional to $\delta t$. Then in the r.h.s. of (2.3) we shall have $\frac{d\rho^{\pm}}{dt}$ , $\frac{d\rho^{\pm}_{\alpha}}{dt}$ , $\frac{dz_{\alpha\pm}}{dt}$ instead of $\delta\rho^{\pm}, \delta\rho^{\pm}_{\alpha}, \delta z_{\alpha\pm}$ .

$$+\frac{dz_{\alpha+}}{dt}\dot{V}^{(\alpha)+}_{ji}(n)(\rho^{+}_{\alpha})_{ji}+[i(\frac{d\rho^{-}_{\alpha}}{dt})_{ij}-\sum_{p} z^{2p}_{\alpha-}(\rho^{(p)-}_{\alpha})_{ij}]V^{(\alpha)-}_{ij}(n)+ \tag{2.9}$$

$$+\frac{dz_{\alpha-}}{dt}\dot{V}^{(\alpha)-}_{ij}(n)(\rho^{-}_{\alpha})_{ij}\},$$

where by $\rho^{(p)\pm}$ and $\rho^{(p)\pm}_{\alpha}$ we have denoted $\rho^{\pm}_{A}$ and $\rho^{\pm}_{\alpha,A}$ with $A=F^{(p)}$, see (2.4). In order that the l.h.s. of (2.7) vanishes it is necessary and sufficient that all the expansion coefficients in (2.9) vanish. Thus we have proved the equivalence of (2.7a) and (2.8). The equivalence of (2.7b) and (2.8) is proved analogously by using the corresponding expansions for w(n) and $\sigma_3\frac{dw}{dt}$ over the system {Y}. The equivalence of (2.7a) and (2.7b) can be verified also directly by noting, that the following relations hold

$$\hat{\Lambda}^{+}_{1}\sigma_3\frac{d\bar{w}}{dt}=\sigma_3\frac{dw}{dt},\quad \hat{\Lambda}^{+}_{1}\bar{w}=w(n),\quad \bar{\Lambda}_{+}=\bar{\Lambda}^{+}_{1}\Lambda_{+}\hat{\Lambda}^{+}_{1}\ . \tag{2.10}$$

Actually the expansions of $\sigma_3 dw/dt$ and w can be obtained from (2.1) and (2.2) by acting with the operator $\hat{\Lambda}^{+}_{1}$ from the left.

In particular, if in (2.7) we choose $F(z^2)=\frac{1}{2}\sigma_3(2-z^2-z^{-2})$ we obtain the discrete NLS eqs. (1.5) and (1.6). In the next paragraph we shall consider in detail the series of higher NLS eqs. for which $F(z^2)=\frac{1}{2}f(z^2)\sigma_3$, $f(z^2)$ being c-number function. These DEE are local and have the maximally possible number of series of conservation laws. Indeed from (2.8) it follows that (see the appendix):

$$i\frac{dS}{dt}+[F(z^2),S(z,t)]=0 \tag{2.11}$$

where $S=\begin{pmatrix} a^{+}, b^{-}\\ b^{+}, a^{-}\end{pmatrix}$ is the transfer matrix of (1.2). If $F=\frac{1}{2}f(z^2)\sigma_3$ from (2.11) there follows, that all the matrix elements of the diagonal blocks of S, $a^{+}$, $a^{-}$ are conserved, i.e., this gives $s^2+p^2-1$ series of conservation laws*. For generic $F(z^2)$ from (2.11) there follows only the conservation of the eigenvalues of S i.e., $s+p-1$ series.

Suppose we have proved the complete integrability of the DEE (2.7) and have calculated the action-angle variables. Then every function of the "action" type variables will be time independent for any choice of $F(z^2)$. As regards the mat-

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*We have taken into account that $\det S=1$, which gives one relationship between them.

rix elements of $a^{\pm}(z)$ for generic $F$ they will depend on $t$. This means that they depend also on the "angle"-type variables, which is confirmed by the fact that the Poisson brackets between $a^{\pm}_{ij}$ and $a^{\pm}_{i'j'}$ do not vanish[7], see also[16].

At last we note also that the DEE (2.7) in general allow boomeron-type solutions, see[17]. The higher NLS eqs. have no such solutions, since $F \sim \sigma_3$.

