объединенный
институт
ядерных
исследований
дубна

E2-81-801

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RENORMALIZATION GROUP, PRINCIPLE OF INVARIANCE AND FUNCTIONAL AUTOMODELITY

Submitted to "ДАН СССР"

1981

1. During more than a quarter of century in quantum field theory (QFT) the functional equations of renormalization group (RG) are known. These equations obtained [1,2] on the basis of very specific ideas and reasonings, connected with the procedure of renormalization of ultraviolet divergences, have a very simple and elegant form. They obey the universality property, i.e., do not depend on details of specific QFT model.

Several years ago it has been discovered [3] that in the one-dimensional problem of radiation transfer there also arise the functional equations identical with RG equations. This means that the universality property has a more wide scope. To describe the general nature of this phenomenon, we introduce a special term - the functional automodelity (FA).

A natural question about the nature of FA arises. We show here that FA reflects the simple property of transitivity of physical variables with respect to the way of specifying their initial or boundary values in the systems with a small number of "effective" degrees of freedom. This conception allows us to find a wide set of physical systems for which the property of FA takes place. For theoretical analysis of such systems the renormalization group method, which is of a wide use in the modern QFT problems, may turn to be fruitful.

2. The key position in the RG formalism is occupied by the so-called invariant coupling constant $\bar{g}$. For the QFT model with one nonzero mass and one dimensionless coupling constant g it can be represented as a function of three agruments $\bar{g}(x, y, g)$, where x and y are dimensionless momentum and mass arguments. The functional Eq. for $\bar{g}$ has the form [2]*

$$\bar{g}(x, y, g) = \bar{g}(x/\xi,\ y/\xi,\ \bar{g}(\xi, y, g)) \,. \qquad (1)$$

From this Eq. there follows directly the normalization condition $\bar{g}(1, y, g) = g$. Hence the third argument serves as an initial or boundary value of the function $\bar{g}$. Equation (1) states that the simultaneous transformation of all three arguments

*For derivation of Eq.(1) and discussion of details of the RG formalism we refer to the chapter "Renormalization Group" of any of editions of the book [4].

$$x \to x/\xi, \quad y \to y/\xi, \quad g \to \bar{g}(\xi, y, g)$$

leaves the invariant coupling $\bar{g}$ unaltered, that justifies its name.

Under definite physical conditions (massless QFT model, ultraviolet asymptotic behaviour of massive model) the mass argument can be put equal to zero: $y = 0$. The corresponding functional equation

$$\bar{g}(x, g) = \bar{g}(x/\xi, \ \bar{g}(\xi, g)) \tag{2}$$

will be referred to as massless. The system of equations equivalent to Eq.(2) was obtained in paper /1/.

Functional equations (1), (2) obey the evident group structure. In applications of RG to QFT problems the important role is played by differential group equations. By differentiating (1) with respect to x and putting then $\xi = x$, we find

$$\frac{\partial \bar{g}(x, y, g)}{\partial \ln x} = \beta(\frac{y}{x}, \bar{g}(x, y, g)) \ , \tag{3}$$

where

$$\beta(y, g) = \frac{\partial \bar{g}(\xi, y, g)}{\partial \xi}\Big|_{\xi = 1} \ . \tag{4}$$

Differential group equation (3) first obtained in /2/ describes the momentum evolution of the invariant coupling $\bar{g}$. We shall call it the evolution equation.

The main source of dynamical information in QFT is provided by an infinite system of coupled integral-differential equations for Green functions and vertices, i.e., quantities like $\bar{g}$. This system can be approximately solved by the perturbation method that leads to power expansions like

$$\bar{g}_{p.th.}(x, y, g) = g + g^2 f_1(x, y) + g^3 f_2(x, y) + \ldots$$

with coefficients $f_\ell(x, y)$ given in the explicit form. Despite the smallness of the g parameter, such expansions turn out to be unsufficient in specific regions of x, y variables because of the singular behaviour of f_ℓ. So, at $x \gg 1$, $f_\ell \sim (\ln x)^\ell$. Here the evolution differential equations can help. Using for the definition of β-function the $\bar{g}_{p.th.}$ (in the r.h.s. of Eq. (4)) and solving then evolution equation (3) one obtains an explicit expression for $\bar{g}$, which on the one hand satisfies functional equation (1), and on the other hand under the expansion in powers of g provides a given number of correct terms of the perturbation series. In this way it turns out to be possible to enlarge significantly the domain of x, y arguments in which a new "improved with the help of RG" perturbation expansion can serve for the quantitative description of objects like $\bar{g}$. The procedure of combining the dynamical information from the equation of motion with RG properties is known in QFT as the *renormalization group method.* It has been proposed in papers [2,5] and nowadays provides the only basis of quantitative theoretical calculations in quantum chromodynamics and in the theory of grand unification of interactions.

For the following needs we perform in (1) the change of variables and introduce the notation

$$x \to t = \ln x, \quad y \to T = \ln y, \quad \xi \to r = \ln \xi,$$
$$\bar{g}(e^t, e^T, g) = G(t, T, g).$$

The functional equations take the form

$$G(t+r, T, g) = G(t, T-r, G(r, T, g)), \tag{1a}$$

$$G(t+r, g) = G(t, G(r, g)). \tag{2a}$$

In contrast to the initial Eqs.(1) and (2), in the new equations the transformations of arguments t, T have the additive structure. We shall call these equations additive ones.

