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ON THE SPECTRAL THEORY OF THE OPERATOR, GENERATING NONLINEAR EVOLUTION EQUATIONS

Submitted to "Letters in Mathematical Physics"

1981

The inverse scattering method (ISM)[1] allows one to describe a whole class of exactly soluble nonlinear evolution equations (NLEE)[2-8]. These NLEE are generated by operators Λ, constructed from the auxiliary linear problem L; in some important cases Λ are known explicitly. Let us briefly list those aspects of studies of NLEE, for which the operator Λ is important; i) the description of the NLEE and the interpretation of the ISM as a generalized Fourier transform[6,8-10]; ii) the construction of a hierarchy of Hamiltonian structures[10-13]; iii) the calculation of action-angle variables[9,10]; iv) the construction of the Lagrangian manifold for the NLEE[10,14]. The operator Λ naturally appears also in the abstract algebraical approach to the Lax's scheme[5,15,12,13].

In the present paper, following the ideas in[10] we outline the construction of the spectral theory of the operator Λ, related to the first order matrix linear problem:

$$A(L-\lambda)\psi \equiv \left(-i\frac{d}{dx}+q(x)-\lambda A\right)\psi(x,\lambda)=0, \quad A=\mathrm{diag}(a_1,\ldots,a_n), \tag{1}$$

$$q_{ii}=0, \quad q_{ij}(x)\xrightarrow[|x|\to\infty]{}0; \quad a_1>a_2>\ldots>a_n, \quad \mathrm{tr}\,A=0.$$

This allows for better understanding of why the approaches in[2-4],[6-8] and[5] are equivalent, see[12]. Physically important NLEE, related to the problem (1) for $n\geq 3$ have been studied in[16].

The author is grateful to P.P.Kulish, E.Kh.Khristov and M.A.Semenov-Tian-Shanski for numerous usefull discussions.

Considering the linear problem (1) let us assume for simplicity, that: i) the complex-valued functions $q_{ij}(x)\in \mathcal{S}(\mathbf{C})$ are of Schwartz type; ii) the domain D_L of the operator L (1) is the space of vector-valued functions of Schwartz type, $D_L=\mathcal{S}(\mathbf{C}^n)$; iii) the discrete spectrum of the operator L is finite and simple.

The corresponding Λ-operator, related to (1) is defined by the formal expression[6-8]:

$$\Lambda_{\pm}X=\hat{A}*\left\{i\frac{d}{dx}X-[q,X]^{f}-i\left[q(x),\int_x^{\pm\infty}dy\,[q(y),X(y)]^{d}\right]\right\},$$
$$Z=Z^{d}+Z^{f}, \quad Z^{d}\equiv\mathrm{diag}(Z_{11},\ldots,Z_{nn}); \; (\hat{A}*Z^{f})_{ij}=\frac{Z_{ij}}{a_i-a_j}, \tag{2}$$

where $\hat{Z} \equiv Z^{-1}$ and $X = X^{f}$ is a matrix-valued function. As a domain D_Λ of the operators $\Lambda_\pm$ we choose the space of non-diagonal matrix-valued functions of Schwartz type, $D_\Lambda = \mathcal{S}(\mathbb{C}^{n(n-1)})$; obviously if $X \in D_\Lambda$, then $\Lambda_\pm X \in D_\Lambda$.

In the following it will be crucial to use such solutions $\chi^{\pm}(x,\lambda)$ of the problem (1), which are analytic in λ for $\mathrm{Im}\,\lambda \gtrless 0$, respectively. Such solutions are constructed in [17] (see also [1]) and are related to the Jost solutions of (1)

$$(L-\lambda)\,\phi^{\pm}(x,\lambda) = 0, \qquad \lim_{x\to\pm\infty} \phi^{\pm}(x,\lambda)\, e^{-iA\lambda x} = 1 \tag{3}$$

by

$$\chi^{+} = \phi^{+}S^{+} = \phi^{-}S^{-}, \qquad \chi^{-} = \phi^{+}T^{-} = \phi^{-}T^{+}, \tag{4}$$

