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ON FUNCTIONAL DERIVATIVES<br>OF RENORMALIZED<br>WILSON FUNCTIONALS

## 1. INTRODUCTION

In the last two years the Wilson functional in $\mathrm{QCD} W(\mathrm{C})=$ $=\left\langle\mathrm{P} \exp \operatorname{ig} \oint \mathrm{A}_{\mu} \mathrm{d} \mathrm{x}^{\mu}\right\rangle$ has attracted considerable interest. Being gauge invariant it is a distinguished object suited not only for construction of composite operators but also for discussing confinement in a gauge invariant manner. The first step in the study of $W(C)$ was to investigate its renormalization properties. Now this problem can be considered to be completely understood, both for simple smooth contours ${ }^{/ 1 /}$, and for contours with cusps or double points/2/.

More involved and not yet clarified is the case of field theoretic functional equations which have been derived with the intention to obtain non-perturbative solutions for $W(C))^{/ 3,4,5 /}$ These equations appear in two forms which, albeit equivalent in regularized theory, may differ with respect to removing the regularization:

$$
\begin{align*}
& \frac{\delta^{2} \mathrm{~W}}{\delta \mathrm{x}_{\mu}(\eta) \delta \mathrm{x}_{\mu}\left(\eta^{\prime}\right)}=-\left\langle 0!\mathrm{U}\left(\eta_{\mathrm{f}}, \eta\right) \mathrm{g} \mathrm{~F}_{\mu \nu}(\eta) \dot{\mathrm{x}}_{\nu} \mathrm{U}\left(\eta, \eta^{\prime}\right) \mathrm{gF} \mathrm{~F}_{\mu}\left(\eta^{\prime}\right) \dot{\mathrm{x}}_{\lambda} \mathrm{U}\left(\eta^{\prime}, 0\right) \mid 0\right\rangle  \tag{1}\\
& +\mathrm{g}^{2} \delta\left(\eta-\eta^{\prime}\right) \dot{\mathrm{x}}^{2} \int \mathrm{~d} \tau \delta^{(4)}(\mathrm{x}(\tau)-\mathrm{x}(\eta))\langle 0| \mathrm{U}\left(\eta_{\mathrm{f}}, \tau\right) \mathrm{t}_{\mathrm{a}} \mathrm{U}(\tau, \eta) \mathrm{t}_{\mathrm{a}} \mathrm{U}(\eta, 0)|0\rangle \tag{2}
\end{align*}
$$

or $\partial_{\mu} \frac{\delta}{\delta \sigma} \frac{\mathrm{W}}{\mu \nu}=\mathrm{g}^{2} \dot{\mathrm{x}}^{2} \rho \mathrm{~d} r \delta^{(4)}(\mathrm{x}(r)-\mathrm{x}(\eta))\langle 0| \mathrm{U}\left(\eta_{\mathrm{f}}, r\right) \mathrm{t}_{\mathrm{a}} \mathrm{U}(\tau, \eta) \mathrm{t}_{\mathrm{a}} \mathrm{U}(\eta, 0)|0\rangle$
with

$$
\mathrm{U}\left(\eta_{2^{\eta}} \eta_{1}\right)=\mathrm{P} \operatorname{expig} \int_{\eta_{1}}^{\eta_{\mathcal{L}}} \mathrm{d}_{\eta} \mathrm{A}_{\mu}(\mathrm{x}(\eta)) \dot{\mathrm{x}}_{\mu}(\eta)
$$

In our earlier papers $/ 6,7 /$ we have studied the first equation restricted to the disjoint case $\eta \neq \eta^{\prime}$. Besides an estimation of the short-distance behaviour $\langle\mathrm{UgFUgFU}>-| \eta-\left.\eta^{\prime}\right|^{-4}\left[\log \left|\eta-\eta^{\prime}\right|\right]^{-1}$ (obtained from RG, OPE and asymptotic freedom) we noticed, that for smooth contours without double points the operator insertion on r.h.s. of equ. (1) needs no additional Z factors for renormalization. In other words, one obtains

$$
\left\langle\mathrm{UgF} \mathrm{~F}_{\mu} \dot{\mathrm{x}}_{\nu} \mathrm{UgF} \mathrm{~F}_{\mu \lambda} \dot{\mathrm{x}}_{\lambda} \mathrm{U}\right\rangle=\left\langle\mathrm{Ug} \mathrm{~F}_{\mu \nu} \dot{\mathrm{x}}_{\nu} \mathrm{UgF} \mathrm{~F}_{\mu \lambda} \dot{\mathrm{x}}_{\lambda} \mathrm{U}\right\rangle^{\mathrm{ren}} \eta \neq \eta^{\prime}
$$

at least at one-loop level, i.e., up to order $\mathrm{g}^{4}$.