§3. In this paragraph we shall construct the hierarchy of Hamiltonian structures for the higher discrete NLS eqs.:

$$i\sigma_3 \frac{d\bar{w}}{dt} + f(\bar{\Lambda}_+)\bar{w} = 0, \quad i\sigma_3 \frac{dw}{dt} + f(\Lambda_+)w = 0. \tag{3.1}$$

It is natural as a generating functional for the Hamiltonians of eqs. (3.1) to choose $D(z)$,

$$D(z) = \begin{cases} \ln\det a^+(z), & |z| > 1 \\ -\ln\det(a^-(z)v_2), & |z| < 1, \end{cases} \tag{3.2}$$

where $v_2 = \lim_{n\to-\infty} v_2(n)$, $v_2(n)$ being introduced in (1.3). Indeed, from (2.11) it readily follows, that $\frac{d}{dt}D(z) = 0$. Let us now make use of the trace identities, derived in[14]:

$$z\frac{dD}{dz} = -2\sum_{n=-\infty}^{\infty}\sum_{k=n}^{\infty} \mathrm{tr}[\tilde{w}(k)M_+\Lambda_+(\Lambda_+ - z^2)^{-1}w(k)] =$$

$$= -\sum_{n=-\infty}^{\infty}\{\sum_{k=n}^{\infty} 2\,\mathrm{tr}[\tilde{\bar{w}}(k)(\bar{\Lambda}_+ - z^2)^{-1}\bar{w}(k)] + \mathrm{tr}\,\widetilde{\sigma_3\bar{w}(n)}(\bar{\Lambda}_+ - z^2)^{-1}\bar{w}(n)\}, \tag{3.3}$$

$$M_+ = \hat{\Lambda}_1^+\bar{\Lambda}_1^+ = \Lambda_2^+\hat{\bar{\Lambda}}_2^+ .$$

$$\delta D(z) = [\sigma_3\,\delta w,\ M_+\Lambda_+(\Lambda_+ - z^2)^{-1}w(n)] =$$
$$= [\sigma_3\,\delta\bar{w},\ (\bar{\Lambda}_+ - z^2)^{-1}\bar{w}], \tag{3.4}$$

where $[\,,\,]$ is the skew-scalar product introduced in[14]

$$[X,Y] = \sum_{n=-\infty}^{\infty} \mathrm{tr}(\tilde{X}(n)Y(n)), \quad \tilde{X}(n) = (X^{(2)T}(n),\ -X^{(1)T}(n)).$$

The Hamiltonians for (3.1) shall be linear combinations of the coefficients $D_k$, entering in the asymptotic expansions of $D(z)$

$$D(z) = \sum_{k=0}^{\infty} D_{-k}\, z^{2k}, \ |z| \ll 1; \quad D(z) = -\sum_{k=1}^{\infty} D_k\, z^{-2k}, \ |z| \gg 1. \tag{3.5}$$

Using (3.3), we obtain

$$H_f = -\sum_p f_p D_p =$$

$$= \sum_{n=-\infty}^{\infty} \{ \sum_{k=n}^{\infty} \operatorname{tr} \tilde{\bar{w}}(k) g(\bar{\Lambda}_+) \bar{w}(k) + \frac{1}{2} \operatorname{tr}[\widetilde{\sigma_3 \bar{w}(n) g(\bar{\Lambda}_+)} \bar{w}(n)] +$$

$$+ f_0 \operatorname{tr} \ln( I + \bar{q}(n)\bar{r}(n+1)) \} = \tag{3.6}$$

$$= \sum_{n=-\infty}^{\infty} \{ \sum_{k=n}^{\infty} \operatorname{tr} \tilde{w}(k) M_+ \Lambda_+ g(\Lambda_+) w(k) - f_0 \operatorname{tr} \ln ( I - q(n) r(n)) \}$$

$$g(z^2) = \frac{1}{z^2} \int^{z^2} ds (f(s) - f_0 s^{-1}), \quad f(z^2) = \sum_p f_p z^{2(p-1)}.$$

