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B.A.Nikolov, D.A.Trifonov

ON THE DYNAMICS
OF GENERALIZED COHERENT STATES.

I. Exact and Stable Evolution

1981

1. INTRODUCTION

The generalized coherent states (GCS) were introduced in principle by J.Klauder[1] as an example of his overcomplete family of states (OFS), which realize continuous representations of Hilbert space H and are convenient for the study of the general relationship between classical and quantum mechanics. In the second paper[1] an OFS was defined as

$$|\Phi(\ell_a)\rangle = V(\ell_a)|\Phi_0\rangle = \exp(\ell_a L_a)|\Phi_0\rangle, \qquad (1)$$

$$\text{(sum over repeating indexes)}$$

where $|\Phi_0\rangle$ is a fixed ("fiducial") vector in H and $V(\ell_a)$ is a unitary irreducible representation (UIR) in H of a Lie Group G, ℓ_a being the canonical group parameters; and L_a, the generators of the representation $V(\ell_a)$ $(a = 1,2,.....,r)$. In Ref.[1] it was noted that the states (1) form automatically an OFS for UIR of compact groups but the example that has been thoroughly examined is connected with the representation operator

$$V(q,p) = \exp(-iqP + ipQ), \quad [Q,P] = iI \qquad (2)$$

of the Weil-Heisenberg group G_W and

$$|\Phi(q,p)\rangle = V(q,p)|\Phi_0\rangle. \qquad (3)$$

The same OFS was independently studied by Glauber[2] in the notations

$$|a\rangle = D(a)|\Phi_0\rangle, \quad D(a) = \exp(aa^+ - a^*a) = V(p,q), \qquad (4)$$

where to within a factor, the parameters q and p are the real and imaginary parts of the complex number a and a, a^+ are the known lowering and raising operators for the harmonic oscillator. Glauber has called such states "coherent" (CS). The CS (3) (further called usually CS or G_W-CS) possess a number of remarkable physical and mathematical properties and later many authors[3-10] (see also Refs.[11-13] and references therein) have introduced and studied more general CS, generalizing different properties of the usual CS but one of the best generalizations is just the first one, proposed by Klauder by the Eq. (1). The (over)completenesses of the set of vectors (1)

as was shown in Ref.[7] holds for any UIR of a Lie group G.
In the same paper a factorization of the OFS (1) over the
stationary group $K \subset G$ of the fixed vector $|\Phi_0>$ was also pro-
posed since the physical states are defined up to a constant
phase factor:

$$V_h|\Phi_0> = e^{i\alpha(h)}|\Phi_0>, \qquad h \in K \subset G, \qquad (5)$$

$$|\Phi_x> = V_{s(x)}|\Phi_0>, \qquad x \in X = G/K, \qquad (6)$$

where s is a cross-section in the group fibre bundle (G, X, π)[14],
i.e., s(x) is a representative of the coset $x = gK$, $g \in G$. Such fac-
torized OFS have been called systems of GCS. We shall use in
this paper both terms.

The properties and applications of CS and GCS are widely
discussed in the literature[1-18] so we would not list them
here. We shall only elucidate in Sec.2 the geometrical mean-
ing of GCS as images of cross-sections in the fibre bundle
(M, X, ρ) associated with the fibre bundle (G, X, π). The relation
with the X –measurements in sense of Holevo[15] is pointed out
also.

In our paper we discuss two main aspects of the dynamics
of GCS – their exact time evolution and their stable evoluti-
on. (By stable evolution we mean that an initial GCS, when
evolved in time, remains a GCS of the same type at all times,
i.e., the set of GCS is invariant under time evolution).

The dynamics of the usual CS is completely examined[1,5,11,16-21]
Their exact evolution was explicitly found for quadratic sys-
tems[5,20,21] and in Ref.[22] a method was proposed to const-
ruct usual CS for any system. The stable evolution of CS was
considered by several authors[16-19]. The most general form of
the Hamiltonian which preserves all CS stable was found at
first in Refs.[16,17] and later by other method in Refs.[18,19].

The main results of the present paper are the following.
In Sec.3 we give a method for constructing GCS for any Lie
group and any quantum system by the realization of the repre-
sentation V_g of G in the space H of solutions of the Schrö-
dinger equation. For this purpose one may use invariant lower-
ing and raising operators $A(t)$, $A^*(t)$[5,11] and express V_g
in terms of them[23-25]. This method permits one also to con-
clude that the dynamical symmetry group of quantum system may
be any group G, which has UIR in the Hilbert space H. Using
the sufficient conditions (21), (22) we prove that the N –di-
mensional oscillator preserves stable the GCS for any Lie
group G, which representations are generated by the operators
$a_i a_j$, $a_i^+ a_j^+$, $a_i^+ a_j^+$ (i,j=1,2,...N) Some applications of developed
method are considered in the subsequent paper (see Ref.[26]).

