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THRESHOLD EFFECTS AT TWO-LOOP LEVEL AND PARAMETRIZATION OF THE REAL QCD

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1. In the renormalization group (RG) treatment of QCD the parametrization by scale parameter Λ is widely accepted and used in the analysis of data. It seems to be quite natural [1] for quantum field models with asymptotic freedom in the region of energies much larger than all particle masses.

In this note we want to stress that formulas containing Λ, like the "popular" 2-loop approximation for invariant coupling (IC)

$$\bar{g}(Q^2,f) = \frac{1}{\beta_f L} - \frac{b_f \ln L}{(\beta_f L)^2}, \qquad g = \frac{\alpha_s}{4\pi} \tag{1}$$

$$L = \ln\frac{Q^2}{\Lambda^2}, \quad \beta_f = 11 - \frac{2}{3}f, \quad b_f \beta_f = 102 - \frac{38}{3}f$$

and corresponding expressions for matrix elements in the region of the real nowaday application of QCD turn out to be inadequate and propose to "turn back" to a formulation based on more traditional RG parameters: the normalization point μ^2 and value of IC $g = \bar{g}(\mu^2)$ in it (see below Eq. (12)).

The matter is that the effective scale parameter Λ in Eq. (1) is not universal and like numerical coefficients β_f, b_f, depends on the flavour number f. Under the real conditions in the vicinity of points $Q^2 = M_f^2$ (M_f being the threshold energy of an f-th quark pair creation), which represent the "mirror image" of location of threshold singularities at $Q^2 = -M_f^2$, there occurs a smooth change of the number of operating quarks which can be conveniently described by the expression [2] :

$$f(Q^2) = \sum_f \left\{ 1 + \frac{5}{4} \frac{M_f^2}{Q^2} \right\}^{-1} . \tag{2}$$

Correspondingly, the Λ parameter in the course of travelling across the mirror threshold smoothly changes its value from Λ_{f-1} to Λ_f. The relation between these limiting values can be obtained from the continuity condition of $\bar{g}$ at the point $Q^2 = M_f^2$

$$\bar{g}(M_f^2, f-1) = \bar{g}(M_f^2, f). \tag{3}$$

Using, for a qualitative estimate the 1-loop approximation to Eq. (1) and expanding in the small parameter $\delta_f = (\beta_{f-1} - \beta_f)/\beta_f \sim 10^{-1}$ one can get [3]

$$\ln \frac{\Lambda_{f-1}^2}{\Lambda_f^2} = \frac{2}{33-2f} \ln \frac{M_f^2}{\Lambda_f^2} . \tag{4}$$

It follows from this expression that Λ_f decreases with growing f, i.e., with energy and that the relative jump increases in magnitude with growing the threshold number f and threshold mass M_f. The inclusion of the 2-loop term in r.h.s. of Eq. (1) slightly enlarges the jump value

$$\ln \frac{\Lambda_{f-1}^2}{\Lambda_f^2} \Bigg|_{2\text{-}loop} = \frac{\delta_f + C(\xi_f - \delta_f)}{1 + C \dfrac{\ln L - 1}{\ln L}} \ln \frac{M_f^2}{\Lambda_f^2} , \tag{5}$$

$$\delta_f = \frac{\beta_{f-1}}{\beta_f} - 1 \sim \frac{1}{14} , \quad \xi_f = \frac{b_{f-1}}{b_f} - 1 \sim \frac{1}{6} , \quad C = \left(\frac{\beta_f L_f}{b_f \ln L_f} - 1 \right)^{-1} .$$

2. For a more accurate description of threshold effects one has to analyse the RG equations written in the 2-loop approximation with the account of finite masses. Starting with the standard perturbation theory (in the MOM regularization scheme)

$$\begin{aligned} \bar{g}_{p.th}(x, y, g) = g - g^2 \left[J\left(\tfrac{x}{y}\right) - J\left(\tfrac{1}{y}\right) \right] + \\ + g^3 \left[J\left(\tfrac{x}{y}\right) - J\left(\tfrac{1}{y}\right) \right]^2 - g^3 \left[\Psi\left(\tfrac{x}{y}\right) - \Psi\left(\tfrac{1}{y}\right) \right] , \end{aligned} \tag{6}$$

where $x = \frac{Q^2}{\mu^2}$, $y_i = \frac{\mu_i^2}{\mu^2}$, J is the exact sum

$$J(t) = 9\ln t - \frac{2}{3} I_1(\xi_4 t) - \frac{2}{3} I_1(\xi_5 t) - \cdots, \quad \xi_i = \frac{\mu^2}{m_i^2} \quad , \tag{7}$$

of one-loop vacuum polarization contributions

$$I_1(t) = 6\int_0^1 dx\,(1-x)x\ln[1+t\,x(1-x)] \to \ln t \qquad \text{as } t\to\infty$$

and $\Psi(t)$ is the analogous sum of intrinsic 2-loop contributions

$$\Psi(t) = 64\ln t - \frac{38}{3} I_2(\xi_4 t) - \frac{38}{3} I_2(\xi_5 t) - \cdots \quad , \tag{8}$$

we arrive at the RG differential equation

$$y\frac{\partial \bar{g}(x,y,g)}{\partial x} = -J'\left(\frac{x}{y}\right)\bar{g}^2(x,y,g) - \Psi'\left(\frac{x}{y}\right)\bar{g}^3(x,y,g) . \tag{9}$$

