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EXACT SOLUTION OF THE COVARIANT TWO-PARTICLE EQUATION WITH SUPERPOSITION OF ONE BOSON EXCHANGE QUASIPOTENTIALS

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1. INTRODUCTION

The description of a two-particle relativistic system is one of the central problems of quantum field theory. To study this problem, the four-dimensional Bethe-Salpeter equation and covariant three-dimensional equations, derived in the Logunov-Tavkhelidze single-time approach [1] to the problem of the relativistic description of composite systems, are applied.

For the relativistic amplitude of the elastic scattering of two spinless particles with equal masses $m_1=m_2=m$ and the wave function (WF) of their relative motion the Logunov-Tavkhelidze equations can be written in the form [1]:

$$T(\overset{\circ}{\vec p},\overset{\circ}{\vec q})=V(\overset{\circ}{\vec p},\overset{\circ}{\vec q};\overset{\circ}{E})+\frac{1}{4(2\pi)^3}\int V(\overset{\circ}{\vec p},\overset{\circ}{\vec k};\overset{\circ}{E})\frac{d^3\overset{\circ}{k}}{\sqrt{m^2+\overset{\circ}{\vec k}{}^2}}\cdot\frac{T(\overset{\circ}{\vec k},\overset{\circ}{\vec q})}{\overset{\circ}{\vec k}{}^2-\overset{\circ}{\vec q}{}^2-i\epsilon} \qquad (1.1)$$

$$(m^2+\overset{\circ}{\vec p}{}^2-\overset{\circ}{E}{}^2)=\frac{1}{4(2\pi)^3}\frac{1}{\sqrt{m^2+\overset{\circ}{\vec p}{}^2}}\int d^3\overset{\circ}{k}\,V(\overset{\circ}{\vec p},\overset{\circ}{\vec k};\overset{\circ}{E})\Psi(\overset{\circ}{\vec k}). \qquad (1.2)$$

By zeros we denote the vectors $\overset{\circ}{p}$ and $\overset{\circ}{k}$ that are covariant generalizations of the momentum vectors of particles in the c.m.s. introduced in ref. [2]. Thus, if we define $(\overset{\circ}{p}_1)_\mu = (\Lambda^{-1}_{\mathcal{P}}p_1)_\mu$, where $\Lambda_{\mathcal{P}}$ is the Lorentz boost in the c.m.s. of two particles, that moves with the total momentum $\mathcal{P} = p_1+p_2$, $\Lambda_{\mathcal{P}}(M,\vec 0)=(\mathcal{P}_0,\vec{\mathcal{P}})$ then:

$$\overset{\circ}{\vec p}_1=\vec p_1-\frac{\vec{\mathcal{P}}}{M}\left[(p_1)_0-\frac{(\vec p_1\vec{\mathcal{P}})}{\mathcal{P}_0+M}\right],$$
$$(\overset{\circ}{p}_1)_0=(\Lambda^{-1}_{\mathcal{P}}p_1)_0=p_1^{\mu}\mathcal{P}_\mu/M=\text{inv}. \qquad (1.3)$$

For the vectors $\overset{\circ}{\vec p}_1,\overset{\circ}{\vec p}_2$ and $\overset{\circ}{\vec q}_1,\overset{\circ}{\vec q}_2$ the relations $\overset{\circ}{\vec p}_1=-\overset{\circ}{\vec p}_2$ and $\overset{\circ}{\vec q}_1=-\overset{\circ}{\vec q}(\equiv\overset{\circ}{\vec q})$ hold that generalize for an arbitrary system the relations between the momenta of particles characteristic for the c.m.s. The energy components of the vectors do not coincide $\overset{\circ}{p}_0\neq\overset{\circ}{k}_0$ what reflects the fact that in equations (1.1) and (1.2) all the quantities (like in the Lippman-Schwinger and Schrödinger equations) are defined off the covariant "energy" shell $\overset{\circ}{p}_0 = \overset{\circ}{k}_0$. At the same time in (1.1) and (1.2) the momenta of all the particles belong to the mass shell

$$\overset{\circ}{p}{}_0^2 - \overset{\circ}{\vec{p}}{}^2 = m^2 ; \quad \overset{\circ}{k}{}_0^2 - \overset{\circ}{\vec{k}}{}^2 = m^2 . \tag{1.4}$$

Such a picture is an alternative one to the approach based on the Bethe-Salpeter equation where the conservation law of the 4-momenta takes place at each vertex but all the quantities are defined off the mass shell (1.4).

The wave function (WF) in the single-time approach is defined covariantly through the Bethe-Salpeter WF by the following expression[3]

$$\Psi(\overset{\circ}{\vec{p}}) = \int \exp[\tfrac{i}{2}(p_1 - p_2)x]\,\delta(\lambda_{\mathcal{P}} x) \langle 0 | T\{\phi_1(\tfrac{x}{2})\phi_2(-\tfrac{x}{2})\} | \vec{\mathcal{P}}, M \rangle \, d^4 x ,$$

where $x = x_1 - x_2$ is the relative coordinate of two scalar particles, characterized by the field operators $\phi_1(x_1)$ and $\phi_2(x_2)$.