In the present paper we want to extend the investigation of equ. (1) to all values of $\eta, \eta^{\prime}$ not excluding coincident points. The appearence of singularities like $\left|\eta-\eta^{\prime}\right|^{-4}$ or $\delta\left(\eta-\eta^{\prime}\right)$ proposes to consider $\frac{\delta^{2} W}{\delta \times(\eta) \delta \times\left(\eta^{\prime}\right)}$ as a distribution with respect to the contour parameter. This will be outlined in section 2. Then, provided that renormalization of $W$ has already been performed* the question about the validity of equ. (1) for renormalized Green's functions is reduced to that of existence of r.h.s. of (1) as a distribution. Applying in section 3 the method of OPE to dimensionally regularized field theory we show in order $g^{4}$ how the $r . h . s$. of (l) constitutes itself as a distribution without any need for additional subtractions. When, however, looking on the second term of r.h.s. (1) separately, we will observe an infinity $\sim 1 / \epsilon \mathrm{g}^{6}$ (section 4).

Since it appear's to be almost evident that starting from a renormalized $W$ one gets well-defined functional derivatives $\frac{\delta^{2} \mathrm{~W}}{\delta \mathrm{x}(\eta) \delta \mathrm{x}\left(\eta^{\prime}\right)}$ of infinities among the first and the second term of equ. (1) beginning with order $g^{6}$ (of course, this hypothesis has not been proven here). Because there is no possibility for such cancellations in equ. (2) this equation needs additional subtractions besides those guaranteeing a finite $W$. One should mention, however, that equ. (2) in most cases has been applied to the regularized theory only.

## 2. GENERAL ARGUMENTS

As usual we consider functional derivatives as distributions expressing the response of a functional $F$ defined over smooth simple contours to variations within the space of such contours. We start, e.g., with a closed contour $\mathrm{x}_{\mu}(\eta)$ $(0 \leq \eta \leq 1, \quad x(0)=x(1))$, add a small variation ay $y_{\mu}(\eta)$ retaining the resulting contour in the basic space chosen and define the functional derivatives by

$$
\begin{equation*}
\left.\frac{\partial \mathrm{F}(\mathrm{x}+\mathrm{ay})}{\partial \mathrm{a}}\right|_{\mathrm{a}=0}=\int_{0}^{1} \mathrm{~d} \eta \frac{\delta \mathrm{~F}(\mathrm{x}(\eta))}{\partial \mathrm{x}_{\mu}(\eta)} \mathrm{y}_{\mu}(\eta) \tag{3}
\end{equation*}
$$

[^0]\[

$$
\begin{equation*}
\left.\frac{\partial^{2} F(\mathrm{x}+\mathrm{ay})}{\partial \mathrm{a}^{2}}\right|_{\mathrm{a}=0}=\int_{0}^{1} \mathrm{~d} \eta \mathrm{~d} \eta^{\prime} \frac{\delta^{2} \mathrm{~F}(\mathrm{x}(\eta))}{\delta \mathrm{x}_{\mu}(\eta) \delta \mathrm{x}_{\nu}\left(\eta^{\prime}\right)} y_{\mu}(\eta) \mathrm{y}_{\nu}\left(\eta^{\prime}\right) \tag{3}
\end{equation*}
$$

\]

Now for closed smooth simple contours $x(\eta)$ the dimensionally regularized Wilson functional W requires no overall $Z$ factor for renormalization/1/. The limit

$$
\lim _{\epsilon \rightarrow 0} W_{\epsilon}\left(g_{\epsilon}, x(\eta)\right)=W(g, x(\eta))
$$

exists in the sense of analytic continuation (here $g_{\epsilon}=\mu{ }^{\epsilon / 2} Z_{g} g$ denotes the regularized bare coupling constant, compare equ. (12), and $\epsilon=4-d$ ). With respect to derivatives of the renormalized $W(g, x(\eta))$ the questions arise, whether

$$
\left.\frac{\partial W(g, x+a y)}{\partial a}\right|_{a=0} \quad \text { and }\left.\quad \frac{\partial^{2} W(g, x+a y)}{\partial a^{2}}\right|_{a=0} \quad \text { exist }
$$

and whether the relation

$$
\begin{equation*}
\lim _{\epsilon \rightarrow 0}\left\{\left.\frac{\partial^{2} W_{\epsilon}\left(g_{\epsilon}, x+a y\right)}{\partial a^{2}}\right|_{a=0}\right\}=\left.\frac{\partial^{2}}{\partial a^{2}} W(g, x+a y)\right|_{a=0} \tag{4}
\end{equation*}
$$

is fulfilled (together with an analogous relation for the first derivative, of course):