The corresponding simplectic forms $\bar{\Omega}_0$ and $\Omega_0$ we shall choose so, that the Hamiltonian equations of motion

$$\bar{\Omega}_0(\sigma_3 \frac{d\bar{w}}{dt}, .) = \delta H_f(\cdot), \quad \Omega_0(\sigma_3 \frac{dw}{dt}, \cdot) = \delta H_f(\cdot), \tag{3.7}$$

coincide with the DEE (3.1). It is easy to see that this will be the case if

$$\bar{\Omega}_0 = \frac{1}{i} [\sigma_3 \delta \bar{w} \overset{\wedge}{,} \sigma_3 \delta \bar{w}] \overset{\text{def.}}{=}$$

$$= \frac{1}{i} [\sigma_3 \delta_1 \bar{w}, \sigma_3 \delta_2 \bar{w}] - \frac{1}{i} [\sigma_3 \delta_2 \bar{w}, \sigma_3 \delta_1 \bar{w}] \tag{3.8}$$

$$\Omega_0 = \frac{1}{i} [\sigma_3 \delta w \overset{\wedge}{,} M_+ \Lambda_+ \sigma_3 \delta w],$$

where $\delta_1 \bar{w}$ and $\delta_2 \bar{w}$ are two independent variations of $\bar{w}$. Indeed, from (3.5) and (3.4) we easily obtain, that

$$\delta H_f = [\sigma_3 \delta \bar{w}, f(\bar{\Lambda}_+) \bar{w}] = [\sigma_3 \delta w, M_+ \Lambda_+ f(\Lambda_+) w]. \tag{3.9}$$

Inserting (3.9) into the r.h.s. of (3.7) and using (3.8) we immediately obtain (3.1).

Let us briefly discuss the relation between $\Omega_0$ and $\bar{\Omega}_0$. Using (2.10), the property $[X, \Lambda_i^{\pm} Y] = [\bar{\Lambda}_{3-i}^{\mp} X, Y]$ (see formula (3.14) in /14/) and the explicit form of $\Lambda_i^{\pm}$ we obtain

$$\Omega_0 = \bar{\Omega}_0 + \frac{1}{i} \operatorname{tr}[v_2 \delta \hat{v}_2 \wedge \delta( \sum_{n=-\infty}^{\infty} \bar{q}(n) \bar{r}(n))]. \tag{3.10}$$

Now, if we restrict the phase space* related to the system (1.1) by the condition $v_2$=const, then $\Omega_0=\bar{\Omega}_0$. Such restriction is compatible with the dynamics of the NLS eqs., since $v_2$ is motion invariant for all the DEE of the type (3.1).

The 2-forms $\Omega$ and $\bar{\Omega}_0$ are the simplest from the hierarchies of simplectic forms, related to (3.1):

$$\Omega_m=\frac{1}{i}[\sigma_3\delta w \wedge M_+\Lambda_+^{m+1}\sigma_3\delta w],\quad \bar{\Omega}_m=\frac{1}{i}[\sigma_3\delta\bar{w} \wedge \bar{\Lambda}_+^m\sigma_3\delta\bar{w}]. \tag{3.11}$$

The compatibility of these forms is established by recalculating $\Omega_m$ and $\bar{\Omega}_m$ in terms of the scattering data variations. Using the expansion (2.3) and the relation (2.6), we obtain

$$\begin{aligned} i\bar{\Omega}_m=&\frac{i}{2\pi}\int_{S^1} dz\, z^{2m+1}\mathrm{tr}[\delta\sigma^+\alpha^+\wedge\delta\rho^+a^+-\delta\sigma^-\alpha^-\wedge\delta\rho^-a^-]-\\ &-2\sum_{\alpha=1}^{N}\{z_{\alpha+}^{2m+1}[\delta z_{\alpha+}\wedge\delta\zeta_\alpha^+ + \mathrm{tr}\,\delta\sigma_\alpha^+\wedge\delta\xi_\alpha^+]+\\ &+z_{\alpha-}^{2m+1}[\delta z_{\alpha-}\wedge\delta\zeta_\alpha^- + \mathrm{tr}\,\delta\sigma_\alpha^-\wedge\delta\xi_\alpha^-]\}, \end{aligned} \tag{3.12}$$

$$\delta\zeta_\alpha^\pm=\mathrm{tr}[\sigma_\alpha^\pm\dot{\alpha}_\alpha^\pm\delta\rho_\alpha^\pm\dot{a}_\alpha^\pm-\delta\sigma_\alpha^\pm\dot{\alpha}_\alpha^\pm\rho_\alpha^\pm\dot{a}_\alpha^\pm]$$

$$\delta\xi_\alpha^\pm=\alpha_\alpha^\pm(1-P_\alpha^\pm)\delta\rho_\alpha^\pm\dot{a}_\alpha^\pm+\dot{\alpha}_\alpha^\pm\delta\rho_\alpha^\pm a_\alpha^\pm(1-P_\alpha^\pm).$$