The vector $\lambda^{\mu}_{\mathcal{P}} = \mathcal{P}^{\mu}/\sqrt{\mathcal{P}^2}$ is the 4-velocity of the whole system. In the c.m.s. $\vec{\mathcal{P}} = 0$ (i.e., $\vec{\lambda}_{\mathcal{P}} = 0$) $\overset{\circ}{\lambda}_{\mathcal{P}} = 1$ and the presence of the $\delta(\lambda_{\mathcal{P}} x)$ -function under the sign of integration leads to the coincidence of the particle times $x^{\circ}_1 = x^{\circ}_2$.

The quasipotential $V(\overset{\circ}{\vec{p}}, \overset{\circ}{\vec{k}}; \overset{\circ}{E})$ (complex function, in the general case, and parametrically depending on the total energy of a system $2\overset{\circ}{q}_0 = 2\overset{\circ}{E} = M$) is built with the use of a two-time Green function of the considered system or with the use of the scattering amplitude on the mass shell, i.e., like the solution of equation (1.1) with the quasipotential $V(\overset{\circ}{\vec{p}}, \overset{\circ}{\vec{k}}; \overset{\circ}{E})$ taken as an unknown function. The amplitude $T(\overset{\circ}{\vec{p}}, \overset{\circ}{\vec{q}})$ is considered as a given function, defined through the matrix elements of quantum field theory. Let us note that since the relativistic amplitude $T(\overset{\circ}{\vec{p}}, \overset{\circ}{\vec{q}})$ is defined by the quantum field theory on the energy shell $\overset{\circ}{p}_0 = \overset{\circ}{q}_0$, then there appears some arbitrariness in defining the quasipotential $V(\overset{\circ}{\vec{p}}, \overset{\circ}{\vec{k}}; \overset{\circ}{E})$ at $\overset{\circ}{p}_0 \neq \overset{\circ}{k}_0$, i.e., in extrapolating it off the energy shell. In what follows we shall make use of this arbitrariness defining the quasipotential so that, on the one hand, the quasipotential equation for the WF would have a form that would be maximally close to the form of the nonrelativistic Schrödinger equation, and, on the other hand, the quasipotential would be a local function in the Lobachevsky momentum space[4,5], realized on the upper sheet of the mass shell hyperboloid (1.4).

Equation (1.2) by the redefinition of the WF: $\tilde{\Psi}(\overset{\circ}{\vec{p}}) = \sqrt{m^2 + \overset{\circ}{\vec{p}}{}^2}\, \Psi(\overset{\circ}{\vec{p}})$ can be represented by:

$$(\overset{\circ}{\vec{p}}{}^2 + m^2 - \overset{\circ}{E}{}^2)\tilde{\Psi}(\overset{\circ}{\vec{p}}) = \frac{1}{4(2\pi)^3}\int V(\overset{\circ}{\vec{p}},\overset{\circ}{\vec{k}};\overset{\circ}{E})\tilde{\Psi}(\overset{\circ}{\vec{k}})\frac{d^3\overset{\circ}{\vec{k}}}{\sqrt{m^2 + \overset{\circ}{\vec{k}}{}^2}} \,. \tag{1.5}$$

In the present paper we shall consider equation (1.5) with some model quasipotential that is the generalization of the nonrelativistic Coulomb potential.

In Sec.4 we shall find the energy spectrum for such a model of the interaction in the two-particle system and define a form of the WF's in the momentum representation.

2. QUASIPOTENTIAL EQUATION IN THE CASE OF THE "NONRELATIVISTIC" NORMALIZATION OF A SCATTERING AMPLITUDE TO THE CROSS SECTION

On the energy shell $E_{\overset{\circ}{p}} = E_{\overset{\circ}{q}}$ an amplitude $T(\overset{\circ}{\vec{p}},\overset{\circ}{\vec{q}})$ coincides with an invariant amplitude $T(s,t)$, connected with the differential cross section of the elastic scattering through the relation ($s = (p_1 + p_2)^2$; $t = (q_1 - p_1)^2$),

$$\frac{d\sigma}{d\omega} = \frac{|T(s,t)|^2}{(8\pi)^2 s} \,. \tag{2.1}$$

From (1.1) in the case of a real quasipotential $V(\overset{\circ}{\vec{p}},\overset{\circ}{\vec{k}};\overset{\circ}{E})$ there follows the relativistic two-particle unitarity condition ($d\omega_{\overset{\circ}{\vec{k}}} = \sin\theta\, d\theta\, d\phi$):

$$\mathrm{Im}\, T(\overset{\circ}{\vec{p}},\overset{\circ}{\vec{q}}) = \frac{1}{(8\pi)^2}\cdot\frac{|\overset{\circ}{\vec{q}}|}{\overset{\circ}{q}_0}\int d\omega_{\overset{\circ}{\vec{k}}}\, T^*(\overset{\circ}{\vec{p}},\overset{\circ}{\vec{k}})\, T(\overset{\circ}{\vec{k}},\overset{\circ}{\vec{q}}) , \tag{2.2}$$

$|\overset{\circ}{\vec{p}}| = |\overset{\circ}{\vec{q}}| = |\overset{\circ}{\vec{k}}|$.

