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ON THE "FEYNMAN PATHS"

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Let us assume a quantum-mechanical system with state Hilbert space $\mathcal{H}$ and time evolution governed by a self-adjoint Hamiltonian H . Let further $\sigma = \{ \tau_j : 0 = \tau_0 < \tau_1 < \ldots < \tau_n = t \}$ denote some partition of the interval $[0,t]$ with $\delta_j = \tau_{j+1} - \tau_j$ and $\delta(\sigma) = \max_{0 \le j \le n-1} \delta_j$. Having a strongly continuous projection-valued function $E : [0,b] \to \mathcal{B}(\mathcal{H})$, we can introduce

$$U_E(t,0;\sigma) = E(t)\, e^{-iH\delta_{n-1}} E(\tau_{n-1})\, e^{-iH\delta_{n-2}} \ldots e^{-iH\delta_0} E(0) \quad . \tag{1}$$

This expression has clear physical meaning, if $E(\tau)$ is interpreted as a yes-no experiment performed on the system at the time τ . It seems thus reasonable to accept that

$$U_E(t,0) = \underset{\delta(\sigma)\to 0}{\text{s-lim}}\, U_E(t,0;\sigma) \tag{2}$$

(if it exists) represents the evolution operator of the given system subjected during $[0,t]$ to a continual observation characterized by the function $E(.)$. Standard arguments yield some simple properties of $U_E(.,.)$:

$$\| U_E(t,0) \| \le 1 \quad , \quad 0 \le t \le b \quad , \tag{3a}$$

$$(\mathrm{Ker}\ U_E(t,0))^{\perp} = E(0)\mathcal{H} \quad , \quad \mathrm{Ran}\ U_E(t,0) = E(t)\mathcal{H} \quad , \tag{3b}$$

$$U_E(t,\tau)U_E(\tau,0) = U_E(t,0) \quad , \quad 0 \le \tau \le t \le b \quad . \tag{3c}$$

The known results [1-3] about the evolution operator (2) concern mostly the case when the apparatus function is constant, $E(t) = P$ for all $t \in [0,b]$. A special attention has been paid to the situation when $P\mathcal{H}$ represents a state space of some unstable system ; the evolution operators (1) have been studied together with more general ones corresponding to randomly distributed measurements (see [3-6] and references therein). The peculiar feature is that for a semibounded H continual observation prevents decay : it was recognized first in Ref.4 and later formulated as the so-called Zeno's paradox [3]. It should be mentioned, however, that in realistic physical models with large but finite density of measurements the paradoxical situation is avoided [7,8].

As we shall remark below, "Zeno's theorem" of Ref.3 can be strengthened if it is combined with the product formula of Chernoff [9].

Recently, Aharonov and Vardi [10] treated the case of continual observation in which the apparatus function is non-constant and its values are one-dimensional projections. They argue that the observation forces the system to follow the "trajectory" in $\mathcal{H}$ given by $E(.)$; they assigned to it meaning of an operationally determined "Feynman path". Further they show that if H is a Schrödinger operator in $L^2(\mathbb{R})$ and $E(.)$ corresponds to Gaussian wavepacket travelling along a given trajectory $x_0(.)$ in $\mathbb{R}$, then $\tilde{U}_E(.,0)$ (see (4) below) in effect multiplies this wavepacket by the familiar factor $\exp(iS(x_0))$, where $S(x_0)$ is the classical action along x_0 . The main purpose of the present letter is to show that the results of Aharonov and Vardi can be recovered rigorously under some mild smoothness assumptions and (what is more important) that they are not conditioned by particular assumptions made in Ref.10 about the measuring device (rotating Stern-Gerlach apparatus or reduction to Gaussian wavepacket) and the Hamiltonian. Some related problems will be mentioned in concluding remarks.

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We shall start with the following technical remark. Let σ_n be the equidistant partition of $[0,t]$, $\delta_j = t/n$. The operator-valued function

$$\tilde{U}_E : \tilde{U}_E(t,0) = \underset{n\to\infty}{\text{s-lim}}\, U_E(t,0;\sigma_n) \qquad (4)$$

is often studied instead of U_E because the limit here can be manipulated more easily.

