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ON THE DYNAMICS OF GENERALIZED COHERENT STATES.

II. Classical Equations of Motion

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1. INTRODUCTION

In the previous paper[1] (to which we shall refer as I) the exact and stable time evolution of generalized coherent states (GCS) was discussed and a method for constructing GCS related to any Lie group and any quantum system was described. The aim of the present paper is to study the dynamics of GCS on the example of some concrete groups. The exact and stable evolution of SU(1,1) GCS is obtained for nonstationary oscillator, for a motion of a particle in time-dependent magnetic field and for a singular oscillator with time-dependent friction. The physical properties of the constructed GCS are discussed and in particular it is shown that the GCS related to discrete series of unitary irreducible representations (UIR) of SU(1,1) are exactly those quantum states for which the quantum mean values of observables are equal to statistical averages. Using the Klauder approach we obtain the classical Euler equations

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{z}_i} - \frac{\partial \mathcal{L}}{\partial z_i} = 0,$$
$$\mathcal{L} = i\langle\Phi_z|d/dt|\Phi_z\rangle - \langle\Phi_z|H|\Phi_z\rangle, \qquad (1)$$

($|\Phi_z\rangle$ being the GCS(I(25)); z_i, local coordinates for $X = G/K$), as independent of the type of representation, i.e., one and the same classical solution of Euler equations correctly determines the quantum evolution of many different initial states (distinguished by different values of some conserved quantities) and different Hamiltonians.

2. GLAUBER COHERENT STATES

These states are related to the projective unitary representation of the phase space translation group. The representation operators form the so-called Heisenberg-Weil group, which is generated by lowering and raising operators a, a^+ and identity operator $\hat{I}$, ($[a, a^+] = \hat{I}$). The overcomplete family of states (OFS) is given by (see I.(3), I.(4)):

$$|(\beta, z, z^*)\rangle = \exp(i\beta\hat{I} + za^+ - z^*a)|\Phi_0\rangle, \quad z \in C. \qquad (2)$$

The stationary group K (I, Sec.2) is trivially generated by $\hat{I}$. The quotient space G_W/K (G_W - Heisenberg-Weil group) is isomorphic to the complex plane C. Choosing the cross-sections: $(z,z^*) \to (0,z,z^*)$ and $|\Phi_0\rangle = |0\rangle$ $(a|0\rangle = 0)$ we obtain the system of Glauber CS (I, Refs./1,2/). Another choice, say $s':(z,z^*) \to (\beta(z,z^*), z, z^*)$ permits to make manifest the stability of the system CS (I, Ref./1/). The symplectic 2-form, determined by the Kähler potential $f = z^*z$ is $\omega = i dz \wedge dz^*$ and therefore the equation of motion is

$$i\dot{z} = \partial_{z^*} \mathcal{H} \tag{3}$$

(the complex conjugate equations will not be written down). The same equation can be derived making use of the Lagrangian $\mathcal{L} = (i/2)(\dot{z}z^* - \dot{z}^*z) - \mathcal{H}$, according to Klauder method (I,Sec.3).

For the most general Hamiltonian (I.19) which preserves stable the OFS (2), Eq.(3) assumes the form

$$i\dot{z} = \omega(t)z + F(t). \tag{4}$$

The solution of this equation can be explicitly written:

$$z(t) = (z - i \int F(t)dt)\exp(-i\int \omega(t)dt), \quad z = z(0),$$

We see that the phase space trajectory of the considered system is a superposition of a translation and a rotation. This result simply reflects the fact that the Hamiltonian (I.19) belongs to the (projective) representation of Lie algebra e(2) of Euclidean group E(2).

3. SU(2) COHERENT STATES

The generators of the group SU(2) are J_i, $[J_i, J_j] = i\epsilon_{ijk}J_k$ and the Casimir operator is equal to $J^2 = J_1^2 + J_2^2 + J_3^2 = j(j+1)$, $j = 1/2, 1, 3/2, \ldots$. The three parameters usually used are the Euler angles (ϕ, θ, ψ). As in I Ref./12/ we choose $|\Phi_0\rangle = |j,-j\rangle$, where $|j,m\rangle$ are the eigenvectors of J_3: $J_3|j,m\rangle = m|j,m\rangle$, $m = -j, -j+1, \ldots, j$, and such a cross-section in the fibre bundle $(G, G/K, \pi)$(K being the stationary subgroup of $|j,-j\rangle$) that

$$|\Phi(\phi,\theta)\rangle = |j;z\rangle = (1+z^*z)^{-j}\exp(zJ_+)|j,-j\rangle, \tag{5}$$

where $z = -\mathrm{tg}\,\theta/2\, e^{-i\phi}$, $J_+ = J_1 + iJ_2$.