Following [5], let us define

$$T(\overset{\circ}{\vec{p}},\overset{\circ}{\vec{q}}) = 8\pi\cdot 2E_{\overset{\circ}{q}}\,\tilde{T}(\overset{\circ}{\vec{p}},\overset{\circ}{\vec{q}}), \tag{2.3}$$

$$V(\overset{\circ}{\vec{p}},\overset{\circ}{\vec{k}};\overset{\circ}{E}) = -4mE_{\overset{\circ}{k}}\tilde{V}(\overset{\circ}{\vec{p}},\overset{\circ}{\vec{k}};\overset{\circ}{E}). \tag{2.4}$$

Then the equation for the amplitude $\tilde{T}(\overset{\circ}{\vec{p}},\overset{\circ}{\vec{q}})$ takes a "nonrelativistic" form

$$\tilde{T}(\overset{\circ}{\vec{p}},\overset{\circ}{\vec{q}}) = -\frac{m}{4\pi}\tilde{V}(\overset{\circ}{\vec{p}},\overset{\circ}{\vec{q}};\overset{\circ}{E}) - \frac{m}{(2\pi)^3}\int \tilde{V}(\overset{\circ}{\vec{p}},\overset{\circ}{\vec{k}};\overset{\circ}{E})\frac{d\,k}{\overset{\circ}{\vec{k}}{}^2 - \overset{\circ}{\vec{q}}{}^2 - i\epsilon}\tilde{T}(\overset{\circ}{\vec{k}},\overset{\circ}{\vec{q}}). \tag{2.5}$$

The differential cross section can be expressed through the amplitude $\tilde{T}(\overset{\circ}{\vec{p}},\overset{\circ}{\vec{q}})$ at $E_{\overset{\circ}{p}}=E_{\overset{\circ}{q}}$ in the same way as in the nonrelativistic theory:

$$\frac{d\sigma}{d\omega}=|\tilde{T}(\overset{\circ}{\vec{p}},\overset{\circ}{\vec{q}})|^2 . \qquad (2.6)$$

In the case of a real quasipotential $\tilde{V}(\overset{\circ}{\vec{p}},\overset{\circ}{\vec{k}};\overset{\circ}{E})$ from (2.5) there follows a two-particle unitarity condition, that coincides in form with the nonrelativistic one:

$$\mathrm{Im}\,T(\overset{\circ}{\vec{p}},\overset{\circ}{\vec{q}})=\frac{|\overset{\circ}{\vec{q}}|}{4\pi}\int T^*(\overset{\circ}{\vec{p}},\overset{\circ}{\vec{k}})\,T(\overset{\circ}{\vec{k}};\overset{\circ}{\vec{q}})\,d\omega_{\overset{\circ}{\vec{k}}} \qquad (2.7)$$

(at $|\overset{\circ}{\vec{p}}|=|\overset{\circ}{\vec{q}}|=|\overset{\circ}{\vec{k}}|$).

An equation for the WF (1.3) with account of the definition (2.4) can be written in the form

$$(E^2_{\overset{\circ}{p}}-E^2)\tilde{\Psi}(\overset{\circ}{\vec{p}})=-\frac{m}{(2\pi)^3}\int \tilde{V}(\overset{\circ}{\vec{p}},\overset{\circ}{\vec{k}};\overset{\circ}{E})\tilde{\Psi}(\overset{\circ}{\vec{k}})\,d^3\overset{\circ}{\vec{k}} \qquad (2.8)$$

or passing to the binding energy $W=2m-2\overset{\circ}{E}$, in the form

$$[\overset{\circ}{\vec{p}}{}^2+W(m-\frac{W}{4})]\tilde{\Psi}(\overset{\circ}{\vec{p}})=-\frac{m}{(2\pi)^3}\int \tilde{V}(\overset{\circ}{\vec{p}},\overset{\circ}{\vec{k}};\overset{\circ}{E})\,\tilde{\Psi}(\overset{\circ}{\vec{k}})\,d^3\overset{\circ}{\vec{k}} . \qquad (2.9)$$

In the literature a procedure of deriving equations (2.9) and (2.5) with the potential $\tilde{V}(\overset{\circ}{\vec{p}},\overset{\circ}{\vec{k}};\overset{\circ}{E})$ that is no longer a local function in the Euclidean momentum space is named as the "procedure of a minimal relativization" /6/.

3. QUASIPOTENTIAL EQUATION FOR THE WF IN THE RELATIVISTIC CONFIGURATIONAL REPRESENTATION

The locality property of a nonrelativistic potential in the Euclidean momentum space allows one to transform the Schrödinger equation that has the form of an integral equation in the momentum space with the help of the Fourier transformation into a local differential equation in the coordinate space. The relativistic generalization of this procedure was proposed in /5/, where instead of the Fourier transformation, an expansion over the principal series of the unitary representations of the Lorentz group was applied, i.e., an expansion over the functions /7/:

$$\xi(\overset{\circ}{\vec{p}},\vec{r})=\left(\frac{\overset{\circ}{p}_0-\overset{\circ}{\vec{p}}\vec{n}}{m}\right)^{-1-irm} ; \quad \vec{r}=r\vec{n} ; \quad \vec{n}^2=1 . \qquad (3.1)$$

The modulus of the vector $\vec{r}$, i.e., r is the relativistic invariant [8,9] and in the nonrelativistic limit, where $\xi(\overset{\circ}{\vec{p}},\vec{r}) \to e^{i\vec{p}\vec{r}}$, it transforms into the modulus of the relative coordinate, i.e., it can be considered as its invariant generalization [5].

