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GAUGE REPRESENTATIONS OF THE LORENTZ GROUP AND CLASSICAL SOLUTIONS OF THE YANG-MILLS EQUATION

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1. INTRODUCTION

The discovery of new solutions of the classical Yang-Mills equation certainly leads to deeper understanding of gauge theories. In the last few years various methods for solving these equations, both in Minkowski and Euclidean space, have been obtained (many of them are reviewed in ref./1/). Most of the methods impose symmetry conditions on the gauge potentials or fields which simplify the solution. The requirement of a sufficiently large invariance group of the potentials reduces in some cases the problem of solving the Yang-Mills equation to the problem of solving one nonlinear differential equation for a function of one variable. In this way many of the well-known solutions have been obtained, for instance, the famous one-instanton solution /3/ in Euclidean space, whose invariance group is O(5) (see ref. /2/) and the O(4)-invariant solutions in Minkowski space found in refs. /4,5/ by means of the hypertoroidal conformal formalism.

The group of invariance R of the Yang-Mills equation consists of the conformal space-time transformations and the local gauge transformations. In paper /6/ the possible (nonlinear) representations of a given compact group G, contained in R, in the space of solutions of the Yang-Mills equation have been studied and the general notion for a G-invariant gauge field has been introduced. In ref./7/ all translationally invariant gauge potentials have been found.

Here we consider the noncompact Lorentz group as an invariance group G and find some invariant (with respect to G) solutions of the pure SU(2) Yang-Mills equation in Minkowski space.

2. GAUGE REPRESENTATIONS OF THE LORENTZ GROUP

Let us fix the notation. The potentials

$$A_\mu(x) = A_\mu^a(x)\frac{\sigma_a}{2}, \quad a = 1,2,3,$$

where σ_a are the Pauli matrices and $A_\mu^a(x)$ are real, determine the covariant derivative

$$D_\mu = \partial_\mu + iA_\mu$$

and the Yang-Mills field

$$F_{\mu\nu} = \frac{1}{i}[D_\mu , D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu , A_\nu].$$

The Yang-Mills equation is

$$[D_\mu , F^{\mu\nu}] = \partial_\mu F^{\mu\nu} + i[A_\mu , F^{\mu\nu}] = 0 . \qquad (1)$$

The metric tensor $g_{\mu\nu}$ in the Minkowski space M^4 is defined by $(g_{\mu\nu}) = \mathrm{diag}(1,-1,-1,-1)$; and the three-dimensional fully antisymmetric tensor ϵ_{ijk}, by $\epsilon_{123} = 1$. Irreducible representations of the Lorentz group will be denoted by pairs (ℓ_0 , ℓ_1), where $2\ell_0 \in \mathbf{Z}$, $\ell_1 \in \mathbf{C}$ (see ref. /8/).

We shall define the vector gauge representation of the Lorentz group through the equality

$$(T_g A)^\mu (x) = \Lambda(g)^\mu{}_\nu U(g,x) A^\nu (\Lambda^{-1}(g)x) U^{-1}(g,x) + i\partial^\mu U(g,x) U^{-1}(g,x) , \quad g \in SL(2, \mathbf{C}) , \qquad (2)$$

where $\Lambda(g)$ is the matrix of the four-dimensional vector representation (0,2) and $U(g,x)$ are two-dimensional unitary unimodular matrices defined, in general, locally in M^4 which satisfy the conditions

$$U(g_1 g_2 , x) = U(g_1 , x) U (g_2 , \Lambda^{-1}(g_1)x) , \qquad U(e, x) = I . \qquad (3)$$

It can be easily seen that if $A^\mu(x)$ is a solution of equation (1), then $(T_g A)^\mu (x)$ is also a solution of the same equation. To prove this, it is sufficient to note that the transformation (2) results from the Lorentz transformation

$$A'^\mu(x) = \Lambda(g)^\mu{}_\nu A^\nu (\Lambda^{-1}(g)x), \quad g \in SL(2, \mathbf{C}) \qquad (4)$$

followed by the gauge transformation

$$A''^\mu (x) = U(g,x) A'^\mu(x) U^{-1}(g,x) - iU(g,x) \partial^\mu U^{-1} (g,x) \qquad (5)$$

and the Yang-Mills equation (1) is invariant both with respect to (4) and (5). Thus, one can consider the representation (2) in the space of solutions of equation (1). Among these there are some which satisfy the invariance equation

$$(T_g A)^\mu (x) = A^\mu(x) \qquad \forall g \in SL(2, \mathbf{C}) . \qquad (6)$$

The potentials $A^{\mu}(x)$ obeying (6), i.e., invariant with respect to the gauge representation T_g, play the same role as the spherical harmonics do with respect to the corresponding representation of the rotation group.

