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BLOCK DISCRETE ZAKHAROV-SHABAT SYSTEM.

I. Generalized Fourier Expansions

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§1. In a number of important cases it has been proved, that the inverse scattering method (ISM)[1,2] is a generalized Fourier transform[3-9]. The proofs are based on the completeness relations of the "squared" solutions of the auxiliary linear problem. The classes of difference evolution equations solvable through the ISM for different discrete linear systems have been considered in[8,10-15].

In the present paper we obtain the completeness relation for the "squared" solutions of two equivalent linear problems. The first one is the block discrete Zakharov-Shabat system

$$\psi(n+1, z) = (Z + Q(n))\,\psi(n, z),$$

$$Z = \begin{pmatrix} z I_s & 0 \\ 0 & \frac{1}{z} I_p \end{pmatrix}, \quad Q(n) = \begin{pmatrix} 0 & q(n) \\ r(n) & 0 \end{pmatrix}, \qquad (1.1)$$

where $q(n)$ and $r^T(n)$ are rectangular $s\times p$ matrices. The second one is written as an eigenvalue problem

$$\left[\begin{pmatrix} D_+ & 0 \\ 0 & D_- \end{pmatrix} + \bar{Q}(n) - z\right]\bar{\psi}(n, z) = 0, \quad \bar{Q}(n) = \begin{pmatrix} 0, & -\bar{q}(n) \\ \bar{r}(n), & 0 \end{pmatrix}, \qquad (1.2)$$

where $D_\pm f(n) = f(n\pm 1)$ and is related to (1.1) by

$$\bar{q}(n) = q(n)\,\hat{v}_2(n), \quad \bar{r}(n) = v_2(n)\, r(n-1), \quad \hat{v}_2(n) \equiv v_2^{-1}(n),$$

$$\bar{\psi}(n, z) = \begin{pmatrix} I & 0 \\ 0 & v_2(n) \end{pmatrix}\psi(n, z), \quad v_2(n) = \prod_{k=\infty}^{n} (I - r(k)\, q(k)). \qquad (1.3)$$

In §2 we give the necessary facts from the direct and inverse scattering theory of the system (1.1) and (1.2). It is well known, that the inverse scattering problem for (1.1) is equivalent to a Riemanian problem with noncanonical normalization[16]. The inverse scattering problem for (1.2) is equivalent to a canonical Riemanian problem, whose solution is well known[1,15].

In §3, following the approach developed in [6-9] we obtain the expansions for the potentials $w(n) = \begin{pmatrix} q^T(n) \\ r(n) \end{pmatrix}$, $\bar{w}(n) = \begin{pmatrix} \bar{q}^T(n) \\ \bar{r}(n) \end{pmatrix}$ and their variations over the "squared" solutions of (1.1) and (1.2) resp. As coefficients in these expansions there appear the scattering data and their variations resp. We also derive the trace identities for the systems (1.1)-(1.2).

In the next paper [17] we apply these expansions to describe the classes of the difference evolution equations related to (1.1) and (1.2) and to prove their Hamiltonian structure.

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§2. Let us suppose that $\bar{Q}(n)$ (and $Q(n)$) tend to zero fast enough when $n \to \pm\infty$, and that $\det(1 - q(n)r(n)) \neq 0$ for all n. These suppositions ensure the existence and the analyticity properties of the Jost solutions for both (1.1) (see [11]) and (1.2) defined by

$$\lim_{n\to\infty} \psi(n,z)\, Z^{-n} = \lim_{n\to-\infty} \phi(n,z) Z^{-n} = 1,$$
$$\lim_{n\to\infty} \bar{\psi}(n,z)\, Z^{-n} = \lim_{n\to-\infty} \bar{\phi}(n,z)\, Z^{-n} = 1. \tag{2.1}$$

Let us introduce the block notations $\psi = \|\psi^-,\psi^+\|$, $\bar{\psi} = \|\bar{\psi}^-,\bar{\psi}^+\|$, $\phi = \|\phi^+,\phi^-\|$, $\bar{\phi} = \|\bar{\phi}^+,\bar{\phi}^-\|$, where the upper sign $(\pm)$ means analyticity with respect to z for $|z|>1$ ($|z|<1$). Let us consider the solutions $\chi^+ = \|\phi^+,\psi^+\|$, $\bar{\chi}^+ = \|\bar{\phi}^+,\bar{\psi}^+\|$, $\chi^- = \|\psi^-,\phi^-\|$, $\bar{\chi}^- = \|\bar{\psi}^-,\bar{\phi}^-\|$; from (1.3) there follows that $\chi^\pm$ and $\bar{\chi}^\pm$ are simply related to each other. The scattering data are introduced through

$$\bar{\chi}^+ = \bar{\psi}S^- = \bar{\phi}S^+, \qquad \bar{\chi}^- = \bar{\psi}T^+ = \bar{\phi}T^-,$$

$$S^-(z) = \begin{pmatrix} a^+ & 0 \\ b^+ & 1 \end{pmatrix}, \qquad S^+(z) = \begin{pmatrix} 1 & \beta^- \\ 0 & \alpha^+ \end{pmatrix}, \tag{2.2}$$

$$T^+(z) = \begin{pmatrix} 1 & b^- \\ 0 & a^- \end{pmatrix}, \qquad T^-(z) = \begin{pmatrix} \alpha^-, & 0 \\ \beta^+, & 1 \end{pmatrix}.$$

Obviously $S^-\hat{S}^+ = T^+\hat{T}^- = S(z)$, where by $\hat{X}$ here and in what follows we shall denote the matrix inverse to X, $\hat{X} \equiv X^{-1}$ and $S(z)$ is the transfer matrix of (1.2), $S(z) = \hat{\bar{\psi}}\bar{\phi}(n,z)$. The matrix-valued functions $a^{(\pm)}(z)$, $\alpha^{(\pm)}(z)$ are analytic in z for $|z|>1$ ($|z|<1$).