2. GCS AND QUANTUM MEASUREMENTS

In this section we briefly consider the factorization[7] of the OFS (1) and elucidate the geometrical meaning of GCS defined by Eq. (6) as well as the relation of the systems GCS with quantum measurement theory [15,27].

Let K be the stationary subgroup of the vector $|\Phi_0\rangle$, $V_h|\Phi_0\rangle = e^{ia(h)}|\Phi_0\rangle$, $h \in K \subset G$. Then it is clear that two vectors $|\Phi_{g_1}\rangle$ and $|\Phi_{g_2}\rangle$ differ from each other by a phase factor iff $g_2 = g_1 h$, i.e., iff g_1 and g_2 belong to the same coset gK. The state is determined by the point $x \in X = G/K$. One can construct an OFS according to (1) choosing a representative $s(x)$ from every coset gK. The so-obtained OFS factorized in this manner was called a system GCS [7]. The function $s(x)$ is evidently a cross-section in the group fibre bundle (G, X, π), π being the canonical projection. For a given $s(x)$ every element $g \in G$ can be written as a product $g = s(x_g)h_g$, where x_g denotes the coset to which g belongs. Thus

$$|\Phi_g\rangle = e^{ia(h_g)}|\Phi_{x_g}\rangle. \tag{7}$$

The operators of the representation V_g act on $|\Phi_x\rangle$ transitively to within a phase factor:

$$V_g|\Phi_x\rangle = V_g V_{s(x)}|\Phi_0\rangle = \exp(if(g,x))|\Phi_{x'}\rangle, \tag{8}$$

where x' is determined from the equation $gs(x) = s(x')h(g,x)$. Introducing the mappings $\delta: G \times K \to G$, $\delta(g,h) = \delta_h(g) = gh$ and $\sigma: \mathcal{G} \times K \to \mathcal{G}$, $\sigma(|\psi\rangle, h) = \sigma_h(|\psi\rangle) = e^{ia(h)}|\psi\rangle$ ($\mathcal{G}$ being the set of unit vectors in H), where $a(h)$ is defined by Eq. (4), one can consider G and $\mathcal{G}$ as K-manifolds. The following diagram

$$\begin{array}{ccc} G \times K & \xrightarrow{\delta} & G \\ \Phi \times \mathrm{id} \downarrow & & \downarrow \Phi \\ \mathcal{G} \times K & \xrightarrow{\sigma} & \mathcal{G} \end{array} \qquad \Phi: g \to |\Phi_g\rangle \tag{9}$$

is commutative and consequently the map $\Phi: G \to \mathcal{G}$ is a morphism in the category of K-manifolds[14]. The set of such morphisms $\mathrm{Hom}_K(G, \mathcal{G})$ is in one-to-one correspondence with the set of cross-sections $\tilde{\Phi}$ in the associated fibre bundle (M, X, ρ), where $M = (\mathcal{G} \times G)/K$ and $\rho: M \to X$ is the canonical projection: $\rho \circ \gamma = \pi \circ \mathrm{pr}_1$ [14]. (Here $\gamma: G \times \mathcal{G} \to M$ satisfy the following condition $\gamma(g, |\psi\rangle) = \gamma(gh, \sigma_h(|\psi\rangle)), h \in K$). The correspondence $\Phi \to \tilde{\Phi}$ is provided by the commutative diagram:

$$\begin{array}{ccc} G & \xrightarrow{\mathrm{id} \times \Phi} & G \times \mathcal{G} \\ \pi \downarrow & & \downarrow \gamma \\ X & \xrightarrow{\tilde{\Phi}} & M \end{array} \tag{10}$$

From this diagram it is easy to obtain

$$\tilde{\Phi}(x) = \gamma(g, |\Phi_g>), \qquad g \in \pi^{-1}(x) . \tag{11}$$

Defining $\gamma(g, |\psi>) = \exp(-i\alpha(h_g))|\psi>$ we may identify the system GCS $|\Phi_g>$ with the image of the cross-section $\tilde{\Phi}$.