The Hamiltonian

$$H = h_iJ_i = hJ_+ + h^*J_- + h_0J_0, \quad h = h_1 + ih_2, \; h_0 = h_3, \; J_- = (J_+)^+ \tag{6}$$

according to Malkin theorem (I, Sec.3) should preserve all SU(2) GCS (5) stable. The classical solution $z(t)$ obeys the

Euler eqs. for the functional (I.25), which in this case have the form

$$i\dot{z} = (2j)^{-1}(1+z^*z)\,\partial\mathcal{H}/\partial z^* . \qquad (7)$$

The same equation is obtained in Ref. [2], where it arises as equation of the path providing the main contribution in the path integral, which expresses the transition amplitude from a GCS (5) to another one.

By means of the easily verified formulae ($\langle . \rangle_{jz} = \langle jz|(.)|j;z\rangle$):

$$\langle J_+\rangle_{jz} = 2jz^*(1+z^*z)^{-1}, \langle J_-\rangle_{jz} = 2jz(1+z^*z)^{-1}, \langle J_0\rangle_{jz} = -j(1-z^*z)(1+z^*z)^{-1}$$

we can get the classical Hamiltonian

$$\mathcal{H} = j(1+z^*z)^{-1}(2(hz+h^*z^*) - h_0(1-z^*z)). \qquad (8)$$

From (7) and (8) we derive the Euler equations

$$i\dot{z} = h^* + h_0 z - hz^2 . \qquad (9)$$

Now it is worth noticing the important property of Eq.(9), namely it does not depent on the representation (j) of SU(2). One and the same classical function z(t) entirely determines the quantum evolution of all systems GCS ($\langle (j), |j;-j\rangle$) governed by the Hamiltonian (8). Taking different representations for the angular momentum J_i we get different (and nonequivalent) quantum systems whose dynamics is exactly determined by the same classical trajectory in phase space $X = G/K$.

As we have stated in previous paper I any OFS, in particular the system GCS (5), can be realized in the Hilbert space ***H*** of solutions by means of Eq.(I.15) provided the generators L_a are expressed in terms of a, a^+. The most natural representation in ***H*** for the angular momentum $J_i, J^2 = j(j+1)$, is that of $2j+1$ numbers of operators a_i, a_i^+ (I, Ref. [21]), $i = 1,2,\dots 2j+1$

$$J_k = a_i^+(\hat{J}_k)_{i\ell}a_\ell \; , \qquad (10)$$

where $\hat{J}_k$ are $(2j+1)$-dimensional matrices. Formula (10) generalizes the Schwinger representation ($j=1/2$) [3] to the case of arbitrary j. In Ref. [4] another angular momentum boson representation is constructed for arbitrary j in terms of $2s+1$ pairs $a_i, a_i^+, i = -s,\dots+s$, $s = 1/2, 3/2, \dots$ The natural representation (10) is irreducible in Fock space, spanned by the vectors

$$|n\rangle = \sum_{i=1}^{2j+1} (n_i!)^{-1/2}(a_i^+)^{n_i}|0\rangle \qquad (11)$$

with a fixed number $n=\sum_i n_i = 2j$. The one-dimensional systems are in some sense exceptional but it is still possible to express any UIR(j) of SU(2) in terms of one pair a, a^+ (I, Ref. /24/) in the Fock space of vectors $|m\rangle = (m!)^{-1/2}(a^+)^m|0\rangle$, $m=0,1,\ldots 2j$ ($|0\rangle = |j,-j\rangle$, $|m\rangle = |j,-j+m\rangle$) and

$$V^{(j)}|m\rangle = (m!)^{-1/2}(1+z^*z)^{-j}(1+za^+)^{2j-m}(a^+-z)^m|0\rangle .$$

Putting J_k from Eq.(10) into (6) we obtain Hamiltonians for different (quadratic) quantum systems all having the property their dynamics to be determined by one classical function $z(t) = -\mathrm{tg}\,\theta(t)/2 \cdot \exp(-i\phi(t))$, θ and ϕ being the Euler angles for the SU(2) rotations. The N-dimensional oscillator is one of the simplest such systems. In this case $z(t) = z(0)\exp(i\omega t)$.