Although we have no thorough mathematical proof of (4), this relation is almost obvious since the point $a=0$ is in no respect distinguished from other ones in the space of smooth simple closed contours. From general experience the singular case of an unallowed interchange of limits would be a reflection of some distinguished situation in loop space.

This line of arguments also exhibits the striking difference to the area derivative used in equ. (2). There the variation is defined by adding to a given contour a small loop. Then the limiting case is distinguished since it even corresponds to a change of the topology of the contour.

Applying now these general arguments to $W_{\epsilon}\left(g_{\epsilon}, x_{\mu}(\eta)+a \delta_{\mu \nu} y(\eta)\right)$ we get

$$
\begin{align*}
& \left.\left.\frac{\partial W}{\partial \mathrm{~W}}\right|_{\mathrm{a}=0}=\lim _{\epsilon \rightarrow 0} \mathrm{i} \int_{0}^{1} \mathrm{~d} \eta<z(1) \Omega_{\nu}(\eta) \bar{z}(0)\right\rangle_{\epsilon} \mathrm{y}(\eta)  \tag{5}\\
& \left.\sum_{\nu=1}^{4} \frac{\partial^{2} \mathrm{~W}}{\partial \mathrm{a}^{2}}\right|_{\mathrm{a}=0.0}=\lim _{\epsilon \rightarrow 0} \int_{0}^{1} \mathrm{~d} \eta \mathrm{~d}^{\prime}\left[-\left\langle\mathrm{z}(1) \Omega_{\lambda}(\eta) \Omega_{\lambda}\left(\eta^{\prime}\right) \bar{z}(0)\right\rangle_{\epsilon} \mathrm{y}(\eta) \mathrm{y}\left(\eta^{\prime}\right)\right.  \tag{6}\\
& \quad+\mathrm{g}_{\epsilon}^{2} \dot{\mathrm{x}}(\eta) \dot{\mathrm{x}}\left(\eta^{\prime}\right) \delta_{\epsilon}^{\left.(4)\left(\mathrm{x}(\eta)-\mathrm{x}\left(\eta^{\prime}\right)<\mathrm{z}(1)\left(\overline{\mathrm{z}} \mathrm{t}_{\mathrm{a}} \mathrm{z}(\eta)\right)\left(\overline{\mathrm{z}} \mathrm{t}_{\mathrm{a}} \mathrm{z}\left(\dot{\eta}^{\prime}\right)\right) \overline{\mathrm{z}}(0)\right\rangle_{\epsilon} \mathrm{y}^{2}(\eta)\right]}
\end{align*}
$$

In writing down equations (5) and (6) we turned to the very useful formalism of auxiliary $z$ field $/ 3 /$, where $W$ is given by
$\mathrm{W}_{=\sim}\left\langle\mathrm{z}(1) \overline{\mathrm{z}}(0)>\right.$. Furthermore $\Omega_{\lambda}(\eta)=\mathrm{g} \overline{\mathrm{z}}(\eta) \mathrm{F}_{\lambda \rho}(\mathrm{x}(\eta)) \mathrm{z}(\eta) \dot{x}_{\rho}(\eta), \quad \mathrm{t}_{\mathrm{a}}$ are generators of the $\mathrm{SU}(\mathrm{N})$ gauge group, $<>\epsilon$ denotes regularized Green's functions, $\delta_{\epsilon}^{(4)}(. x)=\frac{\epsilon}{2 \pi^{2}} \frac{1}{|x|^{4-\epsilon}} \quad$ the dimensionally regularized $\delta$-function and $\mathrm{x}_{\mu}=\mathrm{dx} \mathrm{x}_{\mu}(\eta) / \mathrm{d}_{\eta}$.

From (5) we conclude that the composite operator $\Omega_{\mu}(\eta)$ requires no $Z$ factor, a point checked by explicit one-loop calculations/6/ already. Equ. (6) gives the $z$ formalism version of the r.h.s. of (1) integrated over with smooth test functions $y(\eta)$. Its limit must exist in the course of removing the regularization. This yields highly non trivial constraints on the uv divergencies and short distance $\left(\eta \rightarrow \eta^{\prime}\right)$ singularities of r.h.s. of equ. (1).