where $S^{+}(\lambda)$, $T^{+}(\lambda)$, $(S^{-}(\lambda), T^{-}(\lambda))$ are upper-(lower-) triangular matrices satisfying $S^{+} = S(\lambda)S^{-}$, $T^{-} = S(\lambda)T^{+}$, $S(\lambda)$ being the transition matrix, $S(\lambda) = \hat{\phi}^{+}\phi^{-}(x,\lambda)$. The simplicity and finiteness of the discrete spectrum of L required above means that the solutions $\chi(x,\lambda)$ may be degenerate only for $\lambda \in \sigma = \sigma^{+} \cup \sigma^{-}$, $\sigma^{\pm} \equiv \{\lambda_a^{\pm},\ \mathrm{Im}\,\lambda_a^{\pm} \gtrless 0,\ a = 1,\dots,N\}$ and $\hat{\chi}^{\pm}(x,\lambda)$ have for $\lambda \in \sigma$ simple pole singularities. In that case $\chi^{\pm}(x,\lambda)$ may be represented as:

$$\chi^{\pm}(x,\lambda) = u_N^{\pm}(x,\lambda)\dots u_1^{\pm}(x,\lambda)\,\tilde{\chi}^{\pm}(x,\lambda), \quad u_a^{\pm}(x,\lambda) = (1 + c_a^{\pm}(\lambda)P_a^{\pm}(x)),$$

$$P_a^{+}(x) = 1 - P_a^{-}(x), \qquad c_a^{\pm} = \frac{\lambda_a - \lambda_a^{\pm}}{\lambda - \lambda_a^{\mp}} \tag{5}$$

where $\tilde{\chi}^{\pm}(x,\lambda)$ are non-degenerate for all λ solutions of a type (1) problem without discrete spectrum. The projectors $P_a^{\pm}(x)$ and $\tilde{\chi}^{\pm}(x,\lambda)$ are constructed from a minimal set of scattering data, which allows also to recover uniquely both the transition matrix $S(\lambda)$ and the potential $q(x)$ of (1), see [1,8]. From the estimates for the solutions $\tilde{\chi}^{\pm}(x,\lambda)$ [17] and from the explicit form of the projectors $P_a^{\pm}(x)$ there follows, that $P_a^{\pm}(x)$ are uniformly bound for all x and therefore $\chi^{\pm}(x,\lambda)$ satisfy estimates analogous to those for $\tilde{\chi}^{\pm}(x,\lambda)$.

Let us introduce now in the space D_Λ the usual scalar product $(X,Y) = \int_{-\infty}^{\infty} dx\,\mathrm{tr}(X^{T}(x)Y(x))$ and the skew-scalar product $[\,,]$:

$$[X,Y] = \int_{-\infty}^{\infty} dx\,\mathrm{tr}(X^{T}(x), \hat{A} * Y(x)), \quad X,\ Y \in D_\Lambda, \tag{6}$$

where the notation $\hat{A} * Y$ was introduced in (2).

Lemma 1. The operators Λ_+ and Λ_- are adjoint to each other with respect to the skew-scalar product $[\cdot,\cdot]$, i.e.:

$$[\Lambda_- X, Y] = [X, \Lambda_+ Y]. \tag{7}$$

Proof. Perform integration by parts.

Lemma 2. If $q(x)$ in (1) is a function of Schwartz type, then the corresponding scattering data $S(\lambda)-1$, $S^{\pm}(\lambda)-1$, $T^{\pm}(\lambda)-1$ are also functions of Schwartz type.

Proof follows from (5), from the uniform boundedness of the projectors $P_\alpha^{\pm}(x)$ and from the estimates in /17/.

Remark 1. Below for convenience we shall write down the elements $X \in D_\Lambda$ as $n(n-1)$-component vectors $X \to \vec{X}^T = (\vec{X}^{(1)T}, \vec{X}^{(2)T})$, $\vec{X}^{(1)T} = (X_{12}, X_{13}, \ldots, X_{1n}, X_{23}, \ldots, X_{n-1,n})$, $\vec{X}^{(2)T} = (X_{21}, X_{31}, \ldots, X_{n1}, X_{32}, \ldots, X_{n\,n-1})$; the corresponding expressions for the operators $\Lambda_\pm$ as $n(n-1)\times n(n-1)$ matrix operators will be denoted by $\vec{\Lambda}_\pm$.

Let us introduce the systems of functions

$$W_{\overset{+}{(-)}} \equiv \{\vec{X}^{+}_{ip}(x,\lambda),\ \vec{X}^{-}_{pi}(x,\lambda),\ \lambda \in \mathbb{R};\quad \vec{X}^{+}_{\alpha,ip}(x),\quad \vec{X}^{-}_{\alpha,pi}(x),$$

$$\dot{\vec{X}}^{+}_{\alpha,ip}(x),\ \dot{\vec{X}}^{-}_{\alpha,pi}(x),\ \alpha = 1,\ldots,N;\ 1 \leq \underset{(p<i)}{i<p} \leq n\},$$

$$\vec{X}^{\pm}_{\alpha,ip}(x) = \lim_{\lambda\to\lambda_\alpha^{\pm}} (\lambda-\lambda_\alpha^{\pm})\vec{X}^{\pm}_{ip}(x,\lambda); \tag{8}$$

$$\dot{\vec{X}}^{\pm}_{\alpha,ip}(x) = \lim_{\lambda\to\lambda_\alpha^{\pm}} \frac{d}{d\lambda}(\lambda-\lambda_\alpha^{\pm})\vec{X}^{\pm}_{ip}(x,\lambda),$$

where the vectors $\vec{X}^{\pm}_{ip}(x,\lambda)$ are constructed according to remark 1 from the off-diagonal elements of the matrices $X^{\pm}_{ip}(x,\lambda) = \chi_i^{\pm}(x,\lambda)\hat{\chi}^{\pm}(x,\lambda)$, $\chi_i^{\pm}(x,\lambda)$ being the i-th column of the solution $\chi^{\pm}(x,\lambda)$ and $\hat{\chi}_p^{\pm}(x,\lambda)$ -the p-th row of $\hat{\chi}^{\pm}(x,\lambda)$.

Lemma 3. The elements of the system $W_+(W_-)$ are eigen- and adjoint functions of the operator $\vec{\Lambda}_+(\vec{\Lambda}_-)$, i.e.,

$$(\vec{\Lambda}_{\overset{+}{(-)}} - \lambda)\vec{X}^{+}_{ip}(x,\lambda) = 0,\ \lambda \in \mathbb{R}\cup\sigma^{+};\ (\vec{\Lambda}_{\overset{+}{(-)}} - \lambda_\alpha^{+})\dot{\vec{X}}_{\alpha,ip} = \vec{X}^{+}_{\alpha,ip},\quad \underset{(i>p)}{i<p}$$

$$(\Lambda_{\overset{+}{(-)}} - \lambda)\vec{X}^{-}_{pi}(x,\lambda) = 0,\ \lambda \in \mathbb{R}\cup\sigma^{-};\ (\vec{\Lambda}_{\overset{+}{(-)}} - \lambda_\alpha^{-})\dot{\vec{X}}^{-}_{\alpha,pi} = \vec{X}^{-}_{\alpha,pi},\quad \underset{(i>p)}{i<p} \tag{9}$$

Proof follows directly from (1), (2) and from the asymptotics (4) of $\chi^{\pm}(x,\lambda)$ for $x\to\pm\infty$.

Theorem 1. (About the completeness of W_+ [8]). For every vector-function $\vec{g}(x)\in D_{\vec{\Lambda}}$ the following expansion holds

$$\vec{g}(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\lambda\sum_{p<i}\{\vec{X}^{+}_{pi}(x,\lambda)[\vec{X}^{+}_{ip},\vec{g}]-\vec{X}^{-}_{ip}(x,\lambda)[\vec{X}^{-}_{pi},\vec{g}]\}-$$
$$-\sum_{\alpha=1}^{N}\sum_{p<i}\{\mathcal{R}(\vec{X}^{+}_{pi}(x,\lambda)[\vec{X}^{+}_{ip},\vec{g}])|_{\lambda=\lambda^{+}_{\alpha}}+\mathcal{R}(\vec{X}^{-}_{ip}(x,\lambda)[\vec{X}^{-}_{pi},\vec{g}])|_{\lambda=\lambda^{-}_{\alpha}}\} \qquad (10)$$

where $[\vec{X},\vec{g}]\equiv[X^{f},g]$ and the operation $\mathcal{R}$ is defined by $\mathcal{R}(X^{\pm}(\lambda)Y^{\pm}(\lambda))|_{\lambda=\lambda^{\pm}_{\alpha}}=\lim_{\lambda\to\lambda^{\pm}_{\alpha}}\frac{d}{d\lambda}((\lambda-\lambda^{\pm}_{\alpha})^{2}X^{\pm}(\lambda)Y^{\pm}(\lambda))$.

Idea of the proof. Apply the contour integration method to the integral $(2\pi i)^{-1}\oint_{\gamma}d\lambda\vec{G}(\vec{g},\lambda)$, where the contour $\gamma=\gamma_+\cup\gamma_-$ is shown on the figure and $\vec{G}(\vec{g},\lambda)$ is given by

$$\vec{G}(\vec{g},\lambda)=\int_{-\infty}^{\infty}dy\,\vec{G}(x,y,\lambda)\vec{g}(y),\quad \vec{G}(x,y,\lambda)=\begin{cases}\vec{G}^{+}(x,y,\lambda), & \mathrm{Im}\,\lambda>0\\ \vec{G}^{-}(x,y,\lambda), & \mathrm{Im}\,\lambda<0\end{cases}$$

$$\vec{G}^{(\stackrel{+}{-})}(x,y,\lambda)=i\{\sum_{\substack{p>i\\(p<i)}}\vec{G}^{(\stackrel{+}{-})}_{pi}(x,y,\lambda)\theta(x-y)-\sum_{\substack{p\le i\\(p\ge i)}}\vec{G}^{(\stackrel{+}{-})}_{pi}(x,y,\lambda)\theta(y-x)\}, \qquad (11)$$

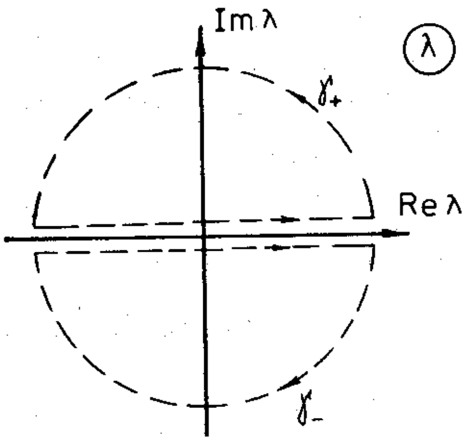
$$\vec{G}^{\pm}_{pi}(x,y,\lambda)=\vec{X}^{\pm}_{pi}(x,\lambda)\tilde{\vec{X}}^{\pm}_{ip}(y,\lambda),\quad \tilde{\vec{X}}=\vec{X}^{T}\vec{A},\quad \vec{A}=\begin{pmatrix}0 & -\vec{a}\\ \vec{a} & 0\end{pmatrix},$$

$$\vec{a}=\mathrm{diag}(a_1-a_2,a_1-a_3,\dots,a_1-a_n,a_2-a_3,\dots,a_{n-1}-a_n).$$

Remark 2. The vectors $\vec{q}^{(k)}(x)$ and $\vec{A}\delta\vec{q}(x)$ related by remark 1 to $[q(x),\Pi^{(k)}]$, $\Pi^{(k)}=\mathrm{diag}(1,\dots,1_k,0\dots0)$ and $A_*\delta q(x)$ may be expanded over the system $W_+(W_-)$. These expansions and also points i) and ii) (see the introduction) for the NLEE related to the system (1) are accomplished in [8].

Lemma 4. The function $\vec{G}(\vec{g},\lambda)$: i) is analytic with respect to λ, $\lambda\notin\mathbb{R}\cup\sigma$ and has poles of second order for $\lambda\in\sigma$; ii) is a function of Schwartz type with respect to x for $\lambda\in\mathbb{R}$; iii) for fixed $\lambda\in\mathbb{R}$ $\vec{G}(x,y,\lambda)$ is uniformly bound with respect to x and y.

Proof follows from the estimates in [17] for $\chi^{\pm}(x,\lambda)$ and from the definition (11).



The contours $\gamma_\pm$.

Theorem 2. The function $\vec{G}(x,y,\mu)$ is the operator $\vec{\Lambda}_+$ resolvents' kernel, i.e.,

$$(\vec{\Lambda}_+-\mu)\vec{G}(\vec{g},\mu)=\vec{G}((\vec{\Lambda}_+-\mu)\vec{g},\mu)=\vec{g}(x), \qquad \vec{g}(x)\in D_{\vec{\Lambda}} \quad . \tag{12}$$

First proof. Let the potential q(x) be on compact support. Then $\phi^{\pm}(x,\lambda)$, $S(\lambda)-1$, $S^{\pm}(\lambda)-1$, $T^{\pm}(\lambda)-1$ are integer functions of λ and relations (4) hold for all λ. Then using (4), (11) and (2) we directly obtain (12). For potentials q(x) of Schwartz type (12) is obtained by limiting procedure.

Below we shall give another proof of theorem 2, based on the following.

Theorem 3 (about the spectral decomposition for $\vec{G}(\vec{g},\mu)$). If the discrete spectrum of the operator L (1) is simple and finite, then $\vec{G}(\vec{g},\mu)$ for $\mu\bar{\in}\mathbf{R}\cup\sigma$, $g\in D_{\vec{\Lambda}}$ may be represented in the form:

$$\vec{G}(\vec{g},\mu)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{d\lambda}{\lambda-\mu}\sum_{i<p}\{\vec{X}^{+}_{ip}(x,\lambda)[\vec{X}^{+}_{pi},\vec{g}]-\vec{X}^{-}_{pi}(x,\lambda)[\vec{X}^{-}_{ip},\vec{g}]\}-$$

$$-i\sum_{\alpha=1}^{N}\sum_{i<p}\{\mathcal{R}(\frac{1}{\lambda-\mu}\vec{X}^{+}_{ip}(x,\lambda)[\vec{X}^{+}_{pi},\vec{g}])|_{\lambda=\lambda^{+}_{\alpha}}+\mathcal{R}(\frac{1}{\lambda-\mu}\vec{X}^{-}_{pi}(x,\lambda)[\vec{X}^{-}_{ip},\vec{g}])|\}_{\lambda=\lambda^{-}_{\alpha}}. \tag{13}$$

For $\mu\in\mathbf{R}$ (13) holds if $\vec{G}(\vec{g},\mu)=\frac{1}{2}(\vec{G}^{+}-\vec{G}^{-})$ and the integral in the r.h.s. of (13) is understood in a sense of principal value.

Idea of the proof. Apply the contour integration method to the integral $(2\pi i)^{-1} \oint_\gamma d\lambda (\mu-\lambda)^{-1} G(\vec{g},\lambda)$.

Second proof of theorem 2 follows directly from (13) applying the operator $\vec{\Lambda}_+ - \mu$ and using (9).

Just like in/8/ we prove

Theorem 4. Let the poles of the rational function $h(\lambda)$ lie outside the spectrum $\boldsymbol{R} \cup \sigma$ of the operator L. Then the action of the operator $h(\vec{\Lambda}_+)$ on $\vec{g}(x) \in D_{\vec{\Lambda}}$ is defined by:

$$h(\vec{\Lambda}_+)\vec{g}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\, h(\lambda) \sum_{i<p} \{ \vec{X}^+_{ip}(x,\lambda)[\vec{X}^+_{pi},\vec{g}] - \vec{X}^-_{pi}(x,\lambda)[\vec{X}^-_{ip},\vec{g}]\} -$$
$$- \sum_{\alpha=1}^{N} \sum_{i<p} \{ \mathcal{R}(h(\lambda)\vec{X}^+_{ip}(x,\lambda)[\vec{X}^+_{pi},\vec{g}])|_{\lambda=\lambda^+_\alpha} + \mathcal{R}(h(\lambda)\vec{X}^-_{pi}(x,\lambda)[\vec{X}^-_{ip},g])|\}_{\lambda=\lambda^-_\alpha} \quad (14)$$

Proof follows directly from theorem 1 and lemmas 3 and 1, which give $[\vec{X}^+_{pi}, h(\vec{\Lambda}_+)\vec{g}] = [h(\vec{\Lambda}_-)\vec{X}^+_{pi},\vec{g}] = h(\lambda)[\vec{X}^+_{pi},\vec{g}]$, $i<p$, etc.