Let us note that the SU(2) CS constructed from Schwinger operators $J_+ = a^+b$, $J_- = b^+a$, $J_0 = (1/2)(a^+a - b^+b)$, namely

$$|N;z\rangle = (1+z^*z)^{-N/2}\sum_{n=0}^{N}\binom{N}{n}^{1/2} z^n |N;n\rangle$$

$$|N;n\rangle = (n!(N-n)!)^{-1/2}(a^+)^n(b^+)^{N-n}|0,0\rangle \qquad (12)$$

$$a|0,0\rangle = b|0,0\rangle = 0, \quad N=0,1,2,\ldots$$

represent an OFS only in the N+1-dimensional subspace of Hilbert space $H \otimes H$. It is possible however to construct such wave-packets from GCS (12) which form OFS in the whole space $H \otimes H$. Indeed, the simple calculation provides ($\alpha \in C$):

$$\sum_{N=1}^{\infty}(N!)^{-1/2}\alpha^N|N;z\rangle = \exp(\alpha(za^+ + b^+)(1+z^*z)^{-1/2})|0,0\rangle . \qquad (13)$$

Introducing the lowering and raising operators

$$A_z = (1+z^*z)^{-1/2}(z^*a+b),$$
$$A_z^+ = (1+z^*z)^{-1/2}(za^+ + b^+) \qquad (14)$$

we see that the wave packets (13) are (apart to a normalization constant) just the Glauber CS for A_z, A_z^+. From the other hand the states (13) are obviously tensor products of the type $|\lambda\rangle|\mu\rangle$ where $\lambda = \alpha z(1+z^*z)^{-1/2}$, $\mu = \alpha(1+z^*z)^{-1/2}$ and consequenly they form an OFS in $H \otimes H$. These wave packets have been called "oscillator-like CS for the rotation group" /5/. From the known result for the Heisenberg-Weil group G_W we obtain that the most general Hamiltonian which preserves stable the OFS (13) has the form $H = \omega A_z^+A_z + f^*A_z + fA_z^+ + \beta$, i.e., the stable OFS of type (13) admit more wide class of Hamiltonians than this one of type (12). The Euler equations for such systems in terms of parameters λ, μ have the form (4).

The physical properties of SU(2) CS will not be discussed here since they have been thoroughly examined (I, Ref./4,10,12/).

4. SU(1,1) COHERENT STATES

The UIR of SU(1,1) are generated by the operators K_i $(i=1,2,3)$ with commutation relations

$$[K_1,K_2]=-iK_3,\ [K_2,K_3]=iK_1,\ [K_3,K_1]=iK_2 \tag{15}$$

and the Casimir operator is $K^2=K_3^2-K_1^2-K_2^2=k(k-1),\ k>0$. (We restrict ourselves with discrete series $V_k^{(+)}$). All UIR have been described by Bargmann/6/.

As in the previous section let us study first the stable evolution of SU(1,1) for the Hamiltonian

$$H=h_0K_3+h_1K_1+h_2K_2=h_0K_0+h^*K_-+hK_+ \tag{16}$$

by making use of Klauder approach. The canonical basis in Hilbert space $\mathcal{H}$ is $|k;m\rangle$, $m=0,1,2,\dots$, $K_0|k;m\rangle=(k+m)|k;m\rangle$. As in I, Ref./12/ we choose the lowest weight vector $|k;0\rangle$ as a fiducial one, whose stationary subgroup is the subgroup of rotations around the third axis. The system GCS is written in the form

$$|k;z\rangle=(1-z^*z)^k\exp(zK_+)|k;0\rangle \tag{17}$$

where $z=-\mathrm{th}\,\tau/2\cdot\exp(-i\phi)$, τ and ϕ being the Euler angles for SU(1,1). Using the formulae

$$\langle K_+\rangle_{kz}=2kz^*(1-z^*z)^{-1},\ \langle K_-\rangle_{kz}=2kz(1-z^*z)^{-1},\ \langle K_0\rangle_{kz}=k(1+z^*z)(1-z^*z)^{-1}$$

we obtain the Euler equations for the functional (I.25):

$$i\dot z=(2k)^{-1}(1-z^*z)^2(\partial\mathcal{H}/\partial z^*), \tag{18}$$

where for the case of linear Hamiltonian (16)

$$\mathcal{H}=k(h_0(1+z^*z)+2hz^*+2h^*z)(1-z^*z)^{-1} \tag{19}$$

and therefore Eq.(18) assumes the form

$$i\dot z=h^*z^2+h_0z+h \tag{20}$$

quite similar to the analogical one (9) for the SU(2) group and again independent of the representation $V_k^{(+)}$.