Below for simplicity we suppose that the discrete spectrum of (1.2) is finite and simple, i.e., that $\bar{\chi}^{\mp}(n,z)$ have only finite number of simple zeroes for $|z| \gtrless 1$. Let us denote the set of these zeroes by $\Delta = \Delta^{+} \cup \Delta^{+}$, $\Delta^{\pm} = \{z_{a\pm}, |z_{a\pm}| \gtrless 1, a=1,\dots,2N\}$ and note, that if $z_{a\pm} \in \Delta^{\pm}$ then $-z_{a\pm} \in \Delta^{\pm}$ also and $\bar{\chi}^{\pm}(n,-z_{a\pm}) = (-1)^{n} \sigma_3 \bar{\chi}^{\pm}(n,z_{a\pm}) \cdot \sigma_3$. Obviously to each eigenvalue $z_{a+}(z_{a-})$ we can relate two vectors $|c_{a+}\rangle, \langle d_{a+}|$ ($|c_{a-}\rangle, \langle d_{a-}|$) such, that

$$\bar{\chi}_a^{\pm}(n)|c_{a\pm}\rangle = 0, \quad \langle d_{a\pm}|\bar{\chi}_a^{\pm}(n) = 0, \quad \bar{\chi}_a^{\pm}(n) = \bar{\chi}^{\pm}(n, z_{a\pm}). \qquad (2.3)$$

The functions $a^{\pm}$ and their inverse $\hat{a}^{\pm}(z)$ in the vicinity of $z_{a\pm}$ have the form (see, e.g., ref. [15]):

$$a^{\pm}(z) = a_a^{\pm}(1 - P_a^{\pm}) + \dot{a}_a^{\pm}(z - z_{a\pm}) + O((z - z_{a\pm})^2),$$

$$\hat{a}^{\pm}(z) = \frac{P_a^{\pm} \hat{a}_a^{\pm}}{(z - z_{a\pm}) \langle d_{a\pm}^{\pm} | \dot{a}_a^{\pm} | c_{a\pm}^{\pm} \rangle} + \dot{\hat{a}}_a^{\pm} + O(z - z_{a\pm}), \qquad (2.4)$$

$$P_a^{\pm} = \frac{|c_{a\pm}^{\pm}\rangle\langle c_{a\pm}^{\pm}|}{\langle c_{a\pm}^{\pm} | c_{a\pm}^{\pm} \rangle},$$

where we have used the notations $|c_{a+}\rangle = \begin{pmatrix} |c_{a+}^{+}\rangle \\ |c_{a+}^{-}\rangle \end{pmatrix}$, $\langle d_{a+}| = (\langle d_{a+}^{+}|, \langle d_{a+}^{-}|)$, etc.; $a_a^{\pm}$ are nondegenerate constant matrices, and $\hat{a}_a^{\pm}$ their inverse. In (2.4) all the upper and lower sign indices should coincide, being simultaneously + or −. From (2.3) and (2.4) there follows, that $\langle d_{a\pm}^{\pm}| = \langle c_{a\pm}^{\pm}| a_a^{\pm}$. The Jost solutions for the discrete spectrum are related by:

$$\bar{\phi}_a^{+}(n)|c_{a+}^{+}\rangle = \bar{\psi}_a^{+}(n) b_a^{+} |c_{a+}^{+}\rangle, \quad \bar{\psi}_a^{-}(n) b_a^{-} |c_{a-}^{-}\rangle = \bar{\phi}_a^{-}(n)|c_{a-}^{-}\rangle, \qquad (2.5)$$

where $\bar{\psi}_a^{\pm}(n) = \bar{\psi}^{\pm}(n, z_{a\pm})$, $\bar{\phi}_a^{\pm}(n) = \bar{\phi}^{\pm}(n, z_{a\pm})$ and $b_a^{\pm}$ are constant matrices, determining the "norms" of the Jost solutions. The analogs of the unitary relation are conveniently written in the form:

$$\hat{a}^{-}\hat{a}^{+}(z) = 1 - \rho^{-}\rho^{+}(z), \quad \hat{a}^{+}\hat{a}^{-}(z) = 1 - \rho^{+}\rho^{-}(z), \qquad (2.6)$$

$$\rho^{\pm} = b^{\pm} \hat{a}^{\pm}(z), \quad |z| = 1.$$

The minimal set of scattering data is defined by

$$\mathcal{T} = \{\rho^{\pm}(z), \ |z| = 1, \ z_{a\pm}, \rho_a^{\pm}, \ |z_{a\pm}| \gtrless 1, \quad a = 1,\dots,N\}$$

$$\rho_a^{\pm} = \frac{b_a^{\pm} |c_{a\pm}^{\pm}\rangle\langle d_{a\pm}^{\pm}|}{\langle d_{a\pm}^{\pm} | \dot{a}_a^{\pm} | c_{a\pm}^{\pm} \rangle}. \qquad (2.7)$$

The inverse scattering problem for the system (1.1) can be reduced to a Riemanian problem with non-canonical normalization /18/. This is related to the fact, that the leading term of the asymptotics of the solutions $\chi^{\pm}$ of (1.1) for $z\to\infty$ and $z\to 0$, resp., depends on $Q(n)$. For the system (1.2) such difficulties do not appear, since

$$\bar{\chi}^{+}(n,z)\,Z^{-n} = I + \frac{1}{z}\overline{Q(n)} + O\left(\frac{1}{z^{2}}\right). \tag{2.8}$$

Thus the inverse scattering problem for the system (1.2) can be reduced to the Riemanian problem:

$$\begin{aligned} &Z^{n}\,\hat{\bar{\chi}}^{-}(n,z)\,\bar{\chi}^{+}(n,z)\,Z^{-n} = Z^{n}\,G(z)\,Z^{-n} \\ &G(z) = \hat{T}^{+}(z)\,S^{-}(z), \qquad \lim_{z\to\infty}\bar{\chi}^{+}(n,z)\,Z^{-n} = I. \end{aligned} \tag{2.9}$$

