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COMPLEX-POTENTIAL DESCRIPTION OF THE DAMPED HARMONIC OSCILLATOR.

II. The One-Dimensional Case

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1. Introduction

In the first part of this paper [1] , we applied the pseudo-Hamiltonian approach [2] to the case of multidimensional damped harmonic oscillator. We set

$$H = -\frac{1}{2}\Delta + x.(A-iW)x \quad , \tag{1}$$

where A, W are strictly positive matrices. By Lie-Trotter formula, we found explicitly the continuous contractive semigroup $V_t = \exp(-iHt)$:

$$(V_t\varphi)(x) = \int_{\mathbb{R}^d} G_t(x,y)\, \varphi(y)\, dy \quad , \quad t>0 \tag{2}$$

for an arbitrary $\varphi \in L^2(\mathbb{R}^d)$, where

$$G_t(x,y) = (2\pi i)^{-d/2}(\det(\Omega^{-1}\sin\Omega t))^{-1/2}\exp\Big\{\frac{i}{2}\big[x.(\Omega\,\mathrm{ctg}\,\Omega t)x + y.(\Omega\,\mathrm{ctg}\,\Omega t)y\big] - iy.(\Omega\,\mathrm{cosec}\,\Omega t)x\Big\} \tag{3}$$

and

$$\Omega = -(A-iW)^{1/2} \quad . \tag{4}$$

In the present paper, we study properties of the above solution. For the sake of simplicity, the discussion is limited essentially to the one-dimensional case.

The first problem concerns the non-damped limit : we show that it gives correct Feynman propagator including the phase factor [3,4] ; thus we find in the present case an alternative and very natural way of finding Maslov correction. Further we shall discuss the classical limit. Let us notice that comparing to common practice [5-9] we did not obtain the pseudo-Hamiltonian (1) by some kind of quantization of the classical damped oscillator (CDO). According to our opinion, such an approach makes sense only if there is a reasonable similarity between the classical and quantum mechanisms of damping. In general, this is not the case ; thus there is no a priori reason why should the classical limit reproduce the exact behaviour of CDO. We shall illustrate it on an example: for our damped oscillator and special Gaussian wavepackets, the classical limit gives trajectories of CDO (with linear damping), but corresponding to changed initial conditions ; the difference vanishes in the weak-damping limit. Finally, we shall find the point spectrum of H , which is of the form of the undamped-oscillator spectrum rotated around the origin to the lower complex halfplane. The eigenvectors, however, are not longer orthogonal because H is not normal.

2. The non-damped limit and Maslov correction

It is known that Feynman's propagator formula for the non-damped harmonic oscillator must be corrected by jumps in phase at every half-period :

$$K_t(x,y) = K_t^F(x,y)M(t) \quad , \tag{5}$$

where

$$K_t^F(x,y) = (2\pi i)^{-1/2}\left(\frac{\omega}{|\sin \omega t|}\right)^{1/2} \exp\left\{\frac{i\omega}{2 \sin \omega t}\left[(x^2+ y^2)\cos \omega t - 2xy\right]\right\} \quad , \tag{5a}$$

$$M(t) = \exp\left\{-\frac{\pi i}{2} \text{ Ent } \frac{\omega t}{\pi}\right\} \tag{5b}$$

if $t \neq \frac{1}{2}k\tau$ (we assume $m = \hbar = 1$) and

$$K_t(x,y) = \exp\left\{-\frac{\pi i}{2}k\right\} \; \delta(x-(-1)^k y) \tag{6}$$

if $t = \frac{1}{2}k\tau$ (see [4] for further references). We shall show that Maslov correction (5b) emerges naturally in non-damped limit of the above results :

Proposition 1 : Let $d = 1$ and $\Omega = \omega - i\nu$ with ω, ν positive. Then, if $\omega t \neq k\pi$, $k = 0,1,2,\dots$, and $\varphi \in L^2(\mathbb{R})$ has a compact support, we have

$$\lim_{\nu \to 0+} (V_t\varphi)(x) = \int_{\mathbb{R}} K_t(x,y)\,\varphi(y)\,dy \quad . \tag{7}$$

On the other hand, it holds

$$\lim_{\nu \to 0+} (V_t\psi)(x) = \exp\left\{-\frac{\pi i}{2}\right\}\psi((-1)^k x) \tag{8}$$

for $t = k\pi/\omega$ and $\psi \in \mathcal{S}(\mathbb{R})$.