The phase space corresponding to the UIR $V_k^{(+)}$ is the Lobachevsky plane represented by the circle $D:|z|<1$ with Kähler potential (see I.(28)) $f=\ln(1-z^*z)^{-2k}$. Hence the symplectic form is

$$\omega=2ik(1-z^*z)^{-2}dz\wedge dz^*$$

and (see I.(29)) $g=(1-z^*z)^2/2ik$. Thus we again arrive to Eq.(18) by means of the Lie bracket (I.(29)).

Now we shall construct the SU(1,1) CS for some concrete quantum systems and investigate some of their properties.

1. Nonstationary Quantum Oscillator with Friction

The Hamiltonian of this system is

$$H=(1/2)(p^2+\omega(t)^2q^2)+(b(t)/2)(qp+pq), \quad [q,p]=iI, \tag{21}$$

Using the following representation of the Lie algebra su(1,1) /7/:

$$K_1=(1/4)(p^2-q^2), \quad K_2=(1/4)(qp+pq), \quad K_3=(1/4)(p^2+q^2) \tag{22}$$

we can write down (21) as a linear combination of generators (22):

$$H=(1-\omega^2)K_1+2bK_2+(1+\omega^2)K_3=h_0K_0+hK_++h^*K_- \quad , \tag{23}$$

where $h=(1-\omega^2)/2-ib$, $h_0=1+\omega^2$, $K_\pm=K_1\pm iK_2$, $K_0=K_3$. Let us find the SU(1,1) CS for UIR, generated by the operators (22). The Casimir operator is $K^2=-3/16$ hence there are two UIR, belonging to discrete series $V_k^{(+)}$: $k=1/4$ and $k=3/4$. The basis for $k=1/4(3/4)$ is formed by the even (odd) states of Fock space, corresponding to the eigenvalue +1(−1) of the parity operator (that is why we shall denote $k=+(-)$). The Fock space is generated by the creation operator $a^+=2^{-1/2}(p+iq)$, then $|+;m\rangle=|2m\rangle$, $|-;m\rangle=|2m+1\rangle$, where $|n\rangle=(n!)^{-1/2}a^{+n}|0\rangle$. The corresponding SU(1,1) CS are

$$|+;z\rangle=(1-z^*z)^{1/4}\exp(z(a^+)^2/2)|0\rangle$$
$$|-;z\rangle=(1-z^*z)^{3/4}\exp(z(a^+)^2/2)|1\rangle \tag{24}$$

or in coordinate representation

$$\langle x|\pm;z\rangle=N_\pm\, x^{(1\mp1)/2}\, e^{-\alpha x^2}, \qquad \alpha=\frac{1}{2}\,\frac{1+z}{1-z} \tag{25}$$

($N_\pm=N_k$ = normalization constants, depending on z). The related probability densities

$$w_\pm(x)=\pi^{-1/2}(2x^2)^{(1\mp1)/2}\lambda^{1\mp1/2}e^{-\lambda x^2}$$
$$\lambda=(1-z^*z)|1-z|^{-2} \tag{26}$$

describe distributions of Gaussian type having maximums at the points $x_\pm$, $x_\pm^2=(1\mp 1)/2\lambda$ and width (distance between the extreme inflex points) - also inversely proportional to $\lambda^{1/2}$. Note to the point that the Gaussian distributions, corresponding to Glauber CS have width, non-depending on the label z of the CS.