The most important property of the system GCS is their overcompleteness, expressed by the equation:

$$\int \mu(dx)|\Phi_x><\Phi_x| = I , \tag{12}$$

where $\mu(dx)$ is an invariant measure on X. Eq. (12) implies that the family of operators

$$M(\Delta) = \int_\Delta \mu(dx)|\Phi_x><\Phi_x| \qquad \Delta \in \mathcal{B}(X)$$

(Δ being a Borel set in X, $\mathcal{B}(X) - \sigma$ - algebra of Borel sets) form a generalized resolution of identity (or positive operator-valued measure on X)[15]. According to a theorem due to Davies and Lewis[27] there exists at least one measurement $\mathcal{E}$, determined by the formula

$$\mathrm{tr}\,\mathcal{E}_\Delta(\rho) = \mathrm{tr}\rho M(\Delta) \tag{13}$$

(recall that the measurement $\mathcal{E}$ on X is a linear continuous map $\mathcal{E}: \mathcal{B}(X) \to \mathrm{Aut}\,\mathcal{J}(H)$, $\mathcal{J}(H)$ being the space of trace class operators in H, such that $\mathrm{tr}\mathcal{E}_X(\rho) = \mathrm{tr}\rho$, and the positive cone $\mathcal{J}(H)^+$ is invariant under $\mathcal{E}_\Delta$, $\Delta \in \mathcal{B}(X))$[28]. Basing on Eq. (13) the resolution of identity $M(\Delta)$ itself can be referred to as X-measurement[15] (Let us note that Holevo definition differs from the one given above). Moreover it is clear from the construction that this measurement is covariant with respect to the UIR V_g, i.e., the following condition is fulfilled[15]:

$$V_g^* M(\Delta)V_g = M(g^{-1}\Delta), \qquad \Delta \in \mathcal{B}(X) . \tag{14}$$

If the state of a quantum system is described by the density operator ρ, then the probability distribution of the results of the measurement $M(\Delta)$ is given by Eq. (13). The covariance property (14) permits one to establish a relation between the physical characteristics of the system and the resolutions of identity in Hilbert space H.

3. EVOLUTION OF GCS

The dynamics of quantum systems is described by the evolution operator S_t:

$$S_t = T \exp(- \int_0^t H(t)\,dt) ,$$

where $H(t)$ is the Hamiltonian of the system. The evolution of an OFS (1) is given by the relation $|\Phi_g; t> = S_t|\Phi_g>$. The direct

action by S_t on the vectors $|\Phi_g>$ may turn to be difficult even impossible especially in the case of nonstationary Hamiltonians. It is proved to be more effective the method of integrals of motion, developed in Refs.[5,11]. If A is an integral of motion then it commutes with the Schrödinger operator $D_S = i\partial_t - H$, ($[A, D_S] = 0$) and thereby one can get new solutions acting by A on a fixed solution $|\Phi_0; t>$. Thus if the operators of the representation $V(\ell_a) = V_g(t)$ are realized as integrals of motion, then by means of them one can obtain OFS (1) as solutions of the equation of motion according to the formulae

$$|\Phi_g(t)> = V_g(t)|\Phi_0(t)> = \exp(\ell_a L_a)|\Phi_0(t)> =$$
$$= S_t V_g S_t^{-1}|\Phi_0(t)> = S_t|\Phi_g> = |\Phi_g; t>. \qquad (15)$$

Here the generators $L_a(t) = S_t L_a S_t^{-1}$ are formal solutions of equation $[A, D_S] = 0$ and consequently are integrals of motion.

Because of the completeness of system of vectors (15) every solution of the Schrödinger equation may be realized in the carrier space H of UIR $V_g(t)$. The group with such a property is called dynamical symmetry group of the quantum system. For nonstationary systems the dynamical symmetry was studied in refs.[11,29]. Thus if the vectors $|\Phi_g> = V_g|\Phi_0>$ form on OFS in Hilbert space H then the related group G may serve as a group of dynamical symmetry for any quantum system. This assertion is in agreement with the results of Refs.[11,29], where it was shown that for N-dimensional system the noncompact group $U(N,1)$ can describe dynamical symmetry. Now we get from Eq. (15) that dynamical symmetry can be described by any Lie group G, which has UIR in H.

In practical calculations the more efficient way is to solve equation $[A, D_S] = 0$, looking for solutions $A(t)$ of some special form, say linear in generators L_a. Since L_a may be expressed in terms of the lowering and raising boson operators a, a^+, it is convenient to construct first the invariant operators

$$A = S_t a S_t^{-1} \qquad A^+ = S_t a^+ S_t^{-1} \qquad (16)$$

and then to use them for construction of generators $L_a(t)$. We follow this way in the subsequent paper [26] in constructing of exact time evolution of OFS for some groups and quantum systems. Here we would like to say that integrals of the form (16) are constructed explicitly for any quadratic Hamiltonian[20,21] and for some nonquadratic ones[11].