As will be shown in section 4 the second term in equ. (1) has no pole up to orderg ${ }^{4}$. Then the existence of the limit (6) gives a restriction on the possible short distance singularities in $\left\langle z(1) \Omega_{\mu}(\eta) \Omega_{\mu}\left(\eta^{\prime}\right) \vec{z}(0)\right\rangle$. The limit $\epsilon \rightarrow 0$ of the integrated regularized expression should exist, or put in other words, the short distance singularities of the renormalized expression must be well defined as one-dimensional distributions over the $\eta$-parameter space without any additional subtraction. In general such additional subtractions are necessary to define insertions of more than one composite operator. A well-known example is the two-point function of a conserved current where the product of two propagators $\frac{1}{(x-y)^{2}} \frac{1}{(x-y)^{2}}$ requires subtractions to be defined as a distribution in $R_{4}$. As further shown in section 4 the second term developes a pole $1 / \epsilon$ in order ${ }^{6}$. Equ. (6) then demands a conspiracy between this divergency and the short distance singularity of the first term.
3. OPERATOR PRODUCT EXPANSION OF $\dot{\Omega}_{\mu}(\eta) \Omega_{\mu}\left(\eta^{\prime}\right)$

The short distance singularity of $\left\langle z(1) \Omega_{\mu}(\eta) \Omega_{\mu}\left(\eta^{\prime}\right) \overline{z( }\right) \gg$ will
studied with the help of the OPE be studied with the help of the OPE

$$
\begin{equation*}
\Omega_{\mu}(\eta) \Omega_{\mu}\left(\eta^{\prime}\right)=\sum_{\mathrm{i}} \mathrm{e}^{(\mathrm{i})}\left(\eta^{\prime}-\eta\right) \mathrm{O}^{(\mathrm{i})}(\eta) \tag{7}
\end{equation*}
$$

Let us start with listing all the gauge invariant operators $O^{(i)}$ of canonical dimension from zero up to three giving rise to coefficient functions with short distance singularities. To simplify notation we use the special parametrization defined by $\mathrm{x}^{2}=1$ and
$\mathrm{Dz}=\dot{\mathrm{z}}-\mathrm{ig} \cdot \mathrm{A} \dot{\mathrm{x}} \mathrm{z}, \quad \overrightarrow{\mathrm{z}} \stackrel{\leftarrow}{\mathrm{D}}=-\dot{\bar{z}}-\mathrm{ig} \overline{\mathrm{z}} \mathrm{A} \dot{\mathrm{x}}$.
dimension $0: \bar{z} z \quad$ compare $^{/ 6 /}$, dimension 1: $\bar{z} D z, \bar{z} D z$


When putting the OPE into the Green's functions under consideration the operators containing $D z$ or $\vec{z} \stackrel{\rightharpoonup}{\mathrm{D}}$ yield a vanishing contribution for closed contours due to the equation of motion for the $z$ field. Thus we get rid of terms like $\left|\eta-\eta^{\prime}\right|^{\epsilon-3}\langle z(1)(\bar{z} D z) \bar{z}(0)>$. The coefficient functions for the remaining operators are either even or odd functions of $\eta^{\prime \prime}-\eta$. This symmetry can in each case be seen by lowest order calculations or in general by the use of the following symmetry transformation. Let us choose the parametrization of the contour in such a way that instead of (7) we have an expansion in the form

$$
\begin{equation*}
\Omega_{\mu}(-\eta / 2) \Omega_{\mu}(\eta / 2)=\sum_{i} c^{(i)}(\eta) 0^{(\mathrm{i})}(0) \tag{7'}
\end{equation*}
$$

The r.h.s. as well as the Lagrangian of the theory $f \mathrm{dx}_{\mathrm{I}} \mathscr{L}_{\mathrm{YM}}{ }^{+}$ $+\int \mathrm{d} \eta \overline{\mathrm{z}} \mathrm{Dz}$ are invariant with respect to

$$
\begin{align*}
& \eta \rightarrow-\eta, \quad \bar{z}(\eta) \rightarrow \mathrm{i} z(-\eta), \quad \mathrm{z}(\eta) \rightarrow \mathrm{i} \overline{\mathrm{z}}(-\eta), \\
& \mathrm{g} \mathrm{~A}_{\mu}(\mathrm{x}) \rightarrow-\mathrm{g} \mathrm{~A}_{\mu}^{\mathrm{T}}(\mathrm{x}) \tag{9}
\end{align*}
$$