Let us introduce the function $R(x,y,\lambda) = R^{(\pm)}(x,y,\lambda)$ for $\mathrm{Im}\,\lambda > 0$ $(\mathrm{Im}\,\lambda < 0)$,

$$R^{\pm}(x,y,\lambda) = \pm i \chi^{\pm}(x,\lambda)\Theta(\pm(x-y))\hat{\chi}^{\pm}(y,\lambda)A, \quad (15)$$
$$\Theta(z) = \mathrm{diag}(\theta(z),\dots,\theta_\kappa(z),\dots -\theta(-z),\dots,-\theta(-z)),$$

where κ is the number of the positive elements in A, i.e., $a_1 > \dots > a_\kappa > 0 > a_{\kappa+1} > \dots > a_n$.

Lemma 5. The function $R(x,y,\lambda)$ is the operator L resolvent's kernel, i.e.,

$$(L-\lambda)R(g,\lambda) = R((L-\lambda)g,\lambda) = g(x), \qquad g(x) \in D_L \;, \quad (16)$$

where $R(g,\lambda) = \int_{-\infty}^{\infty} dy\, R(x,y,\lambda)\, g(y)$. Besides: i) $R(g,\lambda)$ is an analytic function of λ for $\lambda \bar{\in} \boldsymbol{R} \cup \sigma$, having first order pole singularities for $\lambda \in \sigma$; ii) $R(g,\lambda)$ is a Schwartz type function with respect to x for $\lambda \bar{\in} \boldsymbol{R}$; iii) for fixed $\lambda \in \boldsymbol{R}$ $R(x,y,\lambda)$ is uniformly bounded for all x,y.

Proof of (16) follows directly from (1) and (15). Feature i) is obvious from (15) and ii) and iii) are consequences of the estimates for $\chi^{\pm}(x,\lambda)$.

Let us define the diagonal of the resolvent's kernel as $R(x,\lambda) = \frac{1}{2}(R(x+0,x,\lambda) + R(x,x+0,\lambda))$.

Lemma 6. The diagonal of the resonvent's kernel $R(x,\lambda)$ satisfies the equation

$$-i\frac{d}{dx}R(x,\lambda)\hat{A}+[q(x),R(x,\lambda)\hat{A}]-\lambda[A,R(x,\lambda)\hat{A}]=0. \qquad (17)$$

Proof follows directly from (1).

Remark 3. Equation (17) is satisfied also by $R_P(x,\lambda)=i\chi^{(\pm)}(x,\lambda)P\hat{\chi}^{(\pm)}(x,\lambda)A$ for $\mathrm{Im}\,\lambda>0$ ($\mathrm{Im}\,\lambda<0$), P being constant diagonal matrix.

Lemma 7. For the diagonal and off-diagonal parts of $R_P(x,\lambda)$, $\lambda\in\sigma$ the following representations hold:

$$R_P^f(x,\lambda)\hat{A}=i(\Lambda_\pm-\lambda)^{-1}\hat{A}*[q,P], \qquad (18)$$

$$R_P^d(x,\lambda)\hat{A}=iP-\int_x^{\pm\infty}dy[q(y),(\Lambda_\pm-\lambda)^{-1}\hat{A}*[q,P]]^d. \qquad (19)$$

The same is true also for $\lambda\in\mathbb{R}$ if one puts $R_P(x,\lambda)=\frac{1}{2}(R_P^+-R_P^-)(x,\lambda)$.

Proof. Applying the contour integration method to the integral $I=(2\pi i)^{-1}\oint_\gamma d\mu(\mu-\lambda)^{-1}R_P(x,\mu)$ we obtain an expansion for $R_P^f(x,\lambda)$ over the system W_+ (or W_-), which coincides with the expansion for the r.h.s. of (18), see /8/. Relation (19) follows from (18), (17) and from $\lim_{x\to\pm\infty}R_P(x,\lambda)=iP$.

The analyticity properties of $R_P(x,\lambda)$ enable one to expand the r.h.s. of (18) and (19) in Taylor series, e.g., in powers of λ^{-1}. Thus we obtain compact expressions for the coefficients $R_P^{(k)}(x)$

$$R_P(x,\lambda)\hat{A}=\sum_{k=1}^{\infty}R_P^{(k)}(x)\lambda^{-k}+iP, \qquad (20)$$

through $q(x)$ and the operators $\Lambda_\pm$.