Proof : Let $\omega t = k\pi$ and consider (2),(3) with $d = 1$ and $\Omega =$ $= \omega - i\nu$. We denote

$$h_x(y) = \exp\left\{\frac{i\Omega}{2\sin\Omega t}\left(y^2\cos\Omega t - 2xy\right)\right\} \quad ,$$

then

$$|h_x(y)| = \exp\left\{\frac{\omega\nu t}{2|\sin\Omega t|^2}\left[(y^2\cos\omega t - 2xy\,\mathrm{ch}\,\nu t)\frac{\sin\omega t}{\omega t} - \right.\right.$$
$$\left.\left. - (y^2\,\mathrm{ch}\,\nu t - 2xy\cos\omega t)\frac{\mathrm{sh}\,\nu t}{\nu t}\right]\right\}$$

so that

$$|h_x(y)| \leq \exp\left\{\omega|y|(|y| + 2|x|)\,\mathrm{sh}\,\nu t\,\sin^{-2}\omega t\right\} \quad ,$$

and therefore the dominated convergence theorem can be applied if φ has a compact support. It implies

$$\lim_{\nu \to 0+} (V_t\varphi)(x) = \lim_{\nu \to 0+} \exp\left\{\frac{i}{2}\,g_\nu(t)\right\} \int_{\mathbb{R}} K_t^F(x,y)\,\varphi(y)\,dy \,, \tag{9}$$

where $g_\nu(t) = \arg(\Omega/\sin\Omega t)$, i.e.,

$$g_\nu(t) = \mathrm{arctg}(\mathrm{th}\,\nu t\ \mathrm{ctg}\,\omega t) - \mathrm{arctg}\frac{\nu}{\omega} - k\pi \tag{10a}$$

for $k\pi < \omega t < (k+1)\pi$. The term $-k\pi$ is chosen so that the rhs is continuous in the points $t = k\pi/\omega$ and tends to zero with $t \to 0+$ which certainly must be true for g_ν . It is easy to see that g_ν is decreasing ; its shape for three values of ν/ω is sketched on the Figure. For fixed t, (10a) gives

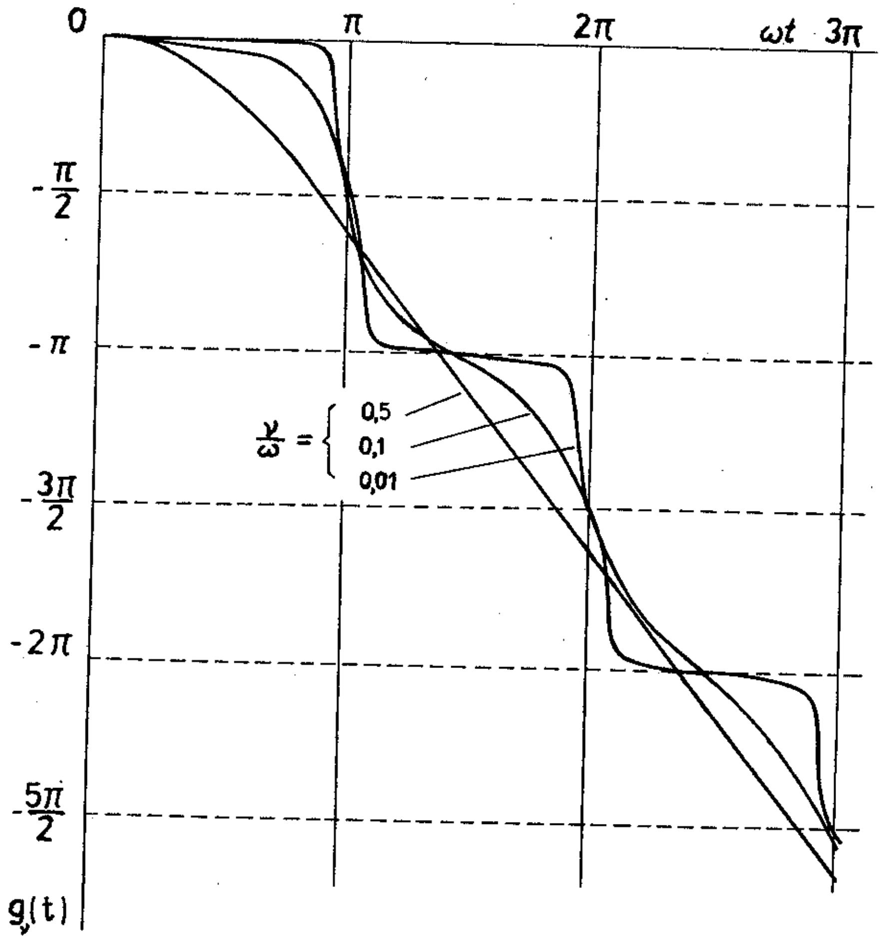

Fig. The function g_ν .

$$\lim_{\nu \to 0+} g_\nu(t) = -k\pi \quad \text{for} \quad k\pi < \omega t < (k+1)\pi \quad ; \tag{10b}$$

this relation together with (5b) and (9) gives (7).