Under this transformation $\bar{Z} \bar{z} \ddot{\mathbf{x}}{ }^{(3)}$ and $\bar{z} \bar{z} \dot{x}{ }^{(4)}$ are odd, but $\bar{z} z, \bar{z} z \ddot{x}^{2}, \bar{z} z \dot{x} x^{(3)}, \bar{z} F_{\mu \nu} z \ddot{x}_{\mu} \dot{x}_{\nu}, \bar{z} D_{\mu} F_{\mu \nu} z \dot{x} \quad$ are even. Therefore the short distance singularities have the structure

$$
\begin{equation*}
\left|\eta^{\prime}-\eta\right|^{\epsilon-4},\left|\eta^{\prime}-\eta\right|^{\epsilon-2},\left|\eta^{\prime}-\eta\right|^{\epsilon-1} \quad \text { and }\left|\eta^{\prime}-\eta\right|^{\epsilon-1} \cdot \operatorname{sgn}\left(\eta^{\prime}-\eta\right) \tag{10}
\end{equation*}
$$

Now $\left|\eta^{\prime}-\eta\right|^{\epsilon-4},\left|\eta^{\prime}-\eta\right|^{\epsilon-2}$ and $\left|\eta^{\prime}-\eta\right|^{\epsilon-1} \operatorname{sgn}\left(\eta^{\prime}-\eta\right)$ are we11 defined one-dimensional distributions including the limit $\xi=0$. The critical case is $\left|\eta^{\prime}-\eta\right|^{\epsilon-1}$ multiplying the operators $\bar{z} F_{\mu \nu} \mathbf{z} \ddot{\mathbf{x}}_{\mu} \dot{\mathbf{x}}_{\nu}$ and $\bar{z} \mathrm{D}_{\mu} \mathrm{F}_{\mu \nu} \mathbf{z} \dot{\mathbf{x}}_{\nu}$. A rather lengthy explicit calculation partly reported in the appendix gives zero for the expansion coefficient of $\bar{z} F_{\mu \nu} Z \ddot{x}_{\mu} \dot{x}_{\nu} \quad u p$ to total order $g^{4}$. To illustrate the non trivial nature of this result we mention that one must take into account not only $\overline{\mathrm{z}} \mathrm{F}_{\mu \nu} \mathrm{z} \ddot{\mathrm{x}}_{\mu} \mathrm{x}_{\nu}$ but also some operators building up $\overline{\mathrm{z}} \mathrm{D}^{3} \mathrm{z}$ and $\overline{\mathrm{z}} \overline{\mathrm{D}}^{3} \mathrm{z}$ (see appendix). The expansion coefficient of $\bar{Z} D_{\mu} F_{\mu \nu} \mathrm{z}_{\nu}$ starts with order $\mathrm{g}^{3}$. Using the equation of motion

$$
\begin{aligned}
& \left\langle z(1)\left(\bar{z} D_{\mu} F_{\mu \nu} z(\eta) \dot{x}_{\nu}\right) \bar{z}(0)\right\rangle_{\epsilon}= \\
& =-i g \int \mathrm{~d} \tau \dot{\mathrm{x}}(r) \dot{\mathrm{x}}(\eta) \delta_{\epsilon}^{(4)}(\mathrm{x}(\eta)-\mathrm{z}(\tau))\left\langle\mathrm{z}(1)\left(\overline{\mathrm{z}} \mathrm{t}_{\mathrm{a}} \mathrm{z}(\eta)\right)\left(\overline{\mathrm{z}} \mathrm{t}_{\mathrm{a}} \mathrm{z}(r)\right) \overline{\mathrm{z}}(0)\right\rangle_{\epsilon}
\end{aligned}
$$

we find the contribution of total order $g^{4}$ multiplied by $\int \mathrm{d} \tau \dot{\mathrm{x}}(\tau) \dot{\mathrm{x}}(\eta) \delta_{\epsilon}^{(4)}(\mathrm{x}(\eta)-\mathrm{x}(\tau)) \quad$ which is zero for dimensionally regularized $\delta_{\epsilon}^{(4)}(x)$ and smooth simple contours $/ 4 /$.