The functions (3.1) obey the relations of orthogonality and completeness on the surface of the mass-shell hyperboloid (1.4). The transformations by them have the form [5]:

$$\tilde{\Psi}(\vec{r}) = \frac{1}{(2\pi)^3} \int \xi(\overset{\circ}{\vec{p}},\vec{r})\, \tilde{\Psi}(\overset{\circ}{\vec{p}}) \frac{m\, d^3\overset{\circ}{\vec{p}}}{\sqrt{m^2 + \overset{\circ}{\vec{p}}^2}}, \tag{3.2}$$

$$\tilde{\Psi}(\overset{\circ}{\vec{p}}) = \int \xi^*(\overset{\circ}{\vec{p}},\vec{r})\, \tilde{\Psi}(\vec{r})\, d^3\vec{r}, \tag{3.3}$$

$$\tilde{V}[(\overset{\circ}{\vec{p}}(-)\overset{\circ}{\vec{k}})^2;\overset{\circ}{E}] = \int \xi^*(\overset{\circ}{\vec{p}},\vec{r})\, \tilde{V}(\vec{r},\overset{\circ}{E})\, \xi(\overset{\circ}{\vec{k}},\vec{r})\, d^3\vec{r}\ . \tag{3.4}$$

In the last line $\overset{\circ}{\vec{p}}(-)\overset{\circ}{\vec{k}}$ is the difference of the vectors in the three-dimensional momentum space realized on the mass-shell hyperboloid (1.4) with the Lobachevsky geometry [5]:

$$\overset{\circ}{\vec{p}}(-)\overset{\circ}{\vec{k}} \equiv \vec{\Delta}_{\overset{\circ}{p},\overset{\circ}{k}} = \overset{\circ}{\vec{p}} - \frac{\overset{\circ}{\vec{k}}}{m}\left(\overset{\circ}{p}_0 - \frac{\overset{\circ}{\vec{k}}\,\overset{\circ}{\vec{p}}}{m + \overset{\circ}{k}_0}\right). \tag{3.5}$$

The dependence of the right-hand side of the formula (3.4) on the vector (3.5) follows from the "addition" theorem for the plane waves $\xi(\overset{\circ}{\vec{p}},\vec{r})$ [5]. These functions $\xi(\overset{\circ}{\vec{p}},\vec{r})$ obey the equation

$$\hat{H}_0 \xi(\overset{\circ}{\vec{p}},\vec{r}) = 2E_{\overset{\circ}{p}}\, \xi(\overset{\circ}{\vec{p}},\vec{r})\ ; \quad E_{\overset{\circ}{p}} = \sqrt{m^2 + \overset{\circ}{\vec{p}}^2}\ , \tag{3.6}$$

where the "free Hamiltonian"

$$\hat{H}_0 = 2m\cosh\left(\frac{i}{m}\frac{\partial}{\partial r}\right) + \frac{2i}{r}\sinh\left(\frac{i}{m}\frac{\partial}{\partial r}\right) + \frac{\Delta_{\theta,\phi}}{r^2}\exp\left(\frac{i}{m}\frac{\partial}{\partial r}\right) \tag{3.7}$$

is a finite-difference operator [5]. The relativistic invariance of the operator $\hat{H}$ was proved in [9]. From (2.8) with the help of (3.2), (3.4) and (3.6) it is easy to find the equation for the WF $\tilde{\Psi}(\vec{r})$

$$\left[\left(\frac{\hat{H}_0}{2}\right)^2 - \overset{\circ}{E}{}^2\right] \tilde{\Psi}(\vec{r}) = \tilde{V}(\vec{r},\overset{\circ}{E})\, \frac{\hat{H}_0}{2}\, \tilde{\Psi}(\vec{r}). \tag{3.8}$$

As was shown earlier[1,10], the quasipotential $\tilde{V}(\mathring{\vec{p}},\mathring{\vec{k}};\mathring{E})$ can be built with the use of $T(\mathring{\vec{p}},\mathring{\vec{k}})$ on the energy shell, taking $T(\mathring{\vec{p}},\mathring{\vec{k}})$ as given by rules of the perturbation theory of quantum field theory. In the lowest approximation we have on the energy shell

$$\tilde{V}(\mathring{\vec{p}},\mathring{\vec{k}};\mathring{E}) = -\frac{1}{4mE_{\mathring{k}}}T(\mathring{\vec{p}},\mathring{\vec{k}}). \tag{3.9}$$

For the amplitude of the scalar meson exchange (of the mass μ) $T(\mathring{\vec{p}},\mathring{\vec{k}}) = 4m^2g^2(\mu^2-(p-k)^2)^{-1}$, where g is a dimensionless coupling constant, we have on the energy shell $E_{\mathring{p}} = E_{\mathring{k}} = \mathring{E}$

$$\tilde{V}(\mathring{\vec{p}},\mathring{\vec{k}};\mathring{E}) = -\frac{m}{\mathring{E}}\frac{g^2}{\mu^2-(p-k)^2} \equiv -\frac{m}{\mathring{E}}\cdot\frac{g^2}{\mu^2-2m^2+\sqrt{m^2+\Delta^2_{\mathring{p},\mathring{k}}}}. \tag{3.10}$$

Let us define, following[11], the quasipotential $\tilde{V}(\mathring{\vec{p}},\mathring{\vec{k}})$ off the energy shell given by formula (3.10). Then the quasipotential would be local in the Lobachevsky space: $\tilde{V}(\mathring{\vec{p}},\mathring{\vec{k}};\mathring{E}) = \tilde{V}(\mathring{\vec{p}}(-)\mathring{\vec{k}};\mathring{E})$.

