сообщения
объединенного
института
ядерных
исследований
дубна

E2-81-605

P.Exner, I.Úlehla

ON THE OPTICAL APPROXIMATION IN TWO-CHANNEL SYSTEMS

1981

1. Introduction

Complexity of many scattering problems in nuclear and particle physics often hinders us from obtaining exact solutions to them. At the same time, approximative solution is possible in some cases when the state Hilbert space $\mathcal{H}$ of the system can be expressed as an orthogonal sum $\mathcal{H}_1 \oplus \mathcal{H}_2$ of subspaces having a definite physical meaning. Familiar example is the optical model of neutron scattering on nuclei /1-4/, where the subspaces refer to direct and compound scattering of the neutron.

We shall study a two-channel system described by a Hamiltonian $H = H_0 + H_1 + \lambda V$ with $P_i H_i \subset H_i P_i$ and $P_i H_j = H_j P_i = 0$ for $i \neq j$, where P_i are projections on $\mathcal{H}_i$, $i = 0,1$; for the sake of simplicity we shall use the symbol H_i for $H_i \upharpoonright \mathcal{H}_i$ too. We denote $V_{ij} = P_i V P_j$, further it is useful to introduce $h_1 = H_{11} + \lambda V_{11}$ and $v = V - V_{11}$, then the considered Hamiltonian can be rewritten as

$$H = H_1 + h_1 + \lambda v \tag{1.1}$$

with $v_{11} = 0$. We addopt the following assumptions :

(S) the operators H_0, h_1 are self-adjoint, reduced by P_i and their parts in $\mathcal{H}_1, \mathcal{H}_0$, respectively, are zero,

(B) the operator λv is Hermitean, i.e.,bounded and self-adjoint.

Further we shall assume H_0 to be the free Hamiltonian of the first channel ($i = 0$), then the first-channel-part of the S-matrix is

$$S_{00} = \underset{t\to\infty}{\text{s-lim}}\, S_{00}^t \quad , \quad S_{00}^t = P_0 e^{iH_0 t} e^{-2iHt} e^{iH_0 t} P_0 \quad . \tag{1.2}$$

The optical approximation consists of choosing a suitable $V_{opt} \in \mathcal{B}(\mathcal{H}_0)$ such that

$$S_{opt} = \underset{t\to\infty}{\text{s-lim}}\, S_{opt}^t \quad , \quad S_{opt}^t = e^{iH_0 t} e^{-2iH_{opt} t} e^{iH_0 t} , \tag{1.3}$$

with $H_{opt} = H_0 + V_{opt}$, is in some sense near to S_{00}. In what follows, we shall be concerned mostly with the Feshbach-type optical potential[2] :

$$V_{opt}(E) = \lambda v_{00} - \lambda^2 \lim_{\varepsilon \to 0+} v_{01}(h_1 - E - i\varepsilon)^{-1} v_{10} \quad . \tag{1.4}$$

Recently Davies has found[5] a class of Hamiltonians (1.1) for which S_{opt} corresponding to the potential (1.4) approximates strongly S_{00}. His argument is based on Kato theory of smooth perturbations[6]. However the latter can be applied only if the spectrum of h_1 is continuous ; thus one has to exhibit conditions under which S is approximated weakly by S' referring to some h_1' with this property. The results of Ref.5 can be interpreted physically as follows : the optical approximation is justified by the existence of a short time scale (besides the standard one and a much longer one corresponding to half-life of the compound nucleus) which characterizes formation of the compound nucleus. In addition, the eigenvalues of h_1 should be numerous and close together.

However, in order to simplify the deductions, Davies assumes $E = 0$ and $v_{00} = 0$. Especially the last assumption is seriously limiting when realistic physical situations are considered. As an example, assume the simplest optical-model situation[2,7] when

$$\mathcal{H} = L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R}^{3A}) \quad , \tag{1.5}$$

$$H = -\Delta_n + h_N + \lambda V(x_n, x_N) \quad . \tag{1.6}$$

Let h_N possess an eigenvector $u_0 \in L^2(\mathbb{R}^{3A})$ referring to the eigenvalue $\mathcal{E}_0$ (ground state of the target nucleus) and choose

$$\mathcal{H}_0 = L^2(\mathbb{R}^3) \otimes \{u_0\}_{lin} \quad , \quad \text{i.e.} \quad P_0 = I_n \otimes P_{u_0} \quad , \tag{1.7}$$

then

$$H = (-\Delta_n + \mathcal{E}_0)P_0 + (-\Delta_n + h_N)P_1 + \lambda \sum_{i,j=0}^{1} v_{ij} \tag{1.8}$$

with

$$(V_{00}\psi)(x_n, x_N) = u_0(x_N)(u_0, V(x_n, .)u_0)_N (u_0, \psi(x_n, .))_N \quad , \tag{1.9}$$

$(.,.)_N$ being the inner product in $L^2(\mathbb{R}^{3A})$; thus V_{00} is non-zero for reasonable potentials V.