Throughout the paper representations of the type (2) will be referred to as gauge representations of the Lorentz group. Different gauge representations are those for which the corresponding matrix functions $U(g,x)$ are different. The necessary and sufficient condition for two gauge representations T_g (defined by $U(g,x)$) and Q_g (defined by $V(g,x)$) to be gauge equivalent, that is, for a matrix function $W(x) \in SU(2)$ to exist, such that

$$(T_g A)^{\mu}(x) = W^{-1}(x)(Q_g A)^{\mu}(x) W(x) - iW^{-1}(x)\partial^{\mu}W(x), \qquad (7)$$

as can be verified directly, is

$$U(g,x) = W^{-1}(x) V(g,x) W(\Lambda^{-1}(g)x). \qquad (8)$$

If $A^{\mu}(x)$ is a solution of equation (6) for a fixed T_g (i.e., for a given function $U(g,x)$), then from (2) it follows that

$$\Lambda(g)^{\mu}{}_{\nu} A^{\nu}(\Lambda^{-1}(g)x) = U^{-1}(g,x) A^{\mu}(x) U(g,x) - iU^{-1}(g,x)\partial^{\mu}U(g,x). \qquad (9)$$

In other words, every Lorentz transformation of the type (4) on $A^{\mu}(x)$ can be compensated by an appropriate gauge transformation of the type (5). Let us fix the point $x_0 \in M^4$ and consider a smooth curve $\gamma(t) = \Lambda(g(t))x_0$, $g(0) = e$, through it. Obviously, $\gamma(t)$ will lie entirely on the orbit $N(x_0)$ of the Lorentz group through x_0. Respectively, $u(t) = U(g(t), x_0)$ will be a smooth curve passing through the identity element e of the group SU(2). We would like to get a solution of (3) for which the curves $u(t)$ are nontrivial, i.e., are not reduced to a point. The reason is that for the known nontrivial invariant classical solutions space-time and gauge transformations are mixed in a nontrivial way (see refs. /2-6/). We may conjecture that nontrivial solutions of (3), for instance $U(g,x)$ that depend essentially on x, will lead to equations of the type (9) with nontrivial solutions for $A_{\mu}(x)$.

3. INVARIANT SOLUTIONS OF THE YANG-MILLS EQUATION

The two-dimensional representation of the Wigner rotations (see, for instance, /9/) provides us with a solution of equation (3) which is defined for all $x \in V_+ = \{y \in M^4, y^2 > 0, y^0 > 0\}$.

Let $V(g)$ be the two-dimensional irreducible representation $(\frac{1}{2},\frac{3}{2})$ of the group $SL(2, \mathbb{C})$:

$$V(g)\,\sigma_\mu\, V^*(g) = \Lambda(g)^\nu{}_\mu \sigma_\nu \qquad (10)$$

and let us define for any vector $x \in V_+$ the two-dimensional hermitean unimodular positive definite boost matrix $V_B(x)$ by

$$V_B(x)\,\sigma_0\, V_B^*(x) = V_B^2(x) = \frac{x^\mu}{\sqrt{x^2}}\sigma_\mu . \qquad (11)$$

Then

$$\begin{aligned} V_B(\Lambda^{-1}(g)x)\,\sigma_0\, V_B^*(\Lambda^{-1}(g)x) &= \Lambda^{-1}(g)^\mu{}_\nu \frac{x^\nu}{\sqrt{x^2}}\sigma_\mu = \\ = V(g^{-1})\,\sigma_\nu\, V(g^{-1})^* \frac{x^\nu}{\sqrt{x^2}} &= V(g^{-1}) V_B(x)\,\sigma_0\, V_B^*(x)\, V(g^{-1})^* \end{aligned} \qquad (12)$$

and, obviously, the matrix

$$U(g, x) = V_B^{-1}(x)\, V(g)\, V_B(\Lambda^{-1}(g)x) \qquad (13)$$

belongs to the two-dimensional representation of $SU(2)$. The condition (3) is fulfilled, as it can be easily checked.