The computation of the mean values of operators q^2 , p^2 yields:

$$\begin{aligned} \langle q^2\rangle_{kz} &= 2k(1-z^*z)^{-1}|1-z|^2 = 2k/\lambda \\ \langle p^2\rangle_{kz} &= 2k(1-z^*z)^{-1}|1+z|^2 . \end{aligned} \tag{27}$$

Multiplying these quantities one obtains the Heisenberg uncertainty product ($\langle q\rangle_{kz}=0$, $\langle p\rangle_{kz}=0$):

$$\langle q^2\rangle_{kz}\langle p^2\rangle_{kz} = 4k^2(1+r^4-2r^2\cos 2\theta)(1-r^2)^{-2} \geq 4k^2 , \tag{28}$$

where $r=|z|$, $\theta=\arg z$.The equality holds iff z is a real, nonnegative number: $z=r\geq 0$. Hence the only minimum uncertainty states (MUS) from the considered system GCS are $|+;r\rangle=(1-r^2)^{1/4}\exp(ra^{+2}/2)|0\rangle$, $r>0$. They are unitarily equivalent to the Glauber CS in accordance with the result obtained by Stoler/8/.

Turning to the time evolution and observing that the Hamiltonian (21) obeys the Malkin theorem conditions (I,Sec.3) we conclude that the OFS (24) are stable and in every moment t the SU(1,1) CS are determined by the same formulae (24), where $z=z(t)$ is a solution of the Euler equations (20).

On the other hand one can obtain the exact solution making use of the integrals of motion method/1,11/. Comparing the two solutions it is easy to express $z(t)$ as fractional linear transformation:

$$z(t)=\frac{az+c}{c^*z+a^*},\quad a=(\rho^2+1)e^{i\gamma} ,\quad c=(\rho^2-1)e^{-i\gamma} , \tag{29}$$

where $\epsilon=\rho e^{i\gamma}$ is a solution of the following classical equation:

$$\ddot{\epsilon}+\Omega^2\epsilon=0,\quad \Omega^2=\omega^2-b^2-\dot{b},\quad \rho^2\dot{\gamma}=1. \tag{30}$$

Eq. (29) explicitly shows that the time evolution is represented as a SU(1,1) transformation in phase space D: $|z|<1$.

2. Generalized Singular Oscillator

The Hamiltonian of this system contains a singular term, which may serve as a model potential describing interaction between particles (see I, Ref.[11], Ref.[9]):

$$H=(1/2)(p^2+\omega^2(t)q^2)+(b(t)/2)(qp+pq)+g/q^2 \quad . \tag{31}$$

We suppose that $0<q<\infty$ since the singular potential prohibits transitions from $(0,\infty)$ to $(-\infty,0)$. Further we restrict ourselves with sufficiently large values of the interaction constant $g\,(g>-1/8)$ because otherwise it is possible a collapse (I, Ref.[11]).

We shall use the method of integrals of motion. The latter are determined in the form[9]:

$$M_-=(1/2)(a^2+g\epsilon^2(t)q^{-2}), \quad M_+=(M_-)^+, \quad 2M_0=[M_-,M_+], \\ a=2^{-1/2}\,i(\epsilon p+(\epsilon b-\dot{\epsilon})q), \quad [q,p]=iI, \tag{32}$$

where ϵ is the solution of Eq.(30). The operators (32) form a representation of the Lie algebra su(1,1) $(M_0, M_\pm^+=\pm M_\pm)$ and commute with the Schrödinger operator $D_S=i\partial/\partial t-\bar{H}$. Thus we have a concrete realization of the dynamical Lie algebra[7] of the system under consideration. The UIR of the group SU(1,1) building by means of (32) are labelled by the number $k=(d+1)/2$, where $d=(1/2)(1+8g)^{1/2}>0$, i.e., the operators (32) generate UIR from the discrete series $V_k^{(+)}$. Making use of the eigenvectors of M_0[9]:

$$<x|k;n>=(2\epsilon^{-2d-2}\Gamma(n+1)/\Gamma(n+d+1))^{1/2}\,x^{d+1/2}\times \\ \times\exp(-2in\gamma+(i/2)(\dot{\epsilon}/\epsilon-b)x^2)L_n^d(\dot{\gamma}x^2), \quad \gamma=\arg\epsilon \tag{33}$$

we construct the SU(1,1) CS for our system:

$$<x|k;zt>=(2/\Gamma(d+1))^{1/2}\epsilon^{-d-1}(1-z^*z)^{(d+1)/2}\,x^{d+1/2} \\ \times(1-s)^{-d-1}\exp\left(-\frac{1}{2\rho^2}\frac{1+s}{1-s}x^2\right), \tag{34}$$

where $s=ze^{-2i\gamma}$, $|z|<1$, $k=(d+1)/2$ (the phase factor was omitted).