Till now we only considered the gauge invariant operators listed in (8). Of course, there appear also noninvariant operators on the r.h.s. of (7). The study of BRS invariance analogous to $/ 7 /$ shows that noninvariant renormalization mixing partners either contain ghosts or the substructures $D_{z}$ and Z D . The relevant matrix elements for ghost contributions start beyond order $g^{4}$. Operators with $\mathrm{Dz}_{\mathrm{z}}$ or $\bar{z}{ }_{\mathrm{D}}^{\mathrm{D}}$ represent no problems as already discussed. Thus we have proved that the limit $\epsilon \rightarrow 0^{\prime}$ of the first term on r.h.s. of equ. (6) exists in order $\mathrm{g}^{4}$.

## 4. UV PROPERTIES OF THE MAKEENKO-MIGDAL TERM

The Green function involved in the second term of equ. (1) or (6) is the expectation value of three composite operators. (For closed contours $z_{a}(1) \bar{z}_{a}(0)$ has to be handled as a composite operator with anomalous dimension zero). For the $Z$ factor of $\bar{z} t_{a}$ we find

$$
\begin{equation*}
\mathrm{Z}=1+\left[(3-\alpha) \mathrm{C}_{\mathrm{A}} \mathrm{~g}^{2}+\mathrm{O}\left(\mathrm{~g}^{4}\right)\right]\left(16 \pi^{2} \epsilon\right)^{-1}+\left[\mathrm{C}_{\mathrm{A}}^{2} \mathrm{~g}^{4}\left(a^{2}-3 / 2 a-13 / 2\right)+\mathrm{O}\left(\mathrm{~g}^{6}\right)\right]\left(256 \pi^{4} \epsilon^{2}\right)^{-1} \tag{11}
\end{equation*}
$$

( $C_{A}=N$, a gauge parameter). This expression results from oneloop calculation and using the general relations between the coefficients of single and double poles in dimensional regulatization (see, e.g., ref. $/ 8 /$ ). With the same type of relations and the well known one-loop expression for the $\beta$ function one further gets $g_{B a}=Z_{g} g$,

$$
\begin{equation*}
Z_{g}=1-\left[11 C_{A} g^{2}+O\left(g^{4}\right)\right]\left(48 \pi^{2} \epsilon\right)^{-1}+\left[121 C_{\dot{A}}^{2} g^{4}+O\left(g^{6}\right)\right]\left(1536 \pi^{4} \epsilon^{2}\right)^{-1} \tag{12}
\end{equation*}
$$

Therefore we find for the second term of r.h.s. of equ. (6)

$$
\begin{align*}
& \left(1-\frac{5}{24} \frac{\mathrm{C}_{A} \mathrm{~g}^{2}+\mathrm{O}\left(\mathrm{~g}^{4}\right)}{\pi^{2} \epsilon}+\frac{65}{1152} \frac{\mathrm{C}_{\mathrm{A}}^{2} \mathrm{~g}^{4}+\mathrm{O}\left(\mathrm{~g}^{6}\right)}{\pi 4_{\epsilon}^{2}}+. .\right) \mathrm{g}^{2} \mathrm{x} \\
& \times \int_{0}^{1} \mathrm{~d} \eta \mathrm{~d} \eta^{\prime} \dot{\mathrm{x}}(\eta) \dot{\mathrm{x}}\left(\eta^{\prime}\right) \frac{\epsilon}{2 \pi^{2}\left|\mathrm{x}(\eta)-\mathrm{x}\left(\eta^{\prime}\right)\right| 4-\epsilon} \mathrm{y}^{2}(\eta)<\ldots \ldots>. \tag{13}
\end{align*}
$$

This is zero in order $\mathrm{g}^{2}$, finite and nonzero in order $\mathrm{g} 4 / 9 /$ but yields a pole term in order $\mathrm{g}^{6}$ :

$$
\begin{equation*}
\frac{65}{1152} \cdot \frac{\mathrm{~N}^{2}-1}{2} \cdot \frac{C_{A}^{2} g^{6}}{2 \pi^{6}} \cdot \frac{1}{\epsilon} \lim _{\epsilon \rightarrow 0} \int_{0}^{1} \mathrm{~d} \eta \mathrm{~d}^{\prime} \frac{\dot{x}(\eta) \dot{\mathrm{x}}\left(\eta^{\prime}\right) \mathrm{y}^{2}(\eta)}{\left|\dot{x}(\eta)-x\left(\eta^{\prime}\right)\right|} \tag{14}
\end{equation*}
$$