4. THE SOLUTION OF THE SINGLE-TIME EQUATION WITH THE MODEL QUASIPOTENTIAL

Let us consider equation (2.8) taking as the quasipotential the following expression

$$\tilde{V}(\mathring{\vec{p}}(-)\mathring{\vec{k}};\mathring{E}) = -\frac{m}{\mathring{E}}\cdot\frac{g^2}{|\mathring{\vec{p}}(-)\mathring{\vec{k}}|^2}. \tag{4.1}$$

In the nonrelativistic Schrödinger equation the Coulomb potential in the momentum space has the form

$$V(\vec{p}-\vec{k}) = -\frac{g^2}{|\vec{p}-\vec{k}|^2}. \tag{4.2}$$

The model quasipotential (4.1) is nothing but the superposition of two quasipotentials of the form (3.10) corresponding to the values of masses of exchanged bosons $\mu=0$ and $\mu=2m$:

$$\tilde{V}(\mathring{\vec{p}}(-)\mathring{\vec{k}};\mathring{E}) = -\frac{m}{\mathring{E}}\cdot\frac{g^2}{|\mathring{\vec{p}}(-)\mathring{\vec{k}}|^2} = -\frac{m}{\mathring{E}}\cdot g^2\cdot\left[\frac{1}{0-(p-k)^2}-\frac{1}{(2m)^2-(p-k)^2}\right]. \tag{4.3}$$

In what follows we shall consider equation (2.8) with the potential (4.1).

$$(E^2_{\overset{\circ}{p}} - \overset{\circ}{E}^2)\tilde{\Psi}(\overset{\circ}{\vec{p}}) = \frac{1}{(2\pi)^3}\cdot\frac{m^2}{\overset{\circ}{E}}\int \frac{g^2}{|\overset{\circ}{\vec{p}}(-)\overset{\circ}{\vec{k}}|^2}\cdot E_{\overset{\circ}{k}}\cdot\tilde{\Psi}(\overset{\circ}{\vec{k}})\frac{d^3\overset{\circ}{\vec{k}}}{\sqrt{m^2+\overset{\circ}{\vec{k}}^2}}. \tag{4.4}$$

The corresponding to (4.1) quasipotential in the relativistic configurational representation can be found with the formula that is inverse to (3.4).

$$\tilde{V}(r,\overset{\circ}{E}) = -\frac{m}{\overset{\circ}{E}}\cdot\frac{g^2}{4\pi r}\tanh\left(\frac{\pi r m}{2}\right). \tag{4.5}$$

It is clear that in the nonrelativistic limit, i.e., at $rm \gg 1$ we have $\frac{g^2}{4\pi r}\tanh\left(\frac{\pi r m}{2}\right) \to \frac{g^2}{4\pi r}$ which means that the potential (4.1) in the configurational representation can be considered as a generalization of a nonrelativistic Coulomb potential.

In the spherical-symmetrical case ($\tilde{\Psi}(\vec{r}) = \tilde{\Psi}(r)$) equation (3.8) for the function $r\tilde{\Psi}(r)$ for the case of the quasipotential (4.5) can be written in the form

$$\left[m^2\cosh^2\left(\frac{i}{m}\frac{\partial}{\partial r}\right) - \overset{\circ}{E}^2\right] r\tilde{\Psi}(r) = \tag{4.6}$$

$$= \frac{m}{\overset{\circ}{E}}\cdot\frac{g^2}{4\pi r}\cdot\tanh\left(\frac{\pi r m}{2}\right) m\cosh\left(\frac{i}{m}\frac{\partial}{\partial r}\right) r\tilde{\Psi}(r).$$

The transformations of the WF with the plane waves (3.1) take in the spherical-symmetrical case the form (here after $\overset{\circ}{p} = |\overset{\circ}{\vec{p}}|$)

$$\overset{\circ}{p}\,\tilde{\Psi}(\overset{\circ}{p}) = 4\pi\int_0^\infty \sin(mr\chi_{\overset{\circ}{p}})\, r\tilde{\Psi}(r)\,dr, \tag{4.7}$$

$$r\tilde{\Psi}(r) = \frac{4\pi}{(2\pi)^3}\int_0^\infty \sin(mr\chi_{\overset{\circ}{p}})\,\overset{\circ}{p}\,\tilde{\Psi}(\overset{\circ}{p})\, m\, d\chi_{\overset{\circ}{p}}, \tag{4.8}$$

where the variable $\chi_{\overset{\circ}{p}}$, called the rapidity, is defined by the parametrization

$$E_{\overset{\circ}{p}} = m\cosh(\chi_{\overset{\circ}{p}});\quad \overset{\circ}{\vec{p}} = m\sinh(\chi_{\overset{\circ}{p}})\,\vec{n}_{\overset{\circ}{p}};\quad \vec{n}_{\overset{\circ}{p}} = \frac{\overset{\circ}{\vec{p}}}{|\overset{\circ}{\vec{p}}|} \tag{4.9}$$

and analogously for other 4-momenta.