Consider first the case of constant apparatus function

$$E(t) = P \quad \text{for all} \quad t \in \mathbb{R} \quad . \tag{5}$$

Essential characterization of the time evolution is then given by the assertion which follows from the Chernoff product formula [11] as a direct generalization to Theorem 2.2 of Ref.2 :

<u>Proposition</u> : Assume (5) with $\dim P < \infty$ and $P\mathcal{H} \subset D(H)$, then

$$\tilde{U}_E(t,0) = e^{-iPHP} P \quad , \quad t \in \mathbb{R} \quad . \tag{6}$$

This assertion generalizes in various ways. The assumption about finite dimension of P can be removed, for instance, if H is positive with PHP densely defined and t is complex with $\mathrm{Im}\, t < 0$ (cf. [1] and also [12], [13], theorem I.2) ; in such a case PHP on the rhs of (6) has to be replaced by the self-adjoint operator H_P associated with the quadratic form h : $h(\varphi) = \|H^{1/2}\varphi\|^2$, $Q(h) = D(H^{1/2}) \cap P\mathcal{H}$. As for the physically interesting case of real t , the following weaker assertion is valid [1] under the assumptions stated above :

$$\lim_{n\to\infty} \|U_E(t,0;\sigma_n)\varphi\| = \|P\varphi\| \quad \text{for all} \ \varphi \in \mathcal{H} \ \text{and almost all} \ t \in \mathbb{R} \ . \tag{7}$$

Thus if $U_E(t,0)$ exists it must be a partial isometry. With one additional assumption, $U_E(.,0) \upharpoonright P\mathcal{H}$ is even a strongly continuous unitary group similarly as in the finite-dimensional case (6) :

<u>Theorem 1</u> : Assume that (5) holds with P being an orthoprojection on $\mathcal{H}$. Let H be a semibounded self-adjoint operator on $\mathcal{H}$ with $D(H)$ dense in $P\mathcal{H}$, and let exist an antiunitary operator Θ such that

$$\Theta P \Theta^{-1} = P \quad , \quad \Theta e^{-iHt} \Theta^{-1} = e^{iHt} \quad \text{for each} \quad t \in \mathbb{R} \ . \tag{8}$$

Suppose that $U_E(t,0)$ exists for $t \in [0,b]$, $b > 0$, then there is a self-adjoint $H_P \supset PHP$ such that

$$U_E(t,0) = \exp(-iH_P t)\, P \quad \text{for all} \quad t \geq 0 \ . \tag{9}$$

Proof : The assumption (5) implies $U_E(t,0) = U_E(t+\tau,\tau)$ so $U_E(t,0)$ exists for each $t \geqslant 0$ due to (3c). It further implies existence of $\widetilde{U}_E(t,0) = U_E(t,0)$ for $t \geqslant 0$. The function F : $F(t) = P\exp(-iHt)P \upharpoonright P\mathcal{H}$ is obviously contraction-valued and strongly continuous with $F(0) = P$, and such that the strong derivative $\dot{F}(+0) \supset -iPHP$ is densely defined in $P\mathcal{H}$. Thus Theorem 2.1 of Ref.9 may be applied according to which $T(.) \equiv \widetilde{U}_E(.,0) \upharpoonright P\mathcal{H}$ is a continuous contractive semigroup and its generator iH_P is an extension of $-\dot{F}(+0) \supset iPHP$. Continuity of $T(.)$ together with semiboundedness of H and (8) make it possible to use the mentioned theorem of Misra and Sudarshan [3], which asserts that $T(.)$ is restriction to $[0,\infty)$ of a strongly continuous unitary group, i.e., that H_P is self-adjoint. ∎

Let us pass now to the case of non-constant apparatus function ; for the sake of simplicity we limit ourselves to the simplest possibility when $\dim E(.) = 1$.