One comment should be added to this result. The renormalized Green function in (13) contains logarithmic short distance singularities. Do they introduce additional divergencies? The integration has to be performed before taking $\epsilon \rightarrow 0$, however. Hence the relevant parts in the regularized expression describing both the $Z$ factor contribution and the terms producing the logarithms are proportional to $\frac{1}{\epsilon}\left|\eta^{\prime}-\eta\right|^{2 \epsilon}$ or $\frac{1}{\varepsilon_{1}^{2}}\left|\eta^{\prime}-\eta\right|^{\text {be }}$. This modification does not influence the result (14), since it gives only an irrelevant shift of the exponent from $4-\epsilon$ to $4-c \epsilon$ with $c$ determined by the unspecified numbers $\mathrm{a}, \mathrm{b}$.

Due to (4) and (6) the pole term (14) has to be canceled by a pole term arising in order $\mathrm{g}^{6}$ by the (in contrast to order $\mathrm{g}^{4}$ ) nonvanishing coefficients in front of the indefined distribution $\left|\eta^{\prime}-\eta\right|^{\epsilon-1}$. Of course an explicit check of this statement is beyond of the scope of this paper.

Closing this section we will add a remark which shows conspiracy between the first and the second terms even at a pure formal level. Looking carefully at the structure of the expansion coefficient $\mathrm{c}^{0}\left(\eta^{\prime}-\eta\right)$ multiplying $\bar{z} z$ in equ. we find

$$
\begin{equation*}
\mathrm{c}^{0}\left(\eta^{\prime}-\eta\right)=\mathrm{C}_{\mathrm{F}} \mathrm{~g}^{2}\left[\dot{\mathrm{x}}^{2} \delta_{\epsilon}^{(\mathrm{d})}\left(\mathrm{x}\left(\eta^{\prime}\right)-\mathrm{x}(\eta)\right)+(2-\mathrm{d})(\dot{\mathrm{x}} \partial)(\dot{\mathrm{x}} \partial) \mathrm{D}_{\epsilon}^{(\mathrm{d})}\left(\mathrm{x}^{\prime}-\mathrm{x}\right)\right]+\mathrm{O}\left(\mathrm{~g}^{4}\right), \tag{15}
\end{equation*}
$$

where we have used $-\partial^{2} D_{\epsilon}^{(d)}(x)=\delta_{\epsilon}^{(d)}(x) ; \quad d$, space-time dimension. Hence the first term in (6) has a contribution

$$
-\mathrm{C}_{\mathrm{F}} \mathrm{~g}^{2} \int_{0}^{1} \mathrm{~d} \eta \mathrm{~d} \eta^{\prime} \mathrm{y}(\eta) \mathrm{y}\left(\eta^{\prime}\right) \dot{\mathrm{x}}^{2} \delta_{\epsilon}^{(4)}\left(\mathrm{x}\left(\eta^{\prime}\right)-\mathrm{x}(\eta)\right)<\mathrm{z}(1) \overline{\mathrm{z}}(0)>.
$$

Forgetting now the regularization and writing $\delta^{4}\left(\mathrm{x}\left(\eta^{\prime}\right)-\mathrm{x}(\eta)\right)=$ $=\delta\left(\eta^{\prime}-\eta\right) \delta^{(3)}(0)$ formally, this term will cancel the Make-enko-Migdal term for simple contours totally. Since in 2 space-time dimensions due to (15) the $\delta$-type singularity is the only short distance singularity in lowest order, this mechanism is a reflection of the formal arguments given in ref/ $10 /$ for the two-dimensional case. Of course this consideration beeing in contrast to our main conclusions based on well behaved regularized perturbation theory should serve as an amusing illustration only.

These conclusions are:
The equation (1) does not develop $1 / \epsilon$ poles when removing the regularization. There are cancellations of the ultra violet divergencies of the second term against subtractions necessary for defining the short distance singularities of the first term as distributions over the parameter space.

We gratefully acknowledge discussions with D. Robaschik.