Let us turn to the question of stable evolution of OFS. The Klauder condition $S_t \mathscr{C} \subset \mathscr{C}$, $\mathscr{C}$ being the manifold of GCS, may be written more explicitly in the form:

$$S_t V(\ell_a) |\Phi_0\rangle = S_t |\Phi(\ell_a)\rangle = |\Phi(\ell_a(t))\rangle = V(\ell_a(t)) |\Phi_0\rangle, \qquad (17)$$

i.e., the whole time-dependence of the vectors from OFS is contained in the group parameters $\ell_a(t)$. It is apparent that Eq. (17) holds for any fixed vector $|\Phi_0\rangle$ iff S_t is an operator of the same representation: $S_t = V_{g(t)}$. Moreover I.A. Malkin has proved the following

Theorem [25]. Arbitrary system of GCS (1) remains stable under time evolution iff the Hamiltonian of the quantum system has the form

$$H = f_a(t) L_a , \qquad (18)$$

where $f_a(t)$ are arbitrary functions.

On the face of it this theorem contradicts to the well known fact [18-19] that the most general form of the Hamiltonian which preserved stable the system of usual CS is

$$H = \omega(t) a^+ a + F(t) a^+ + \bar{F}(t) a + \beta(t) \qquad (19)$$

which is nonlinear on a, a^+. Let us note however that the operators $a^+ a$, a, a^+, I form a projective representation of Lie algebra of the two-dimensional Euclidean group $E(2) = T^2 \textcircled{S} SO(2)$. Then using the relation

$$\exp(\lambda a^+ a + \mu a + \nu a^+) = \exp(u a^+) \exp(z a^+ a) \exp(v a) \exp(w),$$

where z, u, v, w depend on the canonical parameters λ, μ, ν, one can see that GCS for this representation of $E(2)$ and $|\Phi_0\rangle = |0\rangle$, coinsides with the usual CS. The Hamiltonian (18), predicted by Malkin theorem for $E(2)$, has just the form (19).

Let us now consider the special case when the fiducial vector $|\Phi_0\rangle$ is stable under the action of the evolution operator:

$$S_t |\Phi_0\rangle = |\Phi_0\rangle. \qquad (20)$$

Then for the stability of OFS $(V_g, |\Phi_0\rangle)$ the evolution operator S_t ought not to be operator from the representation V_g. S_t can be an (external) automorphism of the group of representation operators:

$$S_t V_g S_t^{-1} = V_{g(t)} , \qquad g = (\ell_a). \qquad (21)$$

The OFS $(V_g, |\Phi_0\rangle)$ for which conditions (20), (21) are satisfied may be called superstable relatively S_t (or corresponding Hamiltonian). Another explanation of the above-mentioned seeming contradiction can be given if one observes that the system of usual CS is superstable relatively to the Hamiltonian (19).

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A superstable OFS may be realized also when the generators of the representation are homogeneous functions of a, a^+ and S_t is the evolution operator corresponding to harmonic oscillator. Then

$$\exp(\ell_a L_a(t)) = \exp(\ell_a(t) L_a) \tag{22}$$

which can be easily proved using the formula[30]:

$$\exp(sa^+a) F(a, a^+) \exp(-sa^+a) = F(ae^{-s}, a^+ e^s). \tag{23}$$

Obviously this result can be easily extended to the case of N-dimensional oscillator. If X_i $(i=1,2,...,p+q)$ are generators of the ~~lowest~~ $(p+q)$-representation of Lie group $U(p,q)$, then the following Hermitean operators

$$L_i = \vec{\phi} X_i \phi, \qquad \phi = (a_1, ..., a_p, a^+_{p+1}, ..., a^+_{p+q})^T \tag{24}$$

$$\vec{\phi} = (a^+_1, ..., a^+_p, -a_{p+1}, ..., -a_{p+q})$$

are homogeneous generators of $U(p,q)$. Thus we derive that N-dimensional oscillator preserves stable GCS $(V_g, |\Phi_n\rangle)$ for any Lie group G, V_g being generated by the corresponding subset of operators (24) and $|\Phi_n\rangle$ any stationary state.