The aim of the present paper is to show that the results of Davies are essentially preserved if one gives up the mentioned assumptions*). This assertion is formulated and proved in Secs.5–8: the method is based again on the smooth-perturbation theory, but technically is more complicated. Before doing that, we discuss briefly two related problems : in Sec.2 we show how the optical potential (1.4) can be derived in a time-dependent way, while Sec.4 is devoted to the approximation of h_1 by h_1' with continuous spectrum. These considerations yield antagonistic requirements on the initial wavepackets which will be mentioned in concluding remarks.

2. The optical potential

Starting from the simplest case referring to (1.5)-(1.9), Feshbach derived the expression (1.4) by a formal time-independent way/2/. This procedure applies to more general situations too, when the selected subspace in the "target" state space is multidimensional, etc./2,3,7/. There are also other time-independent derivations of the optical potential/4/.

In the present section, we shall show how the optical potential (1.4) can be formally obtained within the time-dependent approach. We shall start from the relations

$$e^{-i(A+B)t} = e^{-iAt} - i\int_0^t e^{-i(A+B)s}\, B\, e^{-iA(t-s)}\, ds = \qquad (2.1)$$

$$= e^{-iAt} - i\int_0^t e^{-iA(t-s)}\, B\, e^{-i(A+B)s}\, ds \quad ,$$

which are valid (rigorously) for any self-adjoint A and Hermitean B. We apply these relations repeatedly to expand S_{00}^t and S_{opt}^t, further we pass to the limit $t \to \infty$. Comparing now the resulting expansions of S_{00} and S_{opt} up to second-order terms in v and V_{opt}, respectively, we obtain the following two candidates to the role of the optical potential :

---

*) As for the assumption $E = 0$, it can be always fulfilled by redefinition $H \to H' = H - E$; however, we let mostly E to be non-zero.

$$V_{opt}^{(+)} = \lambda v_{00} - i\lambda^2 \int_0^\infty e^{iH_0 t} v_{01} e^{-ih_1 t} v_{10}\, dt \quad , \tag{2.2a}$$

$$V_{opt}^{(-)} = \lambda v_{00} - i\lambda^2 \int_0^\infty v_{01} e^{-ih_1 t} v_{10} e^{iH_0 t}\, dt \quad . \tag{2.2b}$$

Up to now we have followed Davies[5] ; he rightly points out that besides the ambiguity in choosing the right one, the potentials (2.2) have a more serious defect that neither of them is dissipative : counterexamples can be found even for rank-one operators v_{01} .

Nevertheless, (2.2) can suggest the correct choice in the following way : suppose that the initial state of the system (for t large negative) is a wavepacket φ_E with narrow energy distribution concentrated around the mean E . It lies entirely in $\mathcal{H}_0$, and therefore one can replace $e^{iH_0 t}\varphi_E$ approximately by $e^{iEt}\varphi_E$ obtaining in this way

$$\begin{aligned} V_{opt}\varphi_E &= \lambda v_{00}\varphi_E - i\lambda^2 \int_0^\infty dt\ v_{01} e^{-ih_1 t} v_{10} e^{iEt}\varphi_E = \\ &= \lambda v_{00}\varphi_E - i\lambda^2 \lim_{\varepsilon\to 0+} \int_0^\infty dt\ v_{01} e^{-i(h_1 - E - i\varepsilon)} v_{10}\varphi_E = \\ &= v_{00}\varphi_E - i\lambda^2 \lim_{\varepsilon\to 0+} v_{01}(h_1 - E - i\varepsilon)^{-1} v_{10}\varphi_E \quad . \end{aligned}$$

Thus we have arrived just to Feshbach energy-dependent optical potential (1.4), which can be easily seen to obey the dissipativity condition.

3. Approximation of h_1

As is pointed out in the introduction, application of Kato theory in the following sections is conditioned among others by the fact that h_1 has a purely continuous spectrum. Usualy it is not so (cf., e.g., (1.8)), thus one has first to replace h_1 by h_1' with this property, which is near to h_1 , say, in the operator norm. The question is, whether the related S-matrices are in some sense near on $\mathcal{H}_0$. It can be answered positively if we restrict ourselves to a certain subset of states : let $\mathcal{L}$ be a dense subspace in $\mathcal{H}_0$ endowed by a norm $|.|$ such that

$$\int_R \| v\, e^{-iH_0 t}\varphi \| \,(1 + |t|^\alpha)\, dt \leq |\varphi| \quad , \quad \varphi \in \mathcal{L} \tag{3.1}$$

for some α , $0 < \alpha \leq 1$. The following assertion is valid :

Proposition 1 : Assume (S),(B) and $\|h_1 - h_1'\| < \varepsilon$, then for each $\varphi, \psi \in \mathcal{L}$ and $t \geq 0$ we have

$$|(\varphi, S_t\psi) - (\varphi, S_t'\psi)| \leq 2\lambda^2 \varepsilon^\alpha |\varphi||\psi| , \quad (3.2)$$

$$\|S_t\varphi - S_t'\varphi\| \leq 2\lambda\varepsilon^\alpha(1 + t^\alpha)|\varphi| . \quad (3.3)$$

The proof is essentially the same as in Ref.5 . The only difference caused by non-zero v_{00} is the presence of terms linear in λ in the expansion of $(\varphi, S_t\psi)$ and $(\varphi, S_t'\psi)$. However, these terms do not depend on h_1 and h_1' so they cancel in subtraction. