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ON PROPERTIES OF THE NONLINEAR SCHRÖDINGER EQUATION WITH U(p,q) INTERNAL SYMMETRY

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1. The non-linear Schrödinger Equation (NLS) which appears in the condensed matter theory describes a great body of various physical phenomena: from water waves and spin waves in ferromagnets up to vortices in superfluids and laser beams in glass fibres [1].

There are by now a great amount of papers devoted to it of both physical and mathematical nature [2]. For the simplest version of $U(1)$ symmetry NLS has been studied in detail on the classical level [2] as well as on the quantum one [3]. In the quantum case it describes Bose gas with δ-function pair interaction, the problem that was considered in ref. [4]. Ultimately the complete integrability of $U(1)$ NLS was shown in ref. [5].

The vector generalization of NLS is less studied so far, despite it has a richer internal structure and conserves the integrability. Two-component version (with $U(2)$ isosymmetry) of NLS was also discussed on both the classical [6] and quantum [7] levels. In the first case we have elliptically polarized wave in nonlinear media with dispersion $\omega = k^2 - 2\kappa |\vec{E}|^2$, in the latter a gas of Bose particles possessing an internal degree of freedom. It should be noted that particles may attract or repel one another.

Recently in ref. [8] a new integrable version of NLS was discussed with non-compact isogroup $U(1,1)$. This equation for example describes one-dimensional Hubbard model in long-wave approximation [9] and also the system of two interacting Bose gases "gravitating" and "anti-gravitating" [10]. The properties of such a system are considerably richer even in this simplest variant.

In this note we discuss a generalized vector version of NLS with isogroup $U(p,q)$ which includes all the variants studied as particular cases.

2. Consider column-vector ψ of n complex functions $(\psi)_a = \psi^{(a)}(x,t)$ $(a = 1, \dots, n)$ and Dirac conjugate row-vector $\bar{\psi} = \psi^+ \gamma_0$, where $\gamma_0 = \mathrm{diag}(\underbrace{+1, \dots, +1}_{p}, \underbrace{-1, \dots, -1}_{q})$.

Defining inner product

$$(\bar{\psi}\psi) = \sum_{a=1}^{p} |\psi^{(a)}|^2 - \sum_{a=p+1}^{p+q} |\psi^{(a)}|^2 , \quad p + q = n \tag{1}$$

we write the equation in question

$$i\psi_t + \psi_{xx} + 2\kappa(\bar{\psi}\psi)\psi = 0 . \qquad (2)$$

In terms of canonically conjugate variables $\psi^{(a)}$ and $\bar{\psi}^{(a)}$ the Hamiltonian of system (2) assumes the form:

$$H = \int_{-\infty}^{\infty} dx[(\bar{\psi}_x \psi_x) - \kappa(\bar{\psi}\psi)^2] \qquad (3)$$

and Hamiltonian equations

$$\psi_t^{(a)} = \{H, \psi^{(a)}\} = -i\frac{\delta H}{\delta\bar{\psi}^{(a)}}, \quad \bar{\psi}_t^{(a)} = \{H, \bar{\psi}^{(a)}\} = i\frac{\delta H}{\delta\psi^{(a)}} \qquad (4)$$

coincide with the system (2) and the conjugate one. The Poisson brackets are defined through the canonical variables in the conventional manner.

Corresponding linear problem is the couple of (n+1)-component equations $f_x = \hat{U}_0 f$, $f_t = \hat{V}_0 f$ with their compatibility condition being equivalent to system (2)*. Using the variables

$$f(x,t) = \exp\left(-i\frac{\lambda a x}{1-s^2}\right)\hat{T}\phi(x,t),$$

$$\psi^{(a)} = (1-s^2)^{1/2} q^{(a)}(x,t), \quad \xi = -\frac{2n}{n+1}\left(\frac{\lambda s}{1-s^2}\right), \; \kappa = (1-s^2)^{-1},$$

where

$$\hat{T} = \left(\begin{array}{c|c} (1-s)^{1/2} & 0 \\ \hline 0 & (1+s)^{1/2} I_n \end{array}\right),$$

we come to the following linear problem

$$\phi_x = \hat{U}\phi, \quad \phi_t = \hat{V}\phi$$

and

$$\hat{U}(x,t;\xi) = \left(\begin{array}{c|c} -i\xi & i\bar{q} \\ \hline iq & i\frac{\xi}{n} I_n \end{array}\right),$$

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* The linear problem in the Lax form has been constructed earlier for the system (2) by the authors and Makhaldiani in ref. /8/.

$$\hat{V}(x,t;\xi)=\xi^2\begin{pmatrix} i(\frac{n+1}{n})^2 & 0 \\ 0 & 0\cdot I_n \end{pmatrix}+\xi\begin{pmatrix} 0 & -i\frac{n+1}{n}\bar{q} \\ -i\frac{n+1}{n}q & 0\cdot I_n \end{pmatrix}+\begin{pmatrix} -i(\bar{q}q) & \bar{q}_x \\ -q_x & iq\otimes\bar{q} \end{pmatrix},$$

where $q\otimes\bar{q}$ is the direct (Kronecker) product of the n-component column q and row $\bar{q}$, I_n is the unit $(n\times n)$ matrix. The parameter α was chosen to satisfy the condition $Sp\hat{U}=0$ and we use the freedom in defining matrix $\hat{V}$: $\hat{V}\to\hat{V}+c\hat{I}$. Our $(n+1)\times(n+1)$ linear problem governed by the operator $\hat{U}$ is the n-fold degenerated one, which is connected with nontrivial isotopic properties of system (2)*.

3. Linear transformations $\psi'=R\psi$ when conserve the inner product (1) generate pseudounitary matrix group U(p,q). Matrices $R\in U(p,q)$ are subjected to condition $\bar{R}R=\hat{I}$, where $\bar{R}=\gamma_0 R^+\gamma_0$. Whence the linear problem is transformed with the help of the matrix $\mathcal{R}=\begin{pmatrix}1 & 0\\ 0 & R\end{pmatrix}$ and the operator $\hat{U}$ is transformed as follows $\hat{U}\to\hat{U}'=\bar{\mathcal{R}}\hat{U}\mathcal{R}$. But the last condition is hold only if $\bar{R}R=\hat{I}$, i.e., $R\in U(p,q)$. Therefore, the n-fold degeneracy of the operator $\hat{U}$ implies as isotopic symmetry to be inherent in the system.

The number of independent parameters of U(p,q) group equals $(p+q)^2$ and there are relatively $n^2 (n=p+q)$ conserving local currents J_μ^{ik} $(\mu=0,1)$ with components

$$J_0^{ik}=\bar{q}^i q^k,\qquad J_1^{ik}=i(\bar{q}_x^{\,i}q^k-\bar{q}^{\,i}q_x^k),\quad i,k=1,\dots,n,$$

so that $\partial_\mu J_\mu^{ik}=0$.

The elements of matrix $Q^{ik}=\int_{-\infty}^{\infty}dx\,\bar{q}^i q^k$ are the integrals of motion (IM) and commute with the Hamiltonian, $\{Q^{ik},H\}=0$. They satisfy the commutation relations of the Lie algebra $gl(p+q,R)$ $L\{Q^{ik},Q^{j\ell}\}=\delta_{kj}Q^{i\ell}-\delta_{i\ell}Q^{jk}$ as well as the conjugate conditions: $(Q^{ij})^*=\epsilon_{ij}Q^{ji}$,

$$\epsilon_{ij}=\begin{cases}1 & \text{at } 1\le i,\ j\le p \text{ or } p+1\le i,\ j\le p+q\\ -1 & \text{in all other cases.}\end{cases}$$

Whereby they form the Lie algebra of U(p,q) group. Using them one can construct n^2 Hermitian generators of the same algebra. (See paper[12] for details).

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* The analogous fact was stated independently for the particular case of U(2) symmetry in ref.[11].

Diagonal "charges" $Q^{ii} = \int_{-\infty}^{\infty} \bar{q}^i q^i \, dx$ are the numbers of type "i" particles. Being positive when $1 \le i \le p$, Q^{ii} are related to the particles attracting one another and otherwise when $p+1 \le i \le p+q=n$.