In equation (4.4) in the case $\tilde{\Psi}(\overset{\circ}{\vec{p}}) = \tilde{\Psi}(\overset{\circ}{p}) \equiv \tilde{\Psi}(\chi_{\overset{\circ}{p}})$ after the integration over spherical angles of the vector $\overset{\circ}{\vec{k}}$ one can obtain the one-dimensional equation, written in terms of the rapidities:

$$[m^2\cosh^2(\chi_{\overset{\circ}{p}}) - \overset{\circ}{E}^2]\sinh(\chi_{\overset{\circ}{p}})\tilde{\Psi}(\chi_{\overset{\circ}{p}}) = \frac{g^2}{(2\pi)^3}\cdot\frac{m^2}{\overset{\circ}{E}}\cdot$$

$$\times \int_0^\infty \ln\left|\frac{\coth^2\left(\frac{\chi_{\overset{\circ}{p}}-\chi_{\overset{\circ}{k}}}{2}\right)}{\coth^2\left(\frac{\chi_{\overset{\circ}{p}}+\chi_{\overset{\circ}{k}}}{2}\right)}\right| \sinh(\chi_{\overset{\circ}{k}})\tilde{\Psi}(\chi_{\overset{\circ}{k}})\, m\, d\chi_{\overset{\circ}{k}} . \tag{4.10}$$

Let us integrate formally in the right-hand side of (4.10) by parts. The equation then takes the form

$$(E^2_{\overset{\circ}{p}} - \overset{\circ}{E}{}^2)\tilde{\Psi}(\overset{\circ}{p}) = \frac{g^2}{4\pi^2}\cdot\frac{m^2}{\overset{\circ}{E}}\int_0^\infty \frac{d\overset{\circ}{k}}{\overset{\circ}{p}{}^2-\overset{\circ}{k}{}^2}\cdot\int_{\overset{\circ}{k}}^\infty \tilde{\Psi}(\overset{\circ}{k}{}')\,2\overset{\circ}{k}{}'\,d\overset{\circ}{k}{}'. \tag{4.11}$$

With the help of the free Green function

$$G_0(\overset{\circ}{p},\overset{\circ}{E}) = \frac{1}{E^2_{\overset{\circ}{p}} - \overset{\circ}{E}{}^2} = \frac{1}{m^2\cosh\chi_{\overset{\circ}{p}} - m^2\cos^2 x} \tag{4.12}$$

(here $\overset{\circ}{E} \equiv m\cos x$) equation (4.11) can be represented in the form

$$G_0^{-1}(\overset{\circ}{p},\overset{\circ}{E})\tilde{\Psi}(\overset{\circ}{p}) = $$

$$= \frac{g^2}{4\pi^2}\cdot\frac{m^2}{\overset{\circ}{E}}\int_0^\infty \frac{d\overset{\circ}{k}}{G_0^{-1}(\overset{\circ}{p},\overset{\circ}{E}) - G_0^{-1}(\overset{\circ}{k},\overset{\circ}{E})}\int_{\overset{\circ}{k}}^\infty \tilde{\Psi}(\overset{\circ}{k}{}')\,dG_0^{-1}(\overset{\circ}{k}{}',\overset{\circ}{E}). \tag{4.13}$$

Let us consider as the WF of the ground state the expression

$$\tilde{\Psi}(\overset{\circ}{p}) = G_0^2(\overset{\circ}{p},\overset{\circ}{E}) = \frac{1}{[\overset{\circ}{p}{}^2 + W(m - \frac{W}{4})]^2} =$$

$$= \frac{1}{(\overset{\circ}{p}{}^2 + m^2\sin^2 x)^2} . \tag{4.14}$$

With the help of the algebraic equality

$$[G_0^{-1}(\overset{\circ}{p},\overset{\circ}{E}) - G_0^{-1}(\overset{\circ}{k},\overset{\circ}{E})]^{-1} G_0(\overset{\circ}{k},\overset{\circ}{E}) =$$

$$= \{[G_0^{-1}(\overset{\circ}{p},\overset{\circ}{E}) - G_0^{-1}(\overset{\circ}{k},\overset{\circ}{E})]^{-1} + G_0(\overset{\circ}{k},\overset{\circ}{E})\}\, G_0(\overset{\circ}{p},\overset{\circ}{E}) \tag{4.15}$$

as well as with the use of the relation

$$\int_0^\infty [G_0^{-1}(\overset{\circ}{p},\overset{\circ}{E}) - G_0^{-1}(\overset{\circ}{k},\overset{\circ}{E})]^{-1}\, d\overset{\circ}{k} = 0 \tag{4.16}$$

we come to the conclusion that the equation (4.13) with the WF (4.14) is fulfilled if there holds the equality

$$1 = \frac{g^2}{4\pi^2} \cdot \frac{m^2}{\mathring{E}} \int_0^\infty G_0(\mathring{k}, \mathring{E})\, d\mathring{k} \,. \tag{4.17}$$