In the initial moment $t=0$ Eq.(34) converse to the form: (up to a normalization constant)

$$<x|k;z,0>=(1-z^*z)^{(d+1)/2}\,x^{d+1/2}(1-z)^{-d-1}\exp\left(-\frac{x^2}{2}\frac{1+z}{1-z}\right), \tag{35}$$

where we have put $\dot{\gamma}(0)=1$, $\gamma(0)=0$. The similarity with Eq.(25) is obvious. In fact formula (35) includes (25) as a special case if we put there $d=2k-1$, $k=1/4, 3/4$. Moreover the wave function (34) can be written in the form (35) where z is replaced by

$$z(t)=\frac{az+c}{c^*z+a^*}, \quad a=(1+\rho^2)e^{-i\gamma}, \quad c=(1-\rho^2)e^{i\gamma} \quad . \tag{36}$$

The latter expression essentially coincides with (29), i.e., the addition of the singular term g/q^2 in the Hamiltonian does not exert influence upon the time evolution in phase space D.

The dynamical symmetry group of the quantum oscillator (21) is obviously more large than SU(1,1) (it must include transformations mixing states with different parity). The addition of the singular term to (21) reduces the symmetry to SU(1,1), i.e., this group is a dynamical symmetry group of the singular oscillator. The trajectory in phase space however remains unchanged. Thus the motion in phase space D is not obliged to reflect the dynamics of corresponding classical system. Indeed the quantum mean value of coordinate operator $q^2 = 2M_0\epsilon^*\epsilon - \epsilon^2 M_+ - \epsilon^{*2}M_-$

$$\langle q^2 \rangle_{kz} = 2k(1-z^*z)^{-1}(\epsilon^*\epsilon(1+z^*z) - z^*\epsilon^2 - z\epsilon^{*2})$$

varies in stationary case ($\epsilon = \Omega^{-1/2}\exp(i\Omega t)$, $\Omega = \mathrm{const}$) like the elongation of the usual harmonic oscillator. One can conclude also that the SU(1,1) CS (34), (35) are not close to the classical ones.

3. Motion of a Particle in a Time-Dependent Magnetic Field

Let us consider a particle with unit mass and unit charge ($e=m=1$) moving in a time dependent magnetic field $\mathcal{H}(t)=2\omega(t)$. The Hamiltonian of this system is

$$H = 1/2(p_x^2 + p_y^2) + \omega^2/2(x^2+y^2) + \omega(yp_x - xp_y). \tag{37}$$

Exact solutions of this problem were obtained in Ref. /15/. Two independent integrals of motion can be constructed:

$$A = 1/2(\epsilon(p_x+ip_y) - i\dot{\epsilon}(y-ix))e^{i\phi}$$
$$B = 1/2(\epsilon(p_y+ip_x) - i\dot{\epsilon}(x-iy))e^{-i\phi} \quad , \tag{38}$$

where

$$\phi = 1/2 \int dt\,\omega(t)$$

and ϵ is the solution of Eq.(30) with $\Omega=\omega/2$. The operators A,B obey the relations $[A,A^+]=[B,B^+]=1$, $[A,B]=[A,B^+]=0$ and their eigenvalues determine respectively the running coordinates of the wave-packet centre and the coordinates of the orbital centre in the plane x ,y .The corresponding eigenvectors are the Glauber CS for this system (I, Ref.[11]).

From the operators A,B we build the following representation of Lie algebra su(1,1) (I, Ref.[7]):

$$K_+ = A^+B^+,\ K_- = AB,\ K_0 = 1/2(A^+A + B^+B + 1). \tag{39}$$

The Casimir operator K^2 can be expressed by the third projection of the angular momentum $L_3 = B^+B - A^+A$: $K^2 = 1/4(L_3^2 - 1)$. Denoting the eigenvectors of Hermitean operators A^+A, B^+B with eigenvalues, respectively, n and m by $|n,m;t\rangle$ we obtain that the UIR for the group SU(1,1), generated by operators A,B is $V^{(+)}_{(N+1)/2}$ $(N = 0,1,..)$ and it is spanned by the vectors $|k;n\rangle = |n,n+N;t\rangle$, $n = 0,1,2,..(k = (N+1)/2)$.