Here $\sigma^\pm$, $\sigma_\alpha^\pm$ are the elements of the equivalent to $\mathcal{T}$ set of scattering data $\mathcal{T}$ (see (2.13) in /14/); $\alpha^+,\alpha^-$ are the block-diagonal elements of the inverse transfer matrix $\hat{S}=\begin{pmatrix}\alpha^- & \beta^- \\ \beta^+ & \alpha^+\end{pmatrix}$. The projectors $P_\alpha^\pm$ and $\bar{P}_\alpha^\pm$ determine the degeneracy of $a^\pm(z)$ and $\alpha^\pm(z)$, resp., for $z=z_{\alpha\pm}\in\Delta$.

The calculation of $\Omega_m$ by using the corresponding expansion for $\sigma_3\delta w$ is analogous, and the answer is the same, only now we should insert in the r.h.s. of (3.12) the quantities $\tilde{\rho}^\pm$, $\tilde{\sigma}^\pm$, etc., related to the system (1.1). Comparing the definitions in §2 of /14/ we see that $\tilde{\rho}^\pm=\rho^\pm$, $\tilde{\sigma}^\pm=\sigma^+v_2$, $\tilde{\sigma}^-=\hat{v}_2\sigma^-$. Thus we obtain that $\Omega_m$ differs from $\bar{\Omega}_m$ by terms, con-

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* Analogical restrictions are known in the literature, see ref./18/.

taining under the trace sign the quantity $\delta v_2 \hat{v}_2$. Obviously if $v_2$ = const, then $\Omega_m = \bar{\Omega}_m$.

The Hamiltonians $H_f$ may be also expressed through the scattering data using the dispersion relation:

$$D(z) = \frac{1}{4\pi i} \oint_{S^1} \frac{d\zeta^2}{\zeta^2 - z^2} \ln \det(I - \rho^+ \rho^-) + \frac{1}{2} \sum_{\alpha=1}^{N} \ln \frac{z_{\alpha+}^2 - z^2}{z_{\alpha-}^2 - z^2}. \tag{3.13}$$

From (3.6), (3.5) and (3.13) we readily obtain:

$$H_f = -\frac{1}{4\pi i} \oint_{S^1} d\zeta^2 f(\zeta^2) \ln \det(I - \rho^+ \rho^-) - \sum_{\alpha=1}^{N} (g_{1,\alpha+} - g_{1,\alpha-}),$$

$$g_1(z^2) = \int^{z^2} ds\, f(s), \quad g_{1,\alpha+} = g_1(z_{\alpha+}^2), \quad g_{1,\alpha-} = g_1(z_{\alpha-}^2). \tag{3.14}$$

Let us also write down the four simplest conserved quantities calculated from (3.3):

$$D_0 = \ln \det v_2 = \sum_{n=-\infty}^{\infty} \mathrm{tr} \ln(I - q(n) r(n)) = -\sum_{n=-\infty}^{\infty} \mathrm{tr} \ln(I + \bar{q}(n) \bar{r}(n+1)),$$

$$D_1 = -\sum_{n=-\infty}^{\infty} \mathrm{tr}\, q(n) r(n-1) = -\sum_{n=-\infty}^{\infty} \mathrm{tr}\, \bar{q}(n) \bar{r}(n)$$

$$D_{-1} = -\sum_{n=-\infty}^{\infty} \mathrm{tr}\, q(n-1) r(n) = -\sum_{n=-\infty}^{\infty} \mathrm{tr}(\hat{\bar{h}}_1(n-1) \bar{q}(n-1) \bar{r}(n+1) \hat{\bar{h}}_1(n)),$$

$$D_2 = \sum_{n=-\infty}^{\infty} \mathrm{tr}[-q(n+1) h_2(n) r(n-1) + \frac{1}{2} (q(n) r(n-1))^2] =$$

$$= \sum_{n=-\infty}^{\infty} \mathrm{tr}[-\bar{q}(n+1) \bar{r}(n) + \frac{1}{2} (\bar{q}(n) \bar{r}(n))^2].$$