This Eq. admits exact solution in the one-loop approximation (see [4] page 523):

$$\bar{g}_1(x,y,g) = \frac{g}{1+g[J(\frac{x}{y}) - J(\frac{1}{y})]} \quad .$$

Starting with it we solve Eq. (9) by the successive approximation method and obtain in the second approximation

$$\frac{g}{\bar{g}_2(x,y,g)} = 1 + g[J(\tfrac{x}{y}) - J(\tfrac{1}{y})] + g^2\int_{1/y}^{x/y} \frac{\Psi'(\tau)\,d\tau}{1+g[J(\tau) - J(\frac{1}{y})]} \, . \tag{10}$$

This expression makes the basis for the analysis of mass (threshold) effects at the 2-loop level. It is clear from Eq. (10) that the contributions from "light" and "heavy" quark loops interfere only in terms $\sim g^4$. The error of Eq. (10) is of g^5 order.

To get the transparent formula for numerical estimates we approximate the denominator of the integrand in (10)

$$J(t) \to \frac{9}{64}\Psi(t) = \qquad (11)$$

$$= 9\ln t - \frac{57}{32} I_2(\xi_4 t) - \frac{57}{32} I_2(\xi_5 t) - \frac{57}{32} I_2(\xi_6 t)$$

as to perform the integration. The approximation (11) consists of two ingredients. First, it equates the 1-loop I_1 (t) to 2-loop I_2 (t) contribution of the "polarization" type (from propagators and vertices) . This step can be reasoned by that the considered functions are similarly normalized at infinity $I_{1,2}(t) \to \ln t$ and enter into RG expressions subtracted at the same normalization point. The second, more essential approximation consists of the change of the numerical coefficients: (2/3) $\to$ (57/32) that takes place, however, on the big logarithmic background ($= 9\ln t$). We estimate the error due to this second approximation to be of order 20% in terms $\sim g^4$. This seems to be ruther reasonable as far as in the considered energy interval $g = \alpha_s/4\pi \sim 10^{-2}$.

We get now

$$\frac{g}{\bar{g}_2(x,y,g)} = 1 + g[9\ln x - \frac{2}{3}(I_1(\frac{x}{y}) - I_1(\frac{1}{y}))] + \qquad (12)$$

$$+ \frac{64}{9} g \ln\{1 + g[9\ln x - \frac{57}{32}(I_1(\frac{x}{y}) - I_1(\frac{1}{y}))]\}.$$

This is our final result. Its natural parametrization (similar to that in QED) is the coupling value g referred to the normalization point $Q^2 = \mu^2$.

3. To understand the correspondence with the popular 2-loop Eq. (1) and Λ -parametrization, we apply the numerical comparison. We fix several solutions (12) by choosing g referred to normalization point μ^2 = 10 GeV^2 equal to 100g= 1,2; 1,5; 1,8; 2,0; 2,4 and 2.8. Comparing them with the popular 2-loop Eq. (1) in 3,4 and 5 flavour regions we get Λ_3, Λ_4 and Λ_5 values given in Table 1.

Table I

No. of solution	I	2	3	4	5	6
$100\alpha_S(10\,Gev^2)/\pi$	4.8	6.0	7.2	8.0	9.6	II.2
Λ_3 / Mev	75	I70	300	375	550	760
Λ_4 / Mev	50	I30	235	305	470	670
$100\alpha_S(100\,Gev^2)/\pi$	3.9	4.6	5.2	5.6	6.4	7.0
Λ_5 / Mev	27	75	I50	200	3I5	460
Λ_6 Mev, $M_{\bar{t}t}$=37Gev	II	32	65	88	I46	2I7
Λ_6 Mev, $M_{\bar{t}t}$=100Gev	9	28	60	80	I32	20I

The last two lines in Table 1 contain also Λ_6 values for two different hypotheses on $t\bar{t}$ -pair mass.

Let us stress that the given Λ_f values correspond to the popular Eq.(1) taken at the same integer flavour number f. Therefore, Λ_f values thus calculated are discrete by definition. Under real conditions for analysis of experimental data on deep inelastic scattering within a given limited interval of momentum transfer lying in the intermediate (from f to $f+1$) region one has to use either our more exact formula (12) or the popular Eq. (1) with a continuous number of effective flavours given by the Georgi-Politzer Eq. (2). The Λ value thus obtained will turn out to be intermediate between the corresponding values Λ_f and Λ_{f+1} and can be compared with the value Λ obtained from different experiment by calculating by our Eq. (12).

Table 2 represents the dependence of the parameter Λ_{SLAC} corresponding to the Q^2 interval on the well-known SLAC

Table 2

Λ_{NA4}/Mev	40	80	160	220	340	500
Λ_{SLAC}/Mev	80	150	280	350	510	710

experiment of Λ_{NA4} values corresponding to recently completed NA4 experiment at CERN.

Naturally, the equations for the moments of deep-inelastic structure functions should be appropriately modified according to the solution of RG equations with masses.

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