In the nonrelativistic limit this equation gives the quantization condition for the energy of the ground state. With the help of (4.12) this equation can be represented in the form

$$\frac{g^2}{4\pi} \cdot \frac{m^2}{2\mathring{E}\sqrt{m^2 - \mathring{E}^2}} = \frac{g^2}{4\pi \sin(2x)} = 1 \,. \tag{4.18}$$

By differentiating (4.15) with respect to $\mathring{E}^2$ one can obtain the equality

$$[G_0^{-1}(\mathring{p}, \mathring{E}) - G_0^{-1}(\mathring{k}, \mathring{E})]^{-1} G_0^m(\mathring{k}, \mathring{E}) =$$

$$= [G_0^{-1}(\mathring{p}, \mathring{E}) - G_0^{-1}(\mathring{k}, \mathring{E})]^{-1} G_0^m(\mathring{p}, \mathring{E}) + \tag{4.19}$$

$$+ \sum_{\ell=0}^{m-1} G_0^{\ell+1}(\mathring{k}, \mathring{E})\, G_0^{m-\ell}(\mathring{p}, \mathring{E}) .$$

Let us look for the WF of an n-th radial excitation of the ground state $\tilde{\Psi}^{(n)}(\mathring{p})$ as the polynomial of the $n+1$-th order of the free Green function

$$\tilde{\Psi}^{(n)}(\mathring{p}) = G_0^2(\mathring{p}, \mathring{E}) \sum_{\ell=1}^{n} \ell B_\ell^{(n)} [(m^2 - \mathring{E}^2) G_0(\mathring{p}, \mathring{E})]^{\ell-1} = \tag{4.20}$$

$$= G_0^2(\mathring{p}, \mathring{E}) \sum_{\ell=1}^{n} \ell B_\ell^{(n)} \left[\frac{m^2 \sin^2 x}{\mathring{p}^2 + m^2 \sin^2 x} \right]^{\ell-1},$$

where $B_\ell^{(n)}$ are unknown coefficients of the same dimension. From (4.13) with the help of (4.19) we find

$$\sum_{\ell=1}^{n} \ell B_\ell^{(n)} (m^2 - \mathring{E}^2)^{\ell-1} G_0^\ell(\mathring{p}, \mathring{E}) = \tag{4.21}$$

$$= \sum_{\ell=1}^{n} (m^2 - \mathring{E}^2)^{\ell-1} G_0^\ell(\mathring{p}, \mathring{E}) \sum_{j=\ell}^{n} B_j^{(n)} G_{j-\ell+1}^{(n)}(\mathring{E}),$$

where $F_{j+1}(\mathring{E})$ are defined in the following way

$$F_{j+1}^{(n)}(\mathring{E}) = \frac{g^2}{4\pi^2} \cdot \frac{m^2}{\mathring{E}} (m^2 - \mathring{E}^2)^j \int_0^\infty G_0^{j+1}(\mathring{k}, \mathring{E})\, d\mathring{k} \tag{4.22}$$

The equality (4.21) as well as the equation (4.13) are true if the coefficients $B_j^{(n)}$ obey the following system of equations:

$$n B_n^{(n)} = B_n^{(n)} F_1^{(n)}(\overset{\circ}{E}),$$

$$(n-1) B_{n-1}^{(n)} = B_{n-1}^{(n)} F_1^{(n)}(\overset{\circ}{E}) + B_n^{(n)} F_2^{(n)}(\overset{\circ}{E}),$$

$$(n-2) B_{n-2}^{(n)} = B_{n-2}^{(n)} F_1^{(n)}(\overset{\circ}{E}) + B_{n-1}^{(n)} F_2^{(n)}(\overset{\circ}{E}) + B_n^{(n)} F_3^{(n)}(\overset{\circ}{E}) \tag{4.23}$$

. . . .

$$B_1^{(n)} = B_1^{(n)} F_1^{(n)}(\overset{\circ}{E}) + B_2^{(n)} F_2^{(n)}(\overset{\circ}{E}) + \ldots + B_n^{(n)} F_n^{(n)}(\overset{\circ}{E}).$$

Starting with the definition (4.22) it is easy to show that the following formula

$$F_{j+1}^{(n)}(\overset{\circ}{E}) = \frac{(2j)\,!}{(j!)^2 4^j} F_1^{(n)}(\overset{\circ}{E}) \tag{4.24}$$

takes place. The first equation of the system (4.23) is nothing but the quantization condition

$$n = F_1^{(n)}(\overset{\circ}{E}) = \frac{g^2}{4\pi^2} \cdot \frac{m^2}{\overset{\circ}{E}} \int_0^\infty G_0(\overset{\circ}{k}, \overset{\circ}{E})\, d\overset{\circ}{k} \,. \tag{4.25}$$

From (4.25), denoting the energy of the n-th state by $2\overset{\circ}{E}_n = 2m\cos(x_n)$, we find

$$\frac{g^2 \cdot m^2}{4\pi \cdot 2\overset{\circ}{E}_n \sqrt{m^2 - \overset{\circ}{E}{}_n^2}} = \frac{g^2}{4\pi \sin(2x_n)} = n \,. \tag{4.26}$$

The binding energy of the n-th state W_n can be written in the form

$$W_n = 2m\left[1 - \left(\frac{1}{2} + \frac{1}{2}\sqrt{1 - \left(\frac{g^2}{4\pi n}\right)^2}\right)^{1/2}\right]. \tag{4.27}$$

It is clear that in the case of a small coupling constant g formula (4.27) gives in the first approximation the nonrelativistic coulombic energy levels.