Let now in turn $\omega t = k\pi$. We take $\psi \in \mathscr{S}(\mathbb{R})$ and express $(V_t\psi)(x)$ from Proposition 6 of [1] . Since

$$\left|\exp\left\{-\frac{y^2 t\ \mathrm{th}\ \nu t}{2(k\pi - i\nu t)} + \frac{i(-1)^k xy}{\mathrm{ch}\ \nu t}\right\}\right| \le 1 \quad ,$$

the dominated convergence theorem can be again applied which gives

$$\lim_{\nu\to 0+} (V_t\psi)(x) = (2\pi)^{-1/2}\exp\left\{-\frac{\pi i}{2}k\right\}\int_{\mathbb{R}} \exp\left\{i(-1)^k xy\right\}(F\psi)(y)\,dy \quad ,$$

where $F = F_1$ is the Fourier-Plancherel operator (so $F\psi \in \mathcal{S}(\mathbb{R})$). Using further $(F^2\psi)(x) = \psi(-x)$ for k odd, we arrive at (8).

∎

3. The classical limit

As is mentioned in the introduction,we limit ourselves to the case when the initial wave-packets are Gaussian, especially such obtained by shifting the "ground state". We take $\varphi = \varphi_{L,\alpha,\varkappa}$:

$$\varphi(x) = (\pi l^2)^{-1/4}\exp\left\{-(2L^2)^{-1}(x-\alpha)^2 + \frac{i}{\hbar}\varkappa x\right\} \tag{11a}$$

with L complex, $\mathrm{Re}\ L^2 \geq 0$, $l^{-2} = |L|^{-4}\mathrm{Re}\ L^2$, and $\alpha, \varkappa$ real. Expectations and dispersions of position and momentum are the following

$$\begin{aligned} &\langle Q\rangle_\varphi = \alpha \quad , \quad \langle P\rangle_\varphi = \varkappa \quad , \\ &(\Delta Q)_\varphi = 2^{-1/2}\, l \quad , \quad (\Delta P)_\varphi = 2^{-1/2}\hbar\, l\, |L|^{-2} \quad . \end{aligned} \tag{12}$$

The propagator referring to arbitrary m and $\hbar$ is obtained from (3) by substitutions $t \to \frac{\hbar t}{m}$, $\Omega \to \frac{m\Omega}{\hbar}$. Applying now Theorem 2 and Proposition 4 of [1] with this modification, we obtain

$$\begin{aligned} (V_t\varphi)(x) = {} & (\pi l^2)^{-1/4}(\cos\Omega t + i\Lambda^2 L^{-2}\sin\Omega t)^{-1/2}\exp\Big\{-i(2\Lambda^2)^{-1}\cdot \\ & \cdot\frac{\sin\Omega t - i\Lambda^2 L^{-2}\cos\Omega t}{\cos\Omega t + i\Lambda^2 L^{-2}\sin\Omega t}\Big[x^2 - 2xz\Lambda^2(\sin\Omega t - i\Lambda^2 L^{-2}\cos\Omega t)^{-1} + \\ & + \Lambda^4 z^2(\sin\Omega t - i\Lambda^2 L^{-2}\cos\Omega t)^{-1}\sin\Omega t\Big] - \frac{1}{2}\alpha^2 L^{-2}\Big\} \quad , \end{aligned} \tag{13a}$$

where $\Lambda^2 = \frac{\hbar}{m\Omega}$ and $z = \varkappa\hbar^{-1} - i\alpha L^{-2}$. Further we choose L as follows

$$L^2 = \Lambda^2 = \frac{\hbar}{m\Omega} \quad , \tag{11b}$$

and denote as above $\Omega = \omega - i\nu$, then (13a) can be simplified into the form

$$(V_t\varphi)(x) = (\pi\lambda^2)^{-1/4}\exp\Big\{-\frac{i}{2}\Omega t - \frac{1}{2}\Lambda^{-2}\Big[x-(\alpha+\frac{i}{\hbar}\varkappa\Lambda^2)e^{-i\Omega t}\Big]^2 + \frac{1}{2}\Lambda^{-2}(\alpha+\frac{i}{\hbar}\varkappa\Lambda^2)^2 e^{-i\Omega t}\cos\Omega t - \frac{1}{2}\alpha^2\Lambda^{-2}\Big\} , \tag{13b}$$