Now it is not difficult to write down the SU(1,1) CS using the explicit form of the vectors $|n,m;t\rangle$ $(m-n=N)$ given in I, Ref.[11] (up to a phase factor):

$$\langle w|n,n+N;t\rangle = \pi^{-1/2}(n!/(n+N)!)^{1/2}(-ie^{-2i\gamma})^n \times \rho^{-N-1} r^N \exp(-r^2/2\rho^2) L_n^N(r^2/\rho^2), \tag{40}$$

where

$$w = -2^{-1/2}(x+iy)e^{i\phi} = 2^{-1/2}\, r e^{i\theta} .$$

Then by means of Eq.(17) we obtain

$$\langle w|k;z,t\rangle = (\pi N!)^{-1/2}(2\,\mathrm{Re}\,\alpha)^{(N+1)/2} r^N \exp(-\alpha r^2)$$
$$\alpha = (1+s)/2\rho^2(1-s),\quad s = -ize^{-2i},\quad 2k = N+1. \tag{41}$$

It is clear from the construction that these systems GCS have an important property: all wave-functions (41) with fixed N belong to the eigenspace of the third projection of angular momentum L_3 and corresponds to the eigenvalue $L_3 = N$. Hence they are similar to the GCS with conserved charge, examined recently[10] by B. Skagerstam.

From formula (41) immediately follows that the probability density reaches a maximum at a distance $r = (N/4\,\mathrm{Re}\,\alpha)^{1/2}$, i.e., it is proportional to the amplitude of the classical oscillator (30). It is easy to see also that the trajectory in phase space D is again representable in forms (29), (36), i.e., as SU(1,1) transformation, depending on the motion of the classical oscillator (30).

Using the explicit expression (41) and usual quantum-mechanical rules one can compute the mean values of canonical operators, as follows

$$\langle x^2\rangle_{kz} = (N+1)/4\mathrm{Re}\,\alpha = (N+1)/2\lambda_s^{-2}\ , \quad \lambda_s = (1-s^*s)|1-s|^{-2}$$

$$\langle p_x^2\rangle_{kz} = \mathrm{Re}\,\alpha\,(1+(N+1)(\mathrm{Im}\,\alpha/\mathrm{Re}\,\alpha)^2) \qquad (42)$$

$$= (\lambda_s/2\rho^2)(1+(N+1)(2\,\mathrm{Im}\,s/(1-s^*s))^2)$$

(The same expressions hold for $\langle y^2\rangle_{kz}$ and $\langle p_y^2\rangle_{kz}$). In the stationary case $(\rho=(2/\omega)^{1/2}\ , \gamma=\omega t/2)$ these quantities vary in time according to usual classical equations of motion.

Let us choose in the moment $t=0$ the initial conditions for Eq. (30) in such a manner that $s(0)=z$. Then the mean values (42) for the real states $|k;z,0\rangle$, $\mathrm{Im}\,z=0\ (2k=N+1)$ become

$$\langle x^2\rangle_{kz} = (N+1)(1-z)/2(1+z)\rho^{-2}$$

$$\langle p_x^2\rangle_{kz} = (1+z)/2(1-z)\rho^2 . \qquad (43)$$

Obviously the Heisenberg uncertainty product $\langle x^2\rangle\langle p_x^2\rangle=(N+1)/4$ preserves its value when the time-evolution in the circle D is concentrated on the real axis. The states $|1/2;z\rangle, \mathrm{Im}\,z=0$ are the only MUS. More explicitly they can be written in the form:

$$|1/2;z\rangle = (1-z^2)^{1/2}\sum_{n=0}^{\infty} z^n|n,n\rangle \qquad (44)$$

Let us find now the mean value of the operator $\tilde{A}=A\otimes I$ in state (44) for $z=\exp(-\beta\omega/2)$, $\beta=(kT)^{-1}$ (k being the Boltzmann constant). The operator A acts in Fock space generated by the canonical basis $|n\rangle$. We have

$$\langle 1/2;z|\tilde{A}|1/2;z\rangle = (1-e^{-\beta\omega})\sum_{n=0}^{\infty} e^{-\beta\omega n}\langle n|A|n\rangle = Z^{-1}\ \mathrm{tr}(Ae^{-\beta H_0})$$