After we have fixed the energy of the n-th state by the equation (4.26) we can define completely all the coefficients $F_{j+1}^{(n)}(\overset{\circ}{E}_n)$ of the system (4.23):

$$F_{j+1}^{(n)}(\overset{\circ}{E}_n) = \frac{(2j)\,!}{(j\,!)^2\, 4^j} \cdot n \,. \tag{4.28}$$

It is easy to see that from n-1 equations of the system (4.23) (except for the first one) the coefficients $B^{(n)}_{n-1}, B^{(n)}_{n-2}, \ldots, B^{(n)}_{1}$ can be expressed through the coefficient $B^{(n)}_{n}$ that cannot be defined from the system (4.23), i.e., from the equation (4.13) and can be fixed by an extra condition of the type of the normalization condition. Choosing $n B^{(n)}_{n} = (-1)^{n-1} C_n$ we shall represent the result for all other coefficients

$$\ell B_\ell = (-1)^{\ell-1} \frac{\Gamma(n+\ell)4^\ell}{\Gamma(2\ell)\Gamma(n+1-\ell)} C_n . \tag{4.29}$$

Thus, the wave function of the n-th state can be written in the form

$$\tilde{\Psi}^{(n)}(\overset{\circ}{p}) = C_n G_0^2(\overset{\circ}{p}, \overset{\circ}{E}_n) \times$$
$$\times \sum_{\ell=1}^{n} (-1)^{\ell-1} \frac{\Gamma(n+\ell)\, 4^\ell}{\Gamma(2\ell)\Gamma(n+1\ell)} [(m^2 - E_n^2) G_0 (\overset{\circ}{p}, E_n)]^{\ell-1}, \tag{4.30}$$

where $\overset{\circ}{E}_n$ is defined from the quantization condition (4.26).

It is easy now to define the WF in the relativistic configurational representation. For the WF of the ground state (4.14) with the help of the transformation (4.8) we easily find

$$r\tilde{\Psi}^{(1)}(r) = \tag{4.31}$$
$$= \frac{C_1}{4\pi m^2} \cdot \frac{(-1)}{\sin(2x)} \frac{d}{dx} \{ \frac{1}{\cos x} \frac{\sinh[(\frac{\pi}{2} - x) rm]}{\cosh[\frac{\pi r m}{2}]} \},$$

where after differentiating, the value of x is defined from formula (4.18), i.e., as $\frac{1}{2} \arcsin(\frac{g^2}{4\pi})$. Substituting WF (4.31) into the finite-difference equation (4.6), it is easy to show that (4.31) is really a solution of (4.6) if for the energy $2\overset{\circ}{E} = 2m \cos x$ the quantization condition (4.17) takes place. Analogously, we find in the relativistic configurational representation the WF of the n-th state

$$r\tilde{\Psi}^{(n)}(r) = \frac{1}{4\pi m^2 \sin^2 x} \sum_{\ell=1}^{n} \frac{B^{(n)}_\ell}{(\ell-1)!} \cdot (-\sin^2 x)^\ell \times \tag{4.32}$$
$$\times (\frac{d}{d \sin^2 x})^\ell \cdot \{ \frac{1}{\cos x} \cdot \frac{\sinh[(\frac{\pi}{2} - x) rm]}{\cosh[\frac{\pi r m}{2}]} \},$$

where after performing all the differentiations, the value of x is taken as $x = x_n = \frac{1}{2} \arcsin(\frac{g^2}{4\pi n})$ and the coefficient $B_\ell^{(n)}$ is defined by (4.29).

5. CONCLUSION

We have found the solutions of the relativistic Logunov-Tavkhelidze single-time two-particle equation. The quasipotential is taken in the form of a difference of two propagators of the one-boson exchange (see (3.10) and (4.3)). This quasipotential $\tilde{V} \sim \frac{1}{\Delta^2_{\overset{\circ}{p},\overset{\circ}{k}}}$ can be considered as the relativistic geometrical generalization of the Coulomb potential (4.2) in the sense of the substitution of the difference of two vectors $\vec{p} - \vec{k}$ in the Euclidean momentum space by the difference $\overset{\circ}{\vec{p}}(-)\vec{k}$ (3.5) of two vectors in the Lobachevsky momentum space. This type of the potential was used earlier in the relativistic quark model in paper [12].

The relativistic quantization condition for the energy levels of the composite system (4.24) as well as the wave functions in the momentum (4.30) and configuration (4.31) representations are found.

The questions of normalization of wave functions and applications for describing the models of relativistic two-particle systems will be considered in subsequent publications.

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