3. In paper [3] the property of automodelity in the one-dimensional transfer problem was studied. It reflects the well-known Ambartsumian's principle of invariance and yields the functional equations for the intensity of monochromatic radiation G, penetrating from the vacuum into the semi-infinite medium. The intensity G is considered here as a function of a distance t from the boundary and of intensity g of the radiation falling on the boundary from the vacuum. The remarkable feature of the equation obtained is its identity with the additive massless eq. (2a). In a recent investigation [6], following the line of paper [3], it has been shown that the transition from the semi-infinite medium to the finite layer with thickness yields the massive eq.(1a). At the same time the transition from the monochromatic case to a system of photons with two different energies (when at each act of the interaction of radiation with atoms of the medium quanta of one frequency can transform into quanta of another frequency) leads to the system of coupled functional equations

$$G_i(t+r, g_1, g_2) = G_i(t, G_1(r, g_1, g_2), G_2(r, g_1, g_2)), \quad i = 1,2. \tag{5}$$

This system is just an additive version of RG functional equations for the QFT model with two coupling constants. It was first obtained in paper [7].

The property of identity of functional equations for physical quantities in complex systems from very diverse region of physics, which are described by sufficiently different dynamical equations at the first moment, seems to be rather astonishing. We will show now that the functional equations of the considered type formalize a rather simple and general property of transivity of those physical systems which in a certain sense are equivalent to a mechanical system with a few number of degrees of freedom.

4. Consider the evolution in time t of the simplest system with one degree of freedom. Labelling initial values of the coordinate by g_1, of the velocity by g_2 and their running values by $G_i(t, g_1, g_2)$ $(i=1,2)$, we find that two functions G_1 and G_2 satisfy the system (5). Hence, it follows that Eq. (5) expresses only the transitivity property of running phase variables with respect to their initial values which one can fix at different moments. One coupling constant additive eq. (2a) from such a point of view corresponds to a system with "half of the degree of freedom", the evolution of which is described by the first order differential equation.

This mechanical system, however, in some sense is not pith for our purposes. The problem is that in this example the variable t is the real time and the differential evolution equations (3) coincide with equations of motion.

5. A valuable (or nontrivial) example implies the existence of a physical system in which there is one (or several) physical quantity G that can be represented as a function of the evolution variable t and its boundary value g. The property of transitivity, if it takes place, can be expressed in one of two forms - additive or multiplicative. If the system has no fixed parameter of the same dimensionality as an evolution variable, functional equations are of the "massless" form (2), (2a). Otherwise, we meet with massive equations (1), (1a). The nontriviality of the system is due to the fact that the dynamical equation for G is <u>not</u> the differential equation over t variable.

Consider now several examples and discuss their value in the afore-mentioned sense.

(a) "Rapier." Let us imagine as elastic rod fixed at some point at the nonzero angle with respect to the vertical direction and bent by the force of gravity (the rapier stuck into the floor). By denoting the angle at the point of fixation through g, and the angle at the point at the distance t through $G(t, g)$ we find that the function $G(t, g)$ satisfies the functional equation (2a). To the rod of a finite length T there corresponds the massive equation (1a). However, it turns out that this example is trivial, as far as the shape of the bent rod is defined through the solution of the differential equation with respect to t variable. More interesting is

(b) "Vibrating rapier" - the same rod oscillating around the position of equilibrium. Here it is possible to formulate the nontrivial problem for amplitudes of vibration.

(c) "Symmetrical pond" - a basin with the rotation symmetry filled with some liquid. The FA here takes place for a set of dynamical variables: e.g., the amplitude and energy density of converging and diverging surface waves, including shock waves.

(d) "3-dimensional spherically symmetric system" represents the obvious generalization of the "symmetric pond". Here we have in mind a rather wide scope of physical systems and phenomena from hydrodynamics, aerodynamics, transfer theory, plasma physics and so on.

The use of differential group equations can here turn out to be useful for the study of singularities, as it takes place in the problems of quantum field theory.

The author is indebted to N.N.Bogolubov, M.A.Mnatsakanian and R.M.Muradian for valuable discussions.

REFERENCES

1. Gell-Mann M., Low F. Phys.Rev., 1954, 95, p.1300-1312.
2. Bogoliubov N.N., Shirkov D.V. Doklady AN SSSR, 1955, 103, p.203-206 (see also /8/).
3. Mnatsakanian M.A. Comm. of Biurakan Observatory, 1978, 50, pp.59-78.
4. Bogoliubov N.N., Shirkov D.V. Introduction into the Theory of Quantized Fields. Wiley-Interscience, N.Y., 1st ed., 1959, 3rd ed., 1980.
5. Bogoliubov N.N., Shirkov D.V. Doklady AN SSSR, 1955, 103, p.391-394 (see also).
6. Mnatsakanian M.A. Doklady AN SSSR, 1982, 262, No.4.
7. Shirkov D.V. Doklady AN SSSR, 1955, 105, pp.972-975 (see also /8/).
8. Bogoliubov N.N., Shirkov D.V. Nuovo Cim., 1956, 3, pp.845-863.

Received by Publishing Department
on December 16 1981.