APPENDIX
We want to calculate the OPE coefficient of $\bar{z} F_{\mu \nu} z_{x_{\mu}} \dot{x}_{\nu}$ in lowest order, that means $g^{3}$ for the coefficient itself and $g^{4}$ for the resulting product with the matrix element $\left\langle z(1) \vec{z} F_{\mu \nu} z \ddot{x}_{\mu} \dot{x}{ }_{\nu} \vec{z}(0)\right\rangle$. In this order only the Abelian part of ${ }^{\mu \nu} F_{\mu \nu}{ }_{\mu}{ }^{\text {can }}{ }^{\nu}$ be detected. Defining

$$
\begin{equation*}
\mathrm{O}_{1}=\overline{\mathrm{z}} \ddot{x}_{\mu} \dot{\mathrm{x}} \dot{\partial} \mathrm{~A}_{\mu} \mathrm{z}, \quad \mathrm{O}_{2}=\overline{\mathrm{z}} \dot{x}_{\mu} \ddot{\mathrm{x}} \dot{\partial} \mathrm{~A}_{\mu} z \tag{A1}
\end{equation*}
$$

we find

1. graph $=\frac{\mathrm{ig}^{3} \mathrm{C}_{\mathrm{F}}}{2 \pi^{2}}\left|\eta-\eta^{\prime}\right|^{\epsilon-1}\left(\mathrm{O}_{2}-\mathrm{O}_{1}\right)+\cdots$
2. $\operatorname{graph}=\left.\left.\frac{\operatorname{ig}^{3}\left(\mathrm{C}_{\mathrm{A}} / 2-\mathrm{C}_{\mathrm{F}}\right)}{2 \pi^{2}}\right|_{\eta-\eta^{\prime}}\right|^{\epsilon-1}\left(2 \mathrm{O}_{1}+\mathrm{O}_{2}\right)+\ldots$
3. graph $=\frac{\mathrm{ig}^{3}\left(\mathrm{C}_{\mathrm{A} / 2}-\mathrm{C}_{\mathrm{F}}\right)}{2 \pi^{2}}\left|\cdot \eta-\eta^{\prime}\right|^{\epsilon-1}\left(\mathrm{O}_{2}-\mathrm{O}_{1}\right)+\cdots$
4. graph $=\frac{\mathrm{ig}{ }^{3} \mathrm{C}_{\mathrm{A}^{+}}}{8 \pi^{2}}\left|\eta-\eta^{\prime}\right|^{\epsilon-1}\left(3 \mathrm{O}_{1}-\mathrm{O}_{2}\right)+\ldots$
5. graph: $-\frac{i g^{3} \mathrm{C}_{\mathrm{A}}}{8 \pi^{2}}\left|\eta-\eta^{\prime}\right|^{\epsilon-1}\left(5 \mathrm{O}_{1}+3 \mathrm{O}_{2}\right)+\ldots$.
where possible other operator structures different from $\mathrm{O}_{1}, \mathrm{O}_{2}$ have not been written down. This gives a total contribution

$$
\begin{equation*}
-\frac{\mathrm{ig}^{3} \mathrm{C}_{\mathrm{F}}}{2 \pi^{2}}\left|\eta-\eta^{\prime}\right|^{\epsilon-1}\left(\mathrm{O}_{2}+2 \mathrm{O}_{1}\right) \tag{A3}
\end{equation*}
$$

Now among the operators of dimension 3 listed in (8) there are two other operators containing $\mathrm{O}_{1}$ and $\mathrm{O}_{2}$, namely

$$
\begin{align*}
& \overline{\mathrm{z}} \mathrm{D}^{3} \mathrm{z}=-\mathrm{ig}\left(\mathrm{O}_{2}+2 \mathrm{O}_{1}\right)+\ldots \\
& \overline{\mathrm{z}}^{\leftarrow} \stackrel{\mathrm{D}}{ }^{3} \mathrm{z}=-\mathrm{ig}\left(\mathrm{O}_{2}+2 \mathrm{O}_{1}\right)+\ldots \tag{A4}
\end{align*}
$$

Looking for contributions to gauge invariant operators we therefore have to express an arbitrary combination of $O_{1}$ and $\mathrm{O}_{2}$ as a linear combination of $\mathrm{O}_{2}+2 \mathrm{O}_{1}$ and $\mathrm{O}_{2} \mathrm{O}_{1}=\mathrm{zF} \mathrm{F}_{\mu \nu} \mathrm{z} \ddot{\mathrm{x}}_{\mu} \dot{\mathrm{x}}_{\nu}$. Now from (A3) it is obvious all the graphs contribute to the expansion coefficient of the operators $\bar{z} D^{3} z$ and $\vec{z}^{x} D^{3}$ only. These however pose no problem as argued in the text.

To all of the following graphs a contribution with $\eta \rightarrow \eta^{\prime}$ has to be added:

11/



121


13/


15/


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[^0]:    * For smooth contours without double points to which we always restrict our consideration this is achieved simply by performing renormalization of the coupling constant $/ 1 /$.