Theorem 2 : Let H be a self-adjoint operator on $\mathcal{H}$, and let $E(t)$ for each $t \in [0,b]$ be one-dimensional projection corresponding to a unit vector $\psi_t \in D(H)$. Assume that $t \mapsto \psi_t$ is C^1 on $(0,b)$ and that $t \mapsto H\psi_t$ is C^0 on $[0,b]$, then the operator $U_E(t,0)$ exists and

$$\varphi_t \equiv U_E(t,0)\varphi = \exp\left\{ i \int_0^t L(\tau)\, d\tau \right\} (\psi_0,\varphi)\psi_t \quad , \tag{10a}$$

$$L(t) = -i(\dot{\psi}_t,\psi_t) - (\psi_t,H\psi_t) \tag{10b}$$

for all $t \in [0,b)$ and $\varphi \in \mathcal{H}$. Moreover, $U_E(.,0)\varphi$ obeys the equation

$$i\,\frac{d\varphi_t}{dt} = \left[i\dot{E}(t)E(t) + E(t)HE(t)\right]\varphi_t \quad . \tag{11}$$

Remark : $U_E(t,0)$ depends actually on $E(.)$ only, not on the representing vector-function : if $\widetilde{\psi}_t = \psi_t \exp(i\alpha(t))$ with α absolutely continuous, then $\widetilde{L}(t) = L(t) - \dot{\alpha}(t)$, and consequently $\widetilde{\varphi}_t = \varphi_t$.

Proof : For an arbitrary partition σ of $[0,t]$ and $\varphi \in \mathcal{H}$ we have $U_E(t,0;\sigma) = f(t,0;\sigma)(\psi_0,\varphi)\psi_t$, where

$$f(t,0;\sigma) = \prod_{j=0}^{n-1} g(\tau_{j+1},\tau_j) \quad , \quad g(r,s) = (\psi_r, e^{-iH(r-s)}\psi_s) \quad .$$

By standard limit arguments, the smoothness assumptions of the theorem imply that $g(.,.)$ is continuously differentiable in both arguments and that

$$L : \quad L(t) = -i \left.\frac{\partial g(r,t)}{\partial r}\right|_{r=t}$$

is continuous in $[0,b]$. Since $g(t,t) = 1$, we have also

$$L(t) = -i \left.\frac{\partial \ln g(r,t)}{\partial r}\right|_{r=t} \quad .$$

Further we express $\ln f(t,0;\sigma)$ with the help of Lagrange remainder theorem for the real and imaginary parts of $\ln g(r,s)$:

$$\ln f(t,0;\sigma) = \sum_{j=0}^{n-1} \ln g(\tau_{j+1},\tau_j) =$$
$$= i \sum_{j=0}^{n-1} \left[\mathrm{Re}\, L(\xi_j) + i\, \mathrm{Im}\, L(\xi_j') \right] \delta_j$$

with some $\xi_j, \xi_j' \in (\tau_j, \tau_{j+1})$, $j = 0,1,\ldots,n-1$. But $\mathrm{Re}\, L(.)$ and $\mathrm{Im}\, L(.)$ are continuous and therefore Riemann integrable in $[0,b]$ so we obtain

$$\lim_{\delta(\sigma)\to 0} \ln f(t,0;\sigma) = i \int_0^t \mathrm{Re}\, L(\tau)\, d\tau - \int_0^t \mathrm{Im}\, L(\tau)\, d\tau =$$
$$= i \int_0^t L(\tau)\, d\tau \quad ;$$

it proves (10). Verification of (11) is straightforward. ■

<u>Corollary 1</u> : Let $\psi \in \mathcal{S}(\mathbb{R}^d)$, $x_0 \in C^1[[0,b];\mathbb{R}^d]$ and ψ_t : $\psi_t(x) = \psi(x - x_0(t))$. Let further H be a self-adjoint operator on $\mathcal{H} = L^2(\mathbb{R}^d)$ such that $D(H) \supset \mathcal{S}(\mathbb{R}^d)$ and $t \mapsto H\psi_t$ is continuous on $[0,b]$, then

$$\varphi_t = \exp\left\{ i \int_0^t \left[\sum_{k=1}^{d} \dot{\mathcal{Q}}_k(\tau) \mathcal{P}_k(\tau) - \mathcal{E}(\tau) \right] d\tau \right\} (\psi_0,\varphi)\psi_t \qquad (12a)$$

for all $t \in [0,b)$ and $\varphi \in \mathcal{H}$, where

$$\mathcal{E}(t) = (\psi_t, H\psi_t) \quad , \quad \mathcal{P}_k(t) = (\psi_t, P_k\psi_t) \quad , \quad \mathcal{Q}_k(t) = (\psi_t, Q_k\psi_t) . \qquad (12b)$$