$$Z = \mathrm{tr}\,e^{-\beta H_0},\quad H_0=\omega a^+a \qquad (45)$$

According to Ref. /11/ the states (44) may be considered as ground states of thermodynamical system, corresponding to different temperatures. Quite analogically one can construct the excited states $|N,\beta\rangle=(N!)^{-1/2}a^+(\beta)|0,\beta\rangle$, $(|0,\beta\rangle=|1/2;z\rangle)$, where $a^+(\beta)=U(\beta)a^+U(\beta)^{-1}$, $|0,\beta\rangle=U(\beta)|0\rangle$. The states $|N,\beta\rangle$ form the real subsystem of the SU(1,1) CS related to the UIR $V^{(+)}_{(N+1)/2}$. The possibility of expressing the quantum-statistical averages (45) as usual quantum-mechanical mean values apparently at first was observed by Y.Takahashi and H.Umezawa /11/ without any connection with theory of GCS. Here we establish that the states of thermodynamical systems in thermostat are SU(1,1) CS in the extended Hilbert space $\boldsymbol{H}\otimes\boldsymbol{H}$.

4. U(N+1) COHERENT STATES

Let us consider the N+1 -level system, described by the Hamiltonian:

$$H = \sum_{ij=0}^{N} h_{ij}(t) a_i^+ a_j \quad , \tag{46}$$

where $h_{ij}^* = h_{ji}$ (Hermitean conditions) and $[a_i, a_j^+] = \delta_{ij}$. The Hamiltonian (46) belongs to the (ladder) representation of Lie algebra u(N+1) spanned by the generators /7/:

$$E_{ij}^+ = a_i^+ a_j \quad (i>j), \quad E_{ij}^- = a_i^+ a_j \quad (i<j), \quad E_i = a_i^+ a_i \quad (i,j = 0,1,2,..N) \tag{47}$$

The UIR of U(N+1) are determined by the highest weights $(m_0, m_1, ... m_N)$, $m_0 \geq m_1 \geq ... \geq m_N$, where $m_0, ..., m_N$ are integers. We shall construct the GCS in the carrier space of the UIR (m,0,...,0), spanned by the vectors $|m_0, m_1, ..., m_N\rangle = |m_0\rangle |m_1\rangle ... |m_N\rangle$, where $m_0 + m_1 + ... + m_N = m$ and $|m_i\rangle = (m_i!)^{-1/2} a_i^{+m_i} |0\rangle$. The fiducial vector is chosen to coincide with the weight vector $|m, 0, ..., 0\rangle$. Using a suitable parametrization of the group element we obtain the U(N+1) CS in the form:

$$\begin{aligned} |m; z\rangle &= C_z \exp\Big(\sum_{i>0} z_i E_{i0}^+\Big) |m, 0, ..., 0\rangle \\ &= (1 + z^* z)^{-m/2} \sum_{m_0 + ... + m_N = m} \Big(\frac{m!}{m_0! ... m_N!}\Big)^{1/2} z_1^{m_1} ... z_N^{m_N} |m_0 ... m_N\rangle , \end{aligned} \tag{48}$$

where $z^* z = \sum_{i=1}^{N} z_i^* z_i$. It is seen that in this case the rôle of the phase space is played by the N-dimensional complex space C^N. The states (48) are obviously generalization of the known SU(2) and SU(3) CS (the latter are obtained in Ref. /12/).

Let us turn to the dynamics of U(N+1) CS. Recall that according to general theory /1/ the time evolution of GCS is described by the same formula (48) where one ought to replace z_i by time-dependent functions, determined as solutions of Euler equations. By means of the scalar product

$$\langle m; y | m; z\rangle = (1 + y^* y)^{-m/2} (1 + z^* z)^{-m/2} (1 + y^* z)^m \tag{49}$$

one can obtain the Lagrangian (see (1)):

$$\mathcal{L} = (im/2)(z^* \dot{z} - \dot{z}^* z)(1 + z^* z)^{-1} - \mathcal{H}, \tag{50}$$

where $\mathcal{H} = \langle m; z | H | m; z\rangle$. Hence the Euler equations have the form

$$\dot{z}_i + (\dot{z}_i z_j - z_i \dot{z}_j) z_j = (im)^{-1} (1 + z^* z)^2 \partial \mathcal{H} / \partial z_i^* \; . \tag{51}$$

For N=1 Eqs.(51) are reduced to Eq. (7), concerning to SU(2). Finally using the CS representation one can derive the following formula for the mean value in U(N+1) CS:

$$\langle a_i^+ a_j \rangle = m z_i^* z_j / 2(1 + z^* z). \qquad (52)$$

Since the classical Hamiltonian $\mathcal{H}$ is a linear combination of such expressions we see that Eqs.(51) do not depend on the representation as in all the previous cases.

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