The Hamiltonian structure of the systems (1.5) and (1.6) is given by $(\Omega_0, H_{NLS})$ and $(\Omega_0, H_{NLS})$, where $H_{NLS} = D_0 + D_2 - 2D_1$; the dependence of $H_{NLS}$ on the scattering data is obtained from (3.14) for $f(z^2) = (2 - z^2 - z^{-2})$. The Hamiltonian structure for the discrete NLS eqs. obtained in [12,13] for the case $s = p = 1$ corresponds to the choice $(\Omega_{-1}, H'_{NLS})$, where $H'_{NLS} = -2D_0 + D_1 + D_{-1}$.

At the end let us note that the simplest reductions $q(n) = \pm r^+(n)$, $\bar{q}(n) = \pm \bar{r}^+(n)$ are incompatible with the dynamics of the systems (1.5) and (1.6). Instead we have found the reductions: a) $q(n)=\pm r^{+}(-n)$, $\bar{q}(n)=\pm r^{+}(-n)$ and b) in the case $s=p$, $q(n)=\pm r^{*}(n)$, $\bar{q}(n)=\pm r^{*}(n)$ which are compatible with (1.5) and (1.6) and lead to equations of NLS' type. The detailed analyses of the reduction problem/19/ for the DEE is out of the scope of this paper.

## APPENDIX

Here we shall derive eq. (2.11) from (2.8). In doing this we shall suppose, that the DEE (2.7b) can be interpreted as the compatibility condition of the linear systems (1.1) and

$$i\frac{d\psi(n,z)}{dt}=M_F(n,z)\psi(n,z)-\psi(n,z)C(z), \tag{A.1}$$

where $C(z)$ is a constant (i.e., time and $n$-independent) block diagonal matrix. The recurrent procedure for the construction of $M_F(n,z)$ is a trivial generalization of the one for the case $s=p=1$, see/2/. It steams out from the compatibility condition

$$i\frac{dL(n,z)}{dt}+L(n,z)M_F(n,z)-M_F(n+1,z)L(n,z)=0,$$
$$L(n,z)=Z+Q(n), \tag{A.2}$$

which should hold identically with respect to $z$. In what follows we shall make use only of the fact, that $M_F^{\pm}=\lim\limits_{n\to\pm\infty} M_F(n,z)$ should be block-diagonal matrices; this can be seen, e.g., from (A.2).

We shall obtain the time dependence of the scattering data by inserting the analytic solution, $\chi^{+},\chi^{-}$ in (A.1) and going to the limits $n\to\infty$, $n\to-\infty$. Using the definitions in §2 of/14/ we have

$$i\frac{dS^{-}}{dt}=M_F^{+}S^{-}-S^{-}C(z),$$
$$i\frac{dT^{+}}{dt}=M_F^{-}T^{+}-T^{+}C(z). \tag{A.3}$$

Now let us insert (2.8) in (A.3) remembering, that

$$S^{-}=\begin{pmatrix} a^{+} & 0\\ \rho^{+}a^{+} & 1\end{pmatrix},\quad T=\begin{pmatrix} 1 & \rho^{-}a^{-}\\ 0 & a^{-}\end{pmatrix}. \tag{A.4}$$

It is not difficult to see, that (A.3) and (2.8) are compatible if and only if

$$M_F^+ = M_F^- = C(z) = -F(z^2).$$

Thus we obtain

$$i\frac{dS^-}{dt} + [F(z^2), S^-] = 0, \quad i\frac{dT^+}{dt} + [F(z^2), T^+] = 0,$$

$$i\frac{da^+}{dt} + [F_1(z^2), a^+] = 0, \quad i\frac{da^-}{dt} + [F_2(z^2), a^-] = 0, \qquad (A.5)$$

which together with the "unitarity" condition (2.6) in ref./14/ gives us

$$i\frac{d\alpha^-}{dt} + [F_1(z^2), \alpha^-] = 0, \quad i\frac{d\alpha^+}{dt} + [F_2(z^2), \alpha^+] = 0. \qquad (A.6)$$

From (A.5), (A.6) and $\hat{T}^+ S^- = S^+ \hat{T}^-$ we obtain, that $S^+$ and $T^-$ also satisfy eq. (A.5), which together with $S = S^- \hat{S}^+$ gives us (2.11).

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