where $\lambda^2 = \frac{\hbar}{m\omega}$. The probability density is given by

$$|(V_t\varphi)(x)|^2 = (\pi\lambda^2)^{-1/2}\exp\Big\{-\nu t - \lambda^{-2}(x-x_0(t))^2 + y(t)\Big\} , \tag{14}$$

where

$$x_0(t) = \Big[\alpha\cos\omega t + (m\omega)^{-1}(\varkappa - m\alpha\nu)\sin\omega t\Big]\, e^{-\nu t} \tag{15}$$

and

$$y(t) = \frac{1}{2}\gamma^2\lambda^{-2} - \frac{1}{2}\lambda^{-2}\Big[(\beta^2-\gamma^2)\cos 2\omega t + \frac{\nu}{\omega}(\alpha^2-\gamma^2)\sin 2\omega t - \alpha^2 - \beta^2\Big]e^{-2\nu t}$$

with

$$\beta = (m\omega)^{-1}(\varkappa - m\alpha\nu) \quad , \quad \gamma^{-1} = m|\Omega|\varkappa^{-1} \quad .$$

Thus we have obtained the Gaussian-shaped function with the following properties :

(i) height of the peak decreases with time, for large t approximately as $e^{-\nu t}$,

(ii) its width λ does not change, it is negligible in the classical limit when $\alpha^2 + \beta^2 \gg \lambda^2$,

(iii) the peak travels along $x = x_0(t)$ which is the trajectory of the classical damped oscillator with the initial position $x_0(0) = \alpha$, however, the corresponding initial momentum is $m\dot{x}_0(0) = \varkappa - 2m\alpha\nu$ instead of $\varkappa$. Denoting $x_c(.)$ the trajectory of CDO with initial conditions $(\alpha,\varkappa)$, we have $x_c(t) - x_0(t) = 2\alpha\nu\omega^{-1}\, e^{-\nu t}\sin\omega t$ so that the difference is negligible in the case of weak damping, $\nu \ll \omega$.

4. The point spectrum of H

We put again $\hbar = m = 1$, then $\sigma_p(H)$ is of the following form :

Proposition 2 : Let $d = 1$, $\Omega = \omega - i\nu$, then $H\psi_n = \lambda_n \psi_n$ with

$$\psi_n(x) = N_{nn}^{-1/2} H_n(\sqrt{\Omega}\, x) \exp(-\tfrac{1}{2}\Omega x^2) \quad , \tag{16a}$$

$n = 0,1,2,\dots$, where H_n are Hermite polynomials, and

$$\lambda_n = \Omega(n + \tfrac{1}{2}) \quad . \tag{16b}$$

In general, the eigenvectors are not orthonormal: $(\psi_n, \psi_m) = N_{nn}^{-1/2} N_{mm}^{-1/2} N_{nm}$, where

$$N_{n,n+2s+1} = 0 \quad ,$$

$$N_{n,n+2s} = (\pi/\omega)^{1/2} \frac{n!\,(n+2s)!}{(n+s)!}\, \omega^{-(n+s)} |\Omega|^n \Omega^s \; .$$

$$\cdot \sum_{k=0}^{[\frac{n}{2}]} \sum_{l=0}^{[\frac{n}{2}]+s} (-1)^{k+l} \binom{2(n+s-k-l)}{n-2k} \binom{k+l}{k} \binom{n+s}{k+l} \omega^{k+l} (\bar{\Omega})^{-k} \Omega^{-l} \quad ,$$

with $s = 0,1,2,\dots$, and $[\cdot]$ denotes the entire part.

Proof : By straightforward computation. ∎

In conclusion, let us make some remarks. It is easy to see that $P = \{\psi_n\}_{n=0}^{\infty}$ is complete in $L^2(\mathbb{R})$ so that for $\lambda \neq \lambda_n$, $n = 0,1,2,\dots$, the set $(H - \lambda)P_{lin} = P_{lin}$ is dense and H has no residual spectrum. The problem of absence of continuous spectrum will be considered separately. Proposition 2 determines, of course, also $\sigma_p(H)$ for the multidimensional oscillator in the case when $\Omega^2 = 2(A - iW)$ can be diagonalized. Moreover, some results remain true even if A, W are not simultaneously diagonalizable. For instance, one can check easily that the "ground state" vector

$$\psi_0(x) = \pi^{-d/4} (\det(\operatorname{Re}\Omega))^{1/4} \exp(-\tfrac{1}{2}\, x.\Omega x)$$

corresponds to the eigenvalue $\frac{1}{2}\operatorname{Tr}\Omega$ for any A, W which obey assumptions of Theorems 1,2 of [1] ; notice that it is not a minimum-uncertainity state - cf.(12).

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