Nondiagonal elements generate transformations that mix different "pure" states. They belong to subgroup SU(p, q) and allow us to construct the whole class of solutions to system (2) using a definite particular solution*.

For example, consider single-soliton solution to U(1) NLS:

$$\tilde{\psi}(x,t) = a e^{i\theta} \operatorname{sech} a\bar{x}, \quad \bar{x} = x - vt - x_0, \quad \theta = \frac{v}{2} x - \omega t, \quad \omega = \frac{v^2}{4} - a^2,$$

making an isotopic rotation we get single-soliton solution to U(p, q) NLS:

$$\tilde{\psi}_i = a c_i e^{i\theta} \operatorname{sech} a\bar{x}, \quad (i = 1, \dots, n) \quad \text{and} \quad (\bar{c}c) = \sum_{i=1}^{p} |c_i|^2 - \sum_{i=p+1}^{p+q} |c_i|^2 = 1.$$

For the case $p=2$, $q=0$ we recover the solution which was obtained earlier by Manakov in ref. /6/. The vanishing boundary conditions, $\psi \to 0$ at $x \to \pm\infty$, which the above solution should satisfy are the simplest of the whole set of possible boundary conditions for system (2).

Let matrix Jost solutions $\hat{\phi}(x, \xi)$ and $\hat{\psi}(x, \xi)$ to linear system $\hat{\phi}_x = \hat{U}\hat{\phi}$ be defined by their asymptotics:

$$\hat{\phi}(x, \xi) \to \exp(-i\xi \hat{\Sigma} x), \qquad x \to -\infty \qquad \text{and}$$

$$\hat{\psi}(x, \xi) \to \exp(-i\xi \hat{\Sigma} x), \qquad x \to +\infty,$$

where $\hat{\Sigma} = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{n} I_n \end{pmatrix}$, then one may introduce a transition matrix with the relation $\tilde{\phi}(x, \xi) = \tilde{\psi}(x, \xi) \hat{S}(\xi)$. It satisfies the unimodularity condition $\det \hat{S}(\xi) = 1$ and that of pseudounitarity $\hat{\bar{S}} \hat{S} = I$ as well.

Matrix $\hat{S}(\xi)$ may be shown /12/ to give n^2+1 conserving elements S_{11}, $S_{\alpha\beta}$ $(\alpha, \beta = 2, \dots, n+1)$. Remaining elements $S_{\alpha 1}$ and $S_{1\beta}$ alter in time very simply. The conserving elements generate infinite series of conservation laws (CL). Only S_{11} generates local series. Those conservation laws which are generated by the block $S_{\alpha\beta}$ are nonlocal, barring n^2 local CL associated with J_μ^{ik}.

Local integrals of motion $I_{11}^{(k)}$ $(k=1, \dots)$ form a numerable set. They are in involution with each other $\{I_{11}^{(k)}, I_{11}^{(\ell)}\} = 0$ and with all nonlocal integrals: $\{I_{11}^{(k)}, I_{\alpha\beta}^{(\ell)}\} = 0$. Nonlocal integ-

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* It means that such transformations (preserving the term $(\bar{q}q)$) as though linearize system (2) since certain linear combination of solutions is again the solution to system (2).

rals of motion are not involutative but generate a complex algebraic structure, e.g.,

$$\sum_{m=1}^{k} \{ I_{22}^{(k+1-m)}, I_{23}^{(m)} \} = ik I_{23}^{(k)}, \qquad k = 1, \dots .$$

Only for $k=1$ we have the commutation relations of SU(p,q) isogroup: $\{Q_{11}, Q_{12}\} = Q_{12}$, remainders include higher IM's. Infinite sets of involutative integrals of motion form an infinite-parameter Abelian group. Their existence implies the integrability of system (2) considered.

The role of uncommuting IM's generating more complex non-Abelian transformations remains to understand. Remind only that for the quantum nonlinear σ-model they gave rise to the factorization of scattering matrix /13/.

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