Now let us show, that $R_P(x,\lambda)$ may be considered as a generating functional of the M-operators, taking part in the Lax representation

$$iAL_t=[A(L-\lambda),M] \qquad (21)$$

for the NLEE, related to (1).

Theorem 5. If we choose $M(x,\lambda)$ in the form

$$M_N=\sum_{k=0}^{N}R_F^{(k)}(x)\lambda^{N-k}, \qquad F=\sum_{s=0}^{n}c_s e^{(s)} \qquad (22)$$

where $e^{(s)}_{ij} = \delta_{ij}\delta_{js}$, then the Lax equation (21) is satisfied identically w.r.t. to λ and is equivalent to the NLEE

$$\hat{A} * q_t + \Lambda_+^N \hat{A} * [q, F] = 0. \tag{23}$$

The corresponding time dependence of the scattering data of (1) is given by

$$\frac{dS}{dt} = [F(\lambda), S(\lambda, t)], \qquad F(\lambda) = \lambda^N F . \tag{24}$$

Proof. Note that from (17) and (20) there follows $R_P^{(k+1)f}(x) = \Lambda_\pm R_P^{(k)f}(x)$ and $-i\frac{d}{dx} R_P^{(k)d}(x) + [q, R_P^{(k)f}]^d = 0$. Now insert (23) and (1) in the r.h.s. of (21). This immediately gives $[A(L-\lambda), M_N] = [A, R_F^{(N+1)}]$. At last using the compact expression for $R_F^{(N+1)f}$, which follows from (18) and (20) we obtain the equivalence of (21) and (23). The equivalence of (23) and (24) has been proved earlier (see [6,8]).

Theorem 6. The quantities $R_{\Pi^{(k)}}(x,\lambda)$ with $\Pi^{(k)} = \sum_{s=1}^{k} e^{(s)}$ satisfy the relations

$$\int_{-\infty}^{\infty} dx\, \mathrm{tr}(R_{\Pi^{(k)}}(x,\lambda) - i\Pi^{(k)} A) = -\frac{d}{d\lambda} D^{(k)}(\lambda), \quad D^{(k)}(\lambda) = \begin{cases} \ln \Delta^-_{n-k}(\lambda), & \mathrm{Im}\,\lambda > 0 \\ 1/2 \ln(\Delta^-_{n-k}/\Delta^+_k), & \mathrm{Im}\,\lambda = 0 \\ -\ln \Delta^+_k(\lambda), & \mathrm{Im}\,\lambda < 0 \end{cases} \tag{25}$$

where $\Delta^-_k(\lambda)(\Delta^+_k(\lambda))$ is the upper (lower) principal minor of $S(\lambda)$ of order k. Besides, if $q(x,t)$ is a solution of any NLEE of the type (23), then $\frac{d}{dt} D^{(k)}(\lambda) = 0$.

Proof. Let us use eq. (1) and the following from (1) equations for $\dot{\chi}^\pm = \frac{d}{d\lambda}\chi^\pm(x,\lambda)$ and $\hat{\chi}^\pm(x,\lambda)$. This shows that the l.h.s. of (25) is equal to $\mathrm{tr}(-i\hat{\chi}^\pm\dot{\chi}^\pm \Pi^{(k)} + \Pi^{(k)} A x)\big|_{x=-\infty}^{\infty}$. To prove (25) it is enough to use the asymptotics (4) of $\chi^\pm$ for $x \to \pm\infty$ and the relations between $S^\pm$, $T^\pm$ and $S(\lambda)$ (for details see [8,1]). The fact that $\frac{d}{dt} D^{(k)}(\lambda) = 0$ follows readily from (24).

Remark 4. The general form of the M-operators and the conservation laws of the NLEE has been obtained earlier in [5] by studying the formal expression for the resolvent of (1). A recurrent procedure for the construction of M has been proposed in [2] based on the fact, that M satisfies eq. (17) with iAq_t

in the r.h.s. (insert (1) into (21)). The considerations given above in constructing M and the conservation laws show the equivalence of both approaches. The same conclusion has been obtained in /12/ and /18/ by different methods.

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Received by Publishing Department
on October 20 1981.