<u>Proof</u> : One has just to specify the first term on the rhs of (10b). Using continuity of $\dot{x}_0$ and the fact that $\psi_t \in \mathcal{S}(\mathbb{R}^d) \subset \bigcap_{k=1}^{d} D(P_k)$,

$(P_k\psi_t)(x) = -i(\partial\psi/\partial y_k)_{y=x-x_0(t)}$ together with simple estimates one finds the derivative

$$\dot{\psi}_t = -i \sum_{k=1}^{d} \dot{x}_{0k}(t) P_k \psi_t \quad ;$$

it is continuous due to continuity of $\dot{x}_0$ and of translations in $L^2(\mathbb{R}^d)$. Since $\dot{Q}_k(t) = \dot{x}_{0k}(t)$, the assertion follows. ■

Corollary 2 : Theorem 2 and Corollary 1 remain valid if H is assumed to be a pseudo-Hamiltonian, i.e., a closed operator on $\mathcal{H}$ such that iH generates a continuous contractive semigroup [14].

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It is desirable to generalize Theorem 2 for more general apparatus functions including infinite-dimensional ones. The following case is of particular interest : $H = H_0 + V$ is a Schrödinger operator on $\mathcal{H} = L^2(\mathbb{R}^d)$ and $E(t)$ is the projection on $L^2(M_t) \subset \subset \mathcal{H}$, where $M_t \subset \mathbb{R}^d$. If $H = H_0 = -\frac{1}{2}\Delta$, then

$$(U_E(t,0)\psi)(x) = \chi_{M_t}(x) \int_{\Gamma(M,x)}^{ufn} \psi(\gamma(0))\, D\phi(\gamma) \quad , \qquad (13)$$

where Feynman integral on the rhs of (13) is understood in the sense of Ref.16 and $\Gamma(M,x)$ consists of continuous paths γ : $[0,t] \to \mathbb{R}^d$ with $\gamma(t) = x$ and $\gamma(\tau) \in M_\tau$, $\tau \in [0,t]$. Thus the operator-valued functions U_E can replace in a sense the non-existing Feynman measure (cf.[2] ; compare to the analogous problem for the Wiener measure : [13], Lemma 7.10 and Sec.22).

It would be useful to obtain $U_E(.,0)$, say, by solving a differential equation instead of calculating the limit (2). This problem remains in general open for non-constant $E(.)$. One possibility is represented by the eq.(11) which was first obtained formally by Bloch and Burba [15]. Actually, such an equation holds for $\varphi_t^\sigma = U_E(t,0;\sigma)$ whenever its rhs makes sense ; so it applies to φ_t if the interchange of the derivative with the limit $\delta(\sigma) \to 0$ can be justified. On the other hand, in the above mentioned case with permanent localization of a Schrödinger particle to M_t (when $\dot{E}(t)$ in general does not exist), Friedman proposed [2] the equation

$$i\,\frac{d\varphi_t}{dt} = H_{E(t)}\varphi_t \quad , \qquad (14)$$

where $H_{E(t)}$ is a suitable extension of $E(t)HE(t)$ (multiple of the Laplace-Dirichlet operator for smoothly varying interval $M_t \subset \mathbb{R}$ considered in Ref.2) . He exhibited some conditions under which solution of (14) would exist, however, he gave no proof that it would obey $\varphi_t = U_E(t,0)\varphi$.

The last remark concerns the situation when H is a pseudo-Hamiltonian, i.e., when the undisturbed motion of the system is dissipative. Corollary 2 suggests that the Feynman dynamical formula might hold,e.g., for Schrödinger Hamiltonians with complex absorptive potentials too ; this fact was already established for some particular classes of potentials within various definitions of the Feynman integral [16-18].

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