The corresponding potential $\bar{Q}(n)$ is recovered from the solution $\bar{\chi}^{+}(n,z)$ of (2.9) by the formula

$$\bar{Q}(n) = \frac{1}{2}\lim_{z\to\infty} z\sigma_3\,[\sigma_3\,,\bar{\chi}^{+}(n,z)\,Z^{-n}], \tag{2.10}$$

which follows directly from (2.8). In order to solve the Riemanian problem (2.9) we shall use the contour integration method. Let us apply it to the integral

$$\oint_{\gamma_+}\frac{d\zeta}{\zeta-z}\,\bar{\psi}^{+}(n,\zeta)\,\zeta^{n} - \oint_{\gamma_-}\frac{d\zeta}{\zeta-z}\,\bar{\phi}^{-}(n,\zeta)\,\hat{a}^{-}(\zeta)\,\zeta^{n},$$

where the contours $\gamma_+ = S^1\cup\bar{S}^{\infty}$, $\gamma_- = S^1\cup\bar{S}^{\circ}$, S^1 being the positively oriented unit circle; and $\bar{S}^{\infty}$, $\bar{S}^{\circ}$, the negatively oriented circles with infinitely large and infinitely small radii, respectively. Taking into consideration formulae (2.2), (2.4), (2.5) and (2.8) we arrive at the following representation for $\psi^{+}(n,z)$, $|z|>1$,

$$\begin{aligned} \bar{\psi}^{+}(n,z)\,z^{n} = \begin{pmatrix}0\\ I\end{pmatrix} - \sum_{\alpha=1}^{N}\left(\frac{\bar{\psi}^{-}_{\alpha}(n)}{z_{\alpha-}-z} - \frac{\sigma_3\bar{\psi}^{-}_{\alpha}(n)}{z_{\alpha-}+z}\right)\rho^{-}_{\alpha}\,z^{n}_{\alpha-} + \\ + \frac{1}{2\pi i}\oint_{S^1} d\zeta\left(\frac{\bar{\psi}^{-}(n,\zeta)}{\zeta-z} - \frac{\sigma_3\bar{\psi}^{-}(n,\zeta)}{\zeta+z}\right)\rho^{-}(\zeta)\,\zeta^{n}. \end{aligned} \tag{2.11}$$

Analogically we obtain the representation for $\bar{\psi}^{-}(n,z)$, $|z|<1$

$$\begin{aligned} \bar{\psi}^{-}(n,z)\,z^{-n} = \begin{pmatrix}I\\ 0\end{pmatrix} - \sum_{\alpha=1}^{N}\left(\frac{\bar{\psi}^{+}_{\alpha}(n)}{(z_{\alpha+}-z)} + \frac{\sigma_3\bar{\psi}^{+}_{\alpha}(n)}{z_{\alpha+}+z}\right)\rho^{+}_{\alpha}\,z^{-n}_{\alpha+} - \\ - \frac{1}{2\pi i}\oint_{S^1} d\zeta\left[\frac{\bar{\psi}^{+}(n,\zeta)}{\zeta-z} + \frac{\sigma_3\bar{\psi}^{+}(n,\zeta)}{\zeta+z}\right]\rho^{+}(\zeta)\,\zeta^{-n}. \end{aligned} \tag{2.12}$$

The representations (2.11) and (2.12) allow, starting from the set of scattering data $\mathfrak{T}$ (2.7), to construct a system of singular integral equations for the quantities $\bar{\psi}_{a}^{\pm}(n)$, $\bar{\psi}^{\pm}(n,\zeta)$, $\zeta \in S^1$. By solving this system we are able to reconstruct uniquely the Jost solutions $\bar{\psi}^{+}$, $(\bar{\psi}^{-})$ for all z, $|z| > 1$ $(|z| < 1$. The transfer matrix $S(z)$ and its triangle facrorizations $S^{\pm}$, $T^{\pm}$ can be calculated from the asymptotics of $\bar{\psi}^{\pm}(n,z)_{-+}$for $n \to \pm\infty$ from them we obtain also the Jost solutions $\bar{\phi}^{-}(n,z)$, see formulae (2.2). At last the potential $\bar{Q}(n)$, corresponding to the given set $\mathfrak{T}$ is recovered from $\bar{\chi}^{\pm}(n,z)$ according to (2.10).

This procedure can be performed explicitly in the simple case, when $\rho^{\pm}(z) = 0$, $|z| = 1$. Then from (2.11) and (2.12) we obtain an algebraical system of equations, which is explicitly soluble. Here we give only the answer for the simplest reflectionless potential of the system (1.2).

$$\bar{q}(n) = -\frac{2z_{1-}^{2n}}{1+r_n}\rho_1^{-}, \quad \bar{r}(n) = \frac{2z_{1+}^{-2n}}{1+r_{n+1}}\rho_1^{+},$$

$$r_n = \frac{4z_{1+}^{2}\omega^{2n}\operatorname{tr}\rho_1^{-}\rho_1^{+}}{(z_{1-}^{2}-z_{1+}^{2})^{2}}, \quad \omega = \frac{z_{1-}}{z_{1+}}$$

corresponding to the following set of scattering data

$$\{ z_{1+}, \ z_{1-}, \rho_1^{+}, \rho_1^{-}, \ N = 1 \}.$$

Representations analogical to (2.11) and (2.12) can be derived also for $\bar{\phi}^{+}(n,z)$ and $\bar{\phi}^{-}(n,z)$ They allow one to reconstruct $\bar{\phi}^{\pm}(n,z)$ starting from another, equivalent to the set of scattering data

$$\tilde{\mathfrak{T}} \equiv \{\sigma^{\pm}(z), \ |z| = 1, \ z_{a\pm}, \sigma_a^{\pm}, \ |z_{a\pm}| \gtrless 1, \quad a = 1,\dots,N\},$$

$$\sigma^{\pm} = \beta^{\mp}\hat{a}^{\pm}(z), \quad \sigma_a^{\pm} = \frac{\beta_a^{\mp}|c_{a\pm}^{\mp}\rangle\langle d_{a\pm}^{\mp}|}{\langle d_{a\pm}^{\mp}|\dot{a}_a^{\pm}|c_{a\pm}^{\mp}\rangle}.$$

At last note, that introducing the transformation operator K by $\bar{\psi}(n,z) = \sum_{m=n}^{\infty} K(n,m) Z^{m}$ from (2.11) and (2.12) there follows the Gel'fand-Levitan-Marchenko equation for the system (1.2). The derivation is analogous to the one in ref. /15/ and we omit it.

§3. Let us go now to the expansions over the "squared" solutions of (1.1) and (1.2). Their importance is determined by the fact, that they allow one to perform explicitly the transition from the potentials $Q(n)$, $\bar{Q}(n)$ to the scattering data $\mathfrak{T}$ (2.7). Thus it is possible to prove, that the IST is a generalized Fourier transform. As a guiding tool in determining the "squared" solutions of (1.1) we shall use the relations

$$\hat{\chi}^{\pm}(n,z)A\chi^{\pm}(n,z)\Big|_{n=-\infty}^{\infty} = \sum_{n=-\infty}^{\infty}\hat{\chi}^{\pm}(n+1,z)[A,Q(n)]\chi^{\pm}(n,z),$$
$$\hat{\chi}^{\pm}(n,z)\delta\chi^{\pm}(n,z)\Big|_{n=-\infty}^{\infty} = \sum_{n=-\infty}^{\infty}\hat{\chi}^{\pm}(n+1,z)\delta Q(n)\chi^{\pm}(n,z), \qquad (3.1)$$

which easily follow from (1.1); here $A=\begin{pmatrix}A_1 & 0\\ 0 & A_2\end{pmatrix}$ is a constant block-diagonal matrix. The l.h.s. of (3.1) is easily expressed through the scattering data and their variations by using (2.2). The r.h.s. of (3.1) we shall rewrite so, that it could be interpreted as coefficients of the generalized Fourier transform of $[A,Q]$ and δQ over the "squared" solutions of (1.1); thus it will become clear now to define the "squares". For this let us introduce in the space $\mathfrak{L}_\Lambda$ of block $2s\times p$ fast decreasing* sequences ($\mathfrak{L}_\Lambda \ni X(n)$, $X^T=(\overset{(1)}{X}{}^T,\overset{(2)}{X}{}^T)$, $\overset{(1)}{X}{}^T,\overset{(2)}{X}{}^T$ – $s\times p$ matrices), the skew-scalar product:

$$[X,Y]=\sum_{n=-\infty}^{\infty}\operatorname{tr}\tilde{X}(n)Y(n),\quad \tilde{X}=(\overset{(2)}{X}{}^T,-\overset{(1)}{X}{}^T). \qquad (3.2)$$

Then the i,j-th matrix element of the r.h.s. of (3.1) can be written in the form $[w_A, X^{\pm}_{ji}]$ and $[\sigma_3\delta w, X^{\pm}_{ji}]$ resp., where

$$w_A(n)=\begin{pmatrix}(A_1q(n)-q(n)A_2)^T\\ r(n)A_1-A_2r(n)\end{pmatrix},\quad \sigma_3\delta w(n)=\begin{pmatrix}\delta q^T(n)\\ -\delta r(n)\end{pmatrix} \qquad (3.3)$$

$$X^{\pm}_{ji}(n,z)=\chi^{\pm}_j(n,z)\circ\hat{\chi}^{\pm}_i(n+1,z),$$
$$\chi_j(n,z)\circ\chi_i(m,z) \overset{\text{def}}{=} \begin{pmatrix}(\overset{(1)}{\chi}_j(n)\,\overset{(1)}{\hat{\chi}}_i(m))\\ \overset{(2)}{\chi}_j(n)\,\overset{(2)}{\hat{\chi}}_i(m)\end{pmatrix},\quad \chi_j(n)=\begin{pmatrix}\overset{(1)}{\chi}_j(n)\\ \overset{(2)}{\chi}_j(n)\end{pmatrix},$$
$$\hat{\chi}_i(m)=(\overset{(2)}{\hat{\chi}}_i(m),\ \overset{(1)}{\hat{\chi}}_i(m)). \qquad (3.4)$$

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*It is enough if the elements $X(n)$ decrease like $c^{|n|}$ for $n\to\pm\infty$, where c const, $|c|<1$.

By $\chi_j^{\pm}(n,z)$ in (3.4) we have denoted the j-th column of the solution $\chi^{\pm}(n,z)$ of (1.1) and $\hat{\chi}_i^{\pm}(n)$ is the i-th row of $\tilde{\chi}^{\pm}(n,z)$.

Thus the question of the applicability of the ISM is reduced to the question of the completeness of the system $\{X_{ji}^{\pm}\}$. Following[6-9] let us apply the contour integration method to the integral $\oint_{\gamma_+} \frac{d\zeta}{\zeta(\zeta-z)} G^+(n,m,\zeta) - \oint_{\gamma_-} \frac{d\zeta}{\zeta(\zeta-z)} G^-(n,m,\zeta)$, where

$$G^{\binom{+}{-}}(n,m,z) = \sum_{\substack{i<j\\(i>j)}} G_{ji}^{\binom{+}{-}}(n,m,z)\,\theta(n-m) - \sum_{\substack{i\geq j\\(i\leq j)}} G_{ij}^{\binom{+}{-}}(n,m,z)\,\theta(m-n-1),$$

$$G_{ji}^{\pm}(n,m,z) = Y_{ji}^{\pm}(n,z) \circledast \tilde{X}_{ij}^{\pm}(m,z), \quad Y_{ji}^{\pm}(n,z) = \chi_j^{\pm}(n+1,z) \circ \hat{\chi}_i^{\pm}(n,z).$$

By $\sum_{i<j}$ and $\sum_{i=j}$ here we mean $\sum_{i=1}^{s} \sum_{j=s+1}^{s+p}$ and $\sum_{i,j=1}^{s} + \sum_{i,j=s+1}^{s+p}$, respectively. Omitting the calculational details we give here two equivalent variants of this completeness relation:

$$\delta(n-m)E = \frac{i}{2\pi} \oint_{S^1} \frac{dz}{z} \sum_{i<j} [Y_{ji}^{+}(n) \circledast \tilde{X}_{ij}^{+}(m) - Y_{ij}^{-}(m) \circledast \tilde{X}_{ji}^{-}(m)] -$$

$$-2\sum_{\alpha=1}^{N} \sum_{i<j} \{\mathcal{R}(\tfrac{1}{z} Y_{ji}^{+}(n) \circledast \tilde{X}_{ij}^{+}(m))|_{z=z_{\alpha+}} + \mathcal{R}(\tfrac{1}{z} Y_{ij}^{-}(n) \circledast \tilde{X}_{ji}^{-}(m)|_{z=z_{\alpha-}}\}, \quad (3.5)$$

$$\delta(n-m)E = \frac{i}{2\pi} \oint_{S^1} \frac{dz}{z} \sum_{i<j} \{V_{ji}^{+}(n) \circledast \tilde{V}_{ij}^{+}(m) - V_{ij}^{-}(n) \circledast \tilde{V}_{ji}^{-}(m)] - \quad (3.6)$$

$$-2\sum_{\alpha=1}^{N} \sum_{i<j} \{\mathcal{R}(\tfrac{1}{z} V_{ji}^{+}(n) \circledast \tilde{\tilde{V}}_{ij}^{+}(m)|_{z=z_{\alpha+}} + \mathcal{R}(\tfrac{1}{z} V_{ij}^{-}(n) \circledast \tilde{\tilde{V}}_{ji}^{-}(m))|_{z=z_{\alpha-}}\},$$

where

$$\mathcal{R}(\tfrac{1}{z} X^{\pm}(n,z) \circledast \tilde{Y}^{\pm}(m,z)|_{z_{\alpha\pm}} = \lim_{z\to z_{\alpha\pm}} \frac{d}{dz}[\frac{(z-z_{\alpha\pm})^2}{z} X^{\pm}(n,z) \circledast \tilde{Y}^{\pm}(m,z)],$$

$$U_{ij}^{\pm}(n,z) = \bar{\chi}_i^{\pm}(n,z) \circ \hat{\bar{\chi}}_j^{\pm}(n,z), \quad V_{ji}^{+}(n) = \sum_{k=1}^{s} U_{jk}^{+}(n)\, a_{ik}^{+}(z), \quad (3.7)$$

$$\tilde{V}_{ij}^{+}(n) = \sum_{k=1}^{s} U_{kj}^{+} \hat{a}_{ki}^{+}(z), \quad V_{ij}^{-}(n) = \sum_{k=s+1}^{s+p} U_{ik}^{-} a_{kj}^{-}(z), \quad \bar{V}_{ji}^{-} = \sum_{k=s+1}^{s+p} U_{ki}^{-}(n)\, \hat{a}_{jk}^{-}, \quad i<j$$

If we write down the matrix indeces in the direct product $(Y \circledast \tilde{X})_{\sigma\tau,\gamma\delta} = Y_{\sigma\tau} \tilde{X}_{\gamma\delta}$ then $E_{\sigma\tau,\gamma\delta} = \delta_{\sigma\delta}\,\delta_{\tau\gamma}$. In order to make more explicit the R-operation in (3.5) and (3.6) it is convenient to introduce the expansions of $X^{\pm}$, $Y^{\pm}$, $U^{\pm}$, $V^{\pm}$ in the vicinity of $z_{\alpha\pm}$; from (3.7) and from the simplicity of the discrete spectrum of (1.1) there follows, that

$$Y^{\pm}(n,z) = \frac{1}{z - z_{\alpha\pm}} Y^{(\alpha)\pm} + \dot{Y}^{(\alpha)\pm} + O(z - z_{\alpha\pm}) \tag{3.8a}$$

and analogically for $X^{\pm}$ and $U^{\pm}$. As regards $V^{\pm}$ and $\bar{V}^{\pm}$ from (3.7) and (2.4) we have

$$\begin{aligned} V^{\pm}(n,z) &= V^{(\alpha)\pm}(n) + \dot{V}^{(\alpha)\pm}(z - z_{\alpha\pm}) + O((z - z_{\alpha\pm})^2), \\ \bar{V}^{\pm}(n,z) &= \frac{1}{(z - z_{\alpha\pm})^2}\bar{V}^{(\alpha)\pm}(n) + \frac{1}{z - z_{\alpha\pm}}\dot{\bar{V}}^{(\alpha)\pm}(n) + o(1). \end{aligned} \tag{3.8b}$$

From (3.1) and (3.5) there follow the expansions for $w_A(n)$ and $\sigma_3 \delta w(n)$ over the systems of functions $\{Y_{ij}^{\pm}\}$. Indeed, let us multiply $E_{\sigma\tau,\gamma\delta}\delta(n-m)$ in the l.h.s. of (3.5) by $(w_A)_{\delta\gamma}$ (or by $(\sigma_3 \delta w)_{\delta\gamma}$)) and sum over $1 \leq \delta, \gamma \leq s+p$, $-\infty < m < \infty$. Then the l.h.s. of (3.5) becomes equal to $(w_A)_{\sigma\tau}$ (or $(\sigma_3 \delta w)_{\sigma\tau}$) and the r.h.s. of (3.5) gives us the necessary expansion over $\{Y_{ij}^{\pm}\}$. The expansion coefficients are given by $[X_{ij}^{\pm}, w_A]$ and $[X_{ij}^{\pm}, \sigma_3 \delta w]$, resp., and can easily be expressed through the scattering data of (1.1) by means of relations (3.1). Analogical expansions can be derived for $\bar{w}(n)$ and $\sigma_3 \delta\bar{w}$ starting from (3.6) and using

$$(\hat{\bar{\chi}}^{\pm}(n,z) A \bar{\chi}^{\pm}(n,z))_{ij}\Big|_{n=-\infty}^{\infty} = [U_{ji}^{\pm}, \bar{w}_A], \quad (\hat{\bar{\chi}}^{\pm}(n,z)\delta\bar{\chi}^{\pm}(n,z))_{ij}\Big|_{n=-\infty}^{\infty} = [U_{ji}^{\pm}, \sigma_3 \delta\bar{w}],$$

which follow from (1.2). Thus we obtain:

$$\bar{w}_A(n) = -\frac{i}{2\pi}\oint_{S^1} dz \sum_{i<j} [V_{ji}^{+}(n,z)(\rho_A^{+})_{ji} - V_{ij}^{-}(n,z)(\rho_A^{-})_{ij}] +$$

$$+ 2\sum_{\alpha=1}^{N}\sum_{i<j} [V_{ji}^{(\alpha)+}(n)(\rho_{\alpha,A}^{+})_{ji} + V_{ij}^{(\alpha)-}(n)(\rho_{\alpha,A}^{-})_{ij}],$$

$$\rho_A^{+} = \rho^{+}A_1 - A_2\rho^{+}, \quad \rho_A^{-} = \rho^{-}A_2 - A_1\rho^{-};$$

$$\rho_{\alpha,A}^{+} = \rho_{\alpha}^{+}A_1 - A_2\rho_{\alpha}^{+}; \quad \rho_{\alpha,A}^{-} = \rho_{\alpha}^{-}A_2 - A_1\rho_{\alpha}^{-}; \tag{3.9}$$

$$\sigma_3\delta\bar{w}(n) = \frac{i}{2\pi}\oint_{S^1} dz \sum_{i<j} [V_{ji}^{+}(n,z)(\delta\rho^{+})_{ji} - V_{ij}^{-}(n,z)(\delta\rho^{-})_{ij}] -$$

$$- 2 \sum_{a=1}^{N} \sum_{i<j} \{ V_{ij}^{(a)+}(n)\,(\delta\rho_a^+)_{ji} + \dot{V}_{ji}^{(a)+}(n)\,(\rho_a^+)_{ji}\,\delta z_{a+} +$$

$$+ V_{ij}^{(a)-}(n)\,(\delta\rho_a^-)_{ij} + \dot{V}_{ij}^{(a)-}(n)\,(\rho_a^-)_{ij}\,\delta z_{a-} \}. \qquad (3.10)$$

From (3.9) and (3.10) the interpretation of the ISM as a generalized Fourier transform is obvious. As an analog of the exponent one may choose any of the systems $\{Y^{\pm}\}$, $\{X^{\pm}\}$, $\{V^{\pm}\}$, $\{\bar{V}^{\pm}\}$. The corresponding analogs of the differentiation operator are the nonlocal operators $\Lambda_{\pm}$, $\bar{\Lambda}_{\pm}$, $Л_{\pm}$ defined by the relations:

$$(\Lambda_{\pm} - z^2) Y_{ji}^{\pm} = (\Lambda_{\pm} - z^2) Y_{ij}^{\mp} = 0,$$

$$(\bar{\Lambda}_{\pm} - z^2) V_{ji}^{\pm} = (\bar{\Lambda}_{\pm} - z^2) V_{ij}^{\mp} = 0, \qquad (3.11)$$

$$(Л_{\pm} - z^2) X_{ji}^{\pm} = (Л_{\pm} - z^2) X_{ij}^{\mp} = 0, \quad i<j.$$

They may be represented in the form

$$\Lambda_{\pm} = \bar{\Lambda}_2^{\pm} \bar{\Lambda}_1^{\pm}, \quad \bar{\Lambda}_{\pm} = \Lambda_1^{\pm} \Lambda_2^{\pm} = \bar{\Lambda}_1^{\pm} \bar{\Lambda}_2^{\pm}, \quad Л_{\pm} = \Lambda_2^{\pm} \Lambda_1^{\pm}, \qquad (3.12)$$

where the operators $\Lambda_i^{\pm}$, $\bar{\Lambda}_i^{\pm}$ are defined by

$$\Lambda_1^{\pm} X_{ji}^{\pm} = z U_{ji}^{\pm}, \quad \Lambda_2^{\pm} U_{ji}^{\pm} = z X_{ji}^{\pm},$$

$$\bar{\Lambda}_1^{\pm} Y_{ji}^{\pm} = z U_{ji}^{\pm}, \quad \bar{\Lambda}_2^{\pm} U_{ji}^{\pm} = z Y_{ji}^{\pm}, \quad i<j, \qquad (3.13)$$

and are written explicitly in the appendix. As a domain of definition for the operators $\Lambda_i^{\pm}$, $\bar{\Lambda}_i^{\pm}$ etc. we shall choose the space $\mathfrak{X}_{\Lambda}$; from the subnote on p.6 we see, that if $X \in \mathfrak{X}_{\Lambda}$ then $\Lambda_i^{\pm} X \in \mathfrak{X}_{\Lambda}$ and $\bar{\Lambda}_i^{\pm} X \in \mathfrak{X}_{\Lambda}$. With respect to the skew-scalar product (3.2) the operators $\Lambda_i^{\pm}$, $\bar{\Lambda}_i^{\pm}$, $\Lambda_{\pm}$, $\bar{\Lambda}_{\pm}$, $Л_{\pm}$ satisfy conjugation-like relations:

$$[X, Л_{\pm} Y] = [\Lambda_{\mp} X, Y], \quad [\bar{\Lambda}_{\pm} X, Y] = [X, \Lambda_{\mp} Y],$$

$$[X, \Lambda_i^{\pm} Y] = [\bar{\Lambda}_{3-i}^{\mp} X, Y], \quad i = 1,2, \; X, Y \in \mathfrak{D}_{\Lambda}. \qquad (3.14)$$

(3.14) may be verified either directly, or by using the relations (3.5), (3.6) and (3.11-12). In greater detail the spectral theory of the operators $\Lambda_{\pm}$, $\bar{\Lambda}_{\pm}$, $Л_{\pm}$ will be considered in another paper.

At the end of this paper let us derive the so-called trace identities for the linear problems (1.1) and (1.2). Let us consider the quantity

$$D(z) = \begin{cases} \ln\det a^{+}(z)\,, & |z| > 1\,, \\ -\ln\det(a^{-}(z)\,v_2)\,, & |z| < 1\,, \end{cases} \tag{3.15}$$

where $v_2 = \lim_{n\to-\infty} v_2(n)$, see (1.3). It is not difficult to see,that $z\frac{d}{dz}D(z)$ can be rewritten in the form:

$$z\frac{dD}{dz} = -\sum_{n=-\infty}^{\infty}\sum_{k=n}^{\infty}\,[P(k+1) + P(k-1) - 2P(k)]\,,$$

$$P(k,z) = \begin{cases} P^{+}(k,z)\,, & |z| > 1 \\ P^{-}(k,z)\,, & |z| < 1 \end{cases} \qquad P^{\pm}(k,z) = \operatorname{tr}[\hat{\chi}^{\pm}(k,z)\,\dot{\chi}^{\pm}(k,z)\,\sigma_3 - n\sigma_3]\,. \tag{3.16}$$

Using (1.1) the r.h.s. of (3.16) can be cast into

$$2\sum_{n=-\infty}^{\infty}\sum_{k=n}^{\infty}\operatorname{tr}\tilde{w}(k)\,H(k,z)\,,\quad H(k,z) = \sum_{\sigma=1}^{s} X^{\pm}_{\sigma\sigma}(k,z)\,,\quad \text{for } |z| \gtrless 1\,.$$

If we now apply the contour integration method to the integral

$$\oint_{\gamma_+}\frac{d\zeta^2}{\zeta^2 - z^2}H^{+}(n,\zeta) - \oint_{\gamma_-}\frac{d\zeta^2}{\zeta^2 - z^2}H^{-}(n,\zeta),$$

where the contours $\gamma_\pm$ are introduced in §2, we obtain for

$$H^{+}(n,z) = -M_{+}\Lambda_{+}(\Lambda_{+} - z^2)^{-1}\,w(n) = -\Lambda_2^{+}(\bar{\Lambda}_{+} - z^2)\,\bar{w}(n)\,,$$

$$M_{+} = \hat{\Lambda}_1^{+}\,\bar{\Lambda}_1^{+} = \Lambda_2^{+}\,\hat{\bar{\Lambda}}_2^{+}\,. \tag{3.17}$$

$H^{-}(n,z)$ also equals the r.h.s. of (3.17) for $|z|<1$. Finally we have

$$z\frac{dD}{dz} = -2\sum_{n=-\infty}^{\infty}\sum_{k=n}^{\infty}\operatorname{tr}\tilde{w}(k)\,M_{+}\Lambda_{+}(\Lambda_{+} - z^2)^{-1}\,w(k) =$$

$$= -\sum_{n=-\infty}^{\infty}\Big\{\sum_{k=n}^{\infty} 2\operatorname{tr}\tilde{\bar{w}}(k)\,(\bar{\Lambda}_{+} - z^2)^{-1}\,\bar{w}(k) + \operatorname{tr}\widetrue{\sigma_3\bar{w}}(n)\,(\bar{\Lambda}_{+} - z^2)^{-1}\,w(n)\Big\}. \tag{3.18}$$

To obtain the last line of (3.18) we have used the explicit form of the operators $\Lambda_1^{\pm}$. Quite analogously it is proved, that

$$\delta D(z) = \sum_{n=-\infty}^{\infty}\operatorname{tr}[\widetilde{\sigma_3\delta w}(n)\,H(n,z)] \equiv [\sigma_3\delta w,\ H(n,z)]\,. \tag{3.19}$$

Inserting (3.17) in the r.h.s. of (3.19) and using (3.14) we get

$$\delta D(z) = [\sigma_3 \delta w, M_+ \Lambda_+ (\Lambda_+ - z^2)^{-1} w] = [\sigma_3 \delta \bar{w}, (\bar{\Lambda}_+ - z^2)^{-1} \bar{w}]. \qquad (3.20)$$

The dependence of $D(z)$ on the scattering data $\mathfrak{T}$ is given by the dispersion relation

$$D(z) = \frac{1}{4\pi i} \int_{S^1} \frac{d\zeta^2}{\zeta^2 - z^2} \ln \det(I - \rho^+ \rho^-) + \sum_{a=1}^{N} \ln \frac{z_{a+}^2 - z^2}{z_{a-}^2 - z^2}. \qquad (3.21)$$

Thus the coefficients in the asymptotic expansions of $D(z)$

$$D(z) = \sum_{k=0}^{\infty} D_{-k} z^{2k}, \; |z| \ll 1; \quad D(z) = -\sum_{k=1}^{\infty} D_k z^{-2k}, \; |z| \gg 1 \qquad (3.22)$$

can be expressed both as functionals of the potential $w(n)$ (or $\bar{w}$) (see (3.18)) and as functionals of the scattering data $\mathfrak{T}$, see (3.21). Equating both expressions for D_k we obtain the so-called trace identities for the systems (1.1) and(1.2).

APPENDIX

Here we shall write down explicitly the operators $\Lambda_1^{\pm}$, $\bar{\Lambda}_1^{\pm}$ and their inverse. Below we shall use the notations $\Sigma_n^+ \equiv \sum_{k=n}^{\infty}$, $\Sigma_n^- = \sum_{k=-\infty}^{n-1}$,

$$v_1(n) = \prod_{k=\infty}^{n} h_1(k), \quad v_2(n) = \prod_{k=\infty}^{n} h_2(k), \quad h_1(k) = I - q(k) r(k),$$

$$h_2(k) = I - r(k) q(k)$$

and X, Y, $U \in \mathfrak{L}_\Lambda$ should be considered as arbitrary elements of $\mathfrak{L}_\Lambda$.The possibility to invert the relations (1.3), namely

$$q(n) = \bar{q}(n) \hat{\bar{v}}_2(n), \quad r(n) = \bar{v}_2(n+1) \hat{\bar{r}}(n+1), \quad \bar{v}_1(n) = \prod_{k=\infty}^{n} \bar{h}_1(n),$$

$$\bar{h}_1(k) = I + \bar{q}(k) \bar{r}(k+1), \quad \bar{h}_2(k) = I + \bar{r}(k+1) \bar{q}(k)$$

allows one to express $\Lambda_1^{\pm}$, $\bar{\Lambda}_1^{\pm}$ in terms of q, r (or $\bar{q}, \bar{r}$) only, which we will not do here.

$$\begin{cases} \Lambda_1^{\pm} X(n) = \begin{pmatrix} \hat{v}_2^T(n+1)\; X_1(n) \\ v_2(n-1)\; X_2(n-1) \end{pmatrix} \mp \begin{pmatrix} \bar{q}^T(n)\; K_1^{\pm T}(n) \\ \bar{r}(n)\; K_1^{\pm}(n) \end{pmatrix}; \\ K_1^{\pm}(n) = \Sigma_n^{\pm} [q(k) X_2(k) - X_1^T(k) r(k)], \end{cases}$$

$$\begin{cases} \Lambda_2^{\pm} U(n) = \begin{pmatrix} v_2^T(n+1) & U_1(n+1) \\ \hat{v}_2(n) & U_2(n) \end{pmatrix} \mp \begin{pmatrix} v_2^T(n+1) & K_2^{\pm T}(n)\,\bar{q}^T(n) \\ \hat{v}_2(n) & K_2^{\pm}(n)\,\bar{r}(n+1) \end{pmatrix}, \\ K_2^{\pm}(n) = \Sigma_{n+1}^{\pm}[U_2(k)\,\bar{q}(k) - \bar{r}(k)\,U_1^T(k)], \end{cases}$$

$$\begin{cases} \bar{\Lambda}_1^{\pm} Y(n) = \begin{pmatrix} \hat{v}_2^T(n) & Y_1(n) \\ v_2(n) & Y_2(n-1) \end{pmatrix} \mp \begin{pmatrix} K_3^{\pm T}(n)\,\bar{q}^T(n) \\ K_3^{\pm}(n)\,\bar{r}(n) \end{pmatrix}, \\ K_3^{\pm}(n) = \Sigma_n^{\pm}[v_2(k+1)\,Y_2(k)\,\bar{q}(k) - \bar{r}(k+1)\,Y_1^T(k)\,\hat{v}_2(k)], \end{cases}$$

$$\begin{cases} \bar{\Lambda}_2^{\pm} U(n) = \begin{pmatrix} v_2^T(n) & U_1(n+1) \\ \hat{v}_2(n+1) & U_2(n) \end{pmatrix} \mp \begin{pmatrix} q^T(n)\,K_4^{\pm T}(n) \\ r(n)\,K_4^{\pm}(n) \end{pmatrix} \\ K_4^{\pm}(n) = \Sigma_{n+1}^{\pm}[\bar{q}(k)\,U_2(k) - U_1^T(k)\,\bar{r}(k)], \end{cases}$$

$$\begin{cases} \hat{\Lambda}_1^{\pm} U(n) = \begin{pmatrix} v_2^T(n) & U_1(n) \\ \hat{v}_2(n+1) & U_2(n+1) \end{pmatrix} \pm \begin{pmatrix} q^T(n)\,v_1^T(n+1)\,K_5^{\pm T}(n)\,\hat{v}_1^T(n) \\ r(n)\,\hat{v}_1(n)\,K_5^{\pm}(n)\,v_1(n+1) \end{pmatrix} \\ K_5^{\pm}(n) = \Sigma_{n+1}^{\pm} v_1(k)\,[q(k-1)\,\hat{v}_2(k)\,U_2(k) - U_1^T(k)\,v_2(k)\,r(k)]\,\hat{v}_1(k), \end{cases}$$

$$\begin{cases} \hat{\Lambda}_2^{\pm} X(n) = \begin{pmatrix} v_2^T(n) & X_1(n-1) & h_1^T(n-1) \\ v_2(n) & X_2(n) & h_1(n) \end{pmatrix} \pm \begin{pmatrix} \hat{v}_2^T(n)\,K_6^{\pm T}(n)\,q^T(n-1) \\ v_2(n)\,K_6^{\pm}(n)\,r(n) \end{pmatrix} \\ K_6^{\pm}(n) = \Sigma_n^{\pm}[X_2(k)\,q(k) - r(k)\,X_1^T(k)], \end{cases}$$

$$\begin{cases} \hat{\bar{\Lambda}}^{\pm} U(n) = \begin{pmatrix} v_2^T(n) & U_1(n)\,h_1^T(n) \\ \hat{v}_2(n+1) & U_2(n+1)\,h_1(n) \end{pmatrix} \pm \begin{pmatrix} K_7^{\pm T}(n) & q^T(n) \\ K_7^{\pm}(n) & r(n) \end{pmatrix} \\ K_7^{\pm}(n) = \Sigma_{n+1}^{\pm}[\hat{v}_2(k)\,U_2(k)\,q(k-1) - r(k)\,U_1^T(k)\,v_2(k)], \end{cases}$$

$$\begin{cases} \hat{\bar{\Lambda}}_2^{\pm} Y(n) = \begin{pmatrix} \hat{v}_2^T(n) & Y_1(n-1) \\ v_2(n) & Y_2(n) \end{pmatrix} \pm \begin{pmatrix} \hat{v}_2^T(n)\,q^T(n-1)\,v_1^T(n)\,K_8^{\pm T}(n)\,\hat{v}_1^T(n) \\ v_2(n)\,r(n)\,\hat{v}_1(n)\,K_8^{\pm}(n)\,v_1(n) \end{pmatrix} \\ K_8^{\pm}(n) = \Sigma_n^{\pm} v_1(k+1)\,[q(k)\,Y_2(k) - Y_1^T(k)\,r(k)]\,\hat{v}_1(k). \end{cases}$$

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