From (11) for the boost matrix we have

$$V_B(x) = a^\mu(x)\,\sigma_\mu , \qquad (14)$$

where

$$\begin{aligned} a^0(x) &= \sqrt{\frac{x^0 + \sqrt{x^2}}{2\sqrt{x^2}}} , \\ a^i(x) &= \frac{-x_i}{\sqrt{2\sqrt{x^2}(x^0+\sqrt{x^2})}} , \qquad i = 1,2,3 . \end{aligned} \qquad (15)$$

Of course, $a^\mu(x)\, a_\mu(x) = \det V_B(x) = 1$. The matrix elements of the four-dimensional representation of the boosts $\Lambda(x)^\mu{}_\nu$,

$$V_B(x)\,\sigma_\mu\, V_B(x) = \Lambda(x)^\nu{}_\mu \sigma_\nu \qquad (16)$$

are:

$$\Lambda(x)^0{}_0 = \frac{x^0}{\sqrt{x^2}},$$

$$\Lambda(x)^0{}_j = \Lambda(x)^j{}_0 = -\frac{x_j}{\sqrt{x^2}},$$

$$\Lambda(x)^j{}_k = \delta_{jk} + \frac{x_j x_k}{\sqrt{x^2}(x^\circ + \sqrt{x^2})}. \tag{17}$$

If $g_x \in SL(2, \mathbf{C})$ is defined by

$$V_B(x) = V(g_x) \tag{18}$$

and the vector $(\sqrt{x^2}, \vec{0}) \in V_+$ is denoted by $x_0 = x_0(x)$ (it is clear that x_0 is the same for all the vectors of the orbit $N(x)$), the formulae

$$\begin{aligned} &\Lambda(g_x)x_0 \equiv \Lambda(x)x_0 = x \ , \\ &U(g_x, x) = 1, \quad \forall x \in V_+ \ , \\ &V_B(x_0) = 1 \end{aligned} \tag{19}$$

can be verified.

Now we have to solve the equation (9). Putting in it $\Lambda = \Lambda(x)$, we obtain

$$A^\mu(x) = \Lambda(x)^\mu{}_\nu A^\nu(x_0) + i(\partial^\mu U)(g_x, x) \ , \tag{20}$$

where we have used the simplifications provided by (19).

The use of the representation (20) is obvious. The only arbitrary quantities in the r.h.s. (for a given $U(g, x)$) are the components of the invariant potential at the standard points $x_0 = x_0(x)$, and these depend only on x^2. Denoting $A^{(0)}_\mu(x) = i(\partial_\mu U)(g_x, x)$ and using (13)-(15) we obtain

$$\begin{aligned} &A^{(0)}{}_0(x) = 0 \ , \\ &A^{(0)}{}_j(x) = -\frac{\epsilon_{jk\ell}\, x_k \sigma_\ell}{2\sqrt{x^2}(x^\circ + \sqrt{x^2})}. \end{aligned} \tag{21}$$

It is easy to see that $x_i A^{(0)}_i(x) = \partial_i A^{(0)}_i(x) = 0$. Because $A^{(0)}_\mu(x_0) = 0$, we can put, for example,

$$\begin{aligned} &A_0(x_0) = 0, \\ &A_j(x_0) = \sigma_j f(x^2). \end{aligned} \tag{22}$$

From (19)-(22) we obtain the full expression for the invariant potential:

$$A_0(x) = \frac{x_i \sigma_i}{\sqrt{x^2}} f(x^2) ,$$

$$A_j(x) = -\frac{\epsilon_{jk\ell}\, x_k \sigma_\ell}{2\sqrt{x^2}\,(x^0+\sqrt{x^2})} + \sigma_j f(x^2) + x_j \frac{x_i \sigma_i}{\sqrt{x^2}(x^0+\sqrt{x^2})} f(x^2) , \tag{23}$$

where the function $f(x^2)$ has to be determined from the requirement for $A_\mu(x)$ to be a solution of the Yang-Mills equations.

It turns out that one can arrive at the same expression for $A_\mu(x)$ by quite a different way. One starts with the observation that

$$B_\mu(x) = i\partial_\mu V_B(x) V_B^{-1}(x) \tag{24}$$

with $V_B(x)$ defined by (14), (15), satisfies the invariance condition (9). Unfortunately, $B^a_\mu = \mathrm{Tr}\, B_\mu \sigma_a$ are not real (they would have been real if $V_B(x)$ were unitary). To obtain a real potential, we shall make the following trick. If $V_B(x)$ *is* in the (infinite-dimensional irreducible) unitary representation of $SL(2,C)$ with $(\ell_0, \ell_1) = (½, i\rho), \rho \in R$ (see ref. /8/), then the two-dimensional matrix in the upper left corner (in canonical basis) of $B_\mu(x)$ transforms irreducibly with respect to the subgroup $SU(2)$. The hermitean generators of the representation $(½, i\rho)$ have the form

$$I_j = \begin{pmatrix} \frac{1}{2}\sigma_j & 0 \\ 0 & * \end{pmatrix} \qquad J_j = \begin{pmatrix} \frac{\rho}{3}\sigma_j & * \\ * & * \end{pmatrix} \tag{25}$$

Then

$$B_\mu(x) = \begin{pmatrix} A_\mu(x) & * \\ * & * \end{pmatrix} \tag{26}$$

where for $A_\mu(x)$ one obtains the expression (23) with $f(x^2) = \frac{\rho}{3\sqrt{x^2}}$. But *if* one adds the general solution of the homogeneous invariance equation

$$\Lambda(g)^\mu{}_\nu C^\nu(\Lambda^{-1}(g)x) = U^{-1}(g,x)\, C^\mu(x)\, U(g,x) , \tag{27}$$

one obtains once more the full expression (23).

Inserting $A_\mu(x)$, given by (23), in the Yang-Mills equation leads to the following nonlinear second-order differential equation for $f = f(t)$:

$$4t^2 f'' + 8tf' + 3f + 3tf^3 = 0. \tag{28}$$

This equation is simplified (see [10]) by the substitution

$$f(t) = \frac{z(\ln t)}{\sqrt{t}}, \quad t > 0. \tag{29}$$

From (28), (29) one obtains

$$z'' + \frac{z}{2} + 2z^3 = 0 \tag{30}$$

with a first integral

$$\frac{(z')^2}{2} + V(z) = E, \tag{31}$$

where $V(z)$ is the symmetric anharmonic oscillator potential

$$V(z) = \frac{z^2}{4}(2z^2 + 1). \tag{32}$$

Equation (30) has only one real constant solution, $z = 0$, which leads to $f(t) = 0$ and $A_\mu(x) = A_\mu^{(0)}(x)$. The general solution of (31) with $E > 0$ is expressed in terms of Jacobi elliptic functions (see [11]):

$$z(u) = \sqrt{D - \frac{1}{4}}\, \mathrm{cn}(\sqrt{2D}\,u + c), \tag{33}$$

where $\mathrm{cn}(a)$ is defined by

$$a = \int_1^{\mathrm{cn}(a)} \frac{dy}{\sqrt{(1-y^2)(1-k^2+k^2y^2)}}, \tag{34}$$

c is an arbitrary constant and

$$D = \sqrt{E + \frac{1}{16}} > \frac{1}{4}, \qquad k^2 = \frac{D - 1/4}{2D}. \tag{35}$$

The so obtained invariant solutions of the Yang-Mills equation are defined only for $x^2 > 0$, i.e., inside the light cone. On the cone they have singularities, and for $x^2 < 0$ are not real. Despite this fact they are of some interest because they illustrate the methods for solving the invariance equations. There are good reasons to believe that these methods can be used without great modifications also for obtaining invariant solutions of the $SU(n)$ Yang-Mills equations for $n > 2$.

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