The stable evolution of the OFS is correctly determined by the functions $\ell_a(t)$[1], which are solutions of Euler equations for the functional

$$I(f) = \int dt(i\langle f| d/dt |f\rangle - \langle f|H|f\rangle) \tag{25}$$

whose domain is restricted on the OFS-manifold, i.e., $|f\rangle = |\Phi_g\rangle$, $g = (\ell_a(t))$. Minimizing functional (25) one gets the following equations[1]:

$$(\partial_a R_b - \partial_b R_a)\dot{\ell}_b = \partial_a H, \qquad \partial_a = \partial/\partial\ell_a$$

$$H = H(\ell_a) = \langle\Phi_g|H|\Phi_g\rangle, \qquad g = (\ell_a), \quad a = 1,2,...,r \tag{26}$$

$$R = (1 - \exp(-\ell_a C_a))(\ell_a C_a)^{-1} v, \quad v = (v_a),$$

$$v_a = i\langle\Phi_0|L_a|\Phi_0\rangle, \qquad (C_a)_{bd} = C^d_{ab},$$

where C^d_{ab} are structural constants of the group G. The classical action functional assumes the form:

$$I = \int dt(R_a\dot{\ell}_a - H). \tag{27}$$

If the matrix $\Omega_{ab} = \partial_a R_b - \partial_b R_a$ is symplectic, i.e., the 2-form $\Omega_2 = \Omega_{ab} d\ell_a \wedge d\ell_b$ is nondegenerate, closed $(d\Omega_2 = 0)$ and exact $(\Omega_2 = d\Omega_1)$, then the classical system described by Eqs.(26) was

studied by R.M.Santilli[31] and called by him Birkhoffian system, $\mathcal{H}$ being the Birkhoffian of the system. It is not difficult to see that the matrix Ω_{ab} is symplectic iff it is nondegenerate. The group manifold in this case has symplectic structure and may be regarded as a classical phase space. Involving the matrix Ω^{ab}, inverse to the matrix Ω_{ab} one can write Eqs. (26) in the form

$$\dot{\ell} = (\ell_a, \mathcal{H}), \quad a = 1,2,...,r,$$

where the brackets (,) are defined by

$$(A,B) = \Omega^{ab} \partial_a A \, \partial_b B$$

and apparently are generalized of usual Poisson brackets [31,32].

In the case when Ω_{ab} is singular (e.g., for groups with odd dimensions[1]) the equations of motion (26) do not determine the solution $\ell_a(t)$ uniquely. Then the dynamics of stable OFS can be effectively described by classical equations of motion in the quotient space $X = G/K$, (K being the stability subgroup of $|\Phi_0>$) which can be treated as phase space. The symplectic structure on X was constructed by E.Onofri[33]:

$$\omega = i(\partial^2 f/\partial z_i \, \partial z_j^*) \, dz_i \wedge dz_j^*, \tag{28}$$

where z_i are (complex) local coordinates on X and $f = f(z, z^*) = \ln|<\Phi_0|V_g|\Phi_0>|^{-2}$ is the so-called Kähler potential. Then on X there exists the Poisson bracket

$$(A, B) = g^{ij} (\partial_i A \partial_j^* B - \partial_i^* A \partial_j B)$$

$$\tag{29}$$

$$\partial_i A = \partial A/\partial z_i \qquad \partial_i^* A = \partial A/\partial z_i^*,$$

where g^{ij} is the matrix inverse to the matrix $\||i \partial^2 f/\partial z_i \partial z^*_j|\|$. Consequently the equation of motion for $z_i = z_i(t)$ has the form $\dot{z}_i = (z_i, \mathcal{H}) = g^{ij} \partial_j^* \mathcal{H}$, $\mathcal{H}$ being the classical Hamiltonian.

Finally we shall consider the time evolution of the probability distribution $w_\rho(\Delta) = tr \rho M(\Delta)$ when the density operator ρ evolve in time: $\rho(t) = S_t \rho S_t^{-1}$. Suppose that the system GCS, determining the resolution of identity $M(\Delta)$ (Sec.2) is stable under evolution operator S_t. Then $S_t = V_{g(t)}$ and making use of covariance property (14) one immediately derives

$$w_{\rho(t)}(\Delta) = tr(\rho(t) M(\Delta)) = tr(\rho M(g^{-1}(t)\Delta) = w_\rho(g^{-1}(t)\Delta),$$

i.e., the probability distribution is only translated by means of the group transformation $g^{-1}(t)$, providing some motion on phase space X. Thus the quantum evolution is represented in this case as a classical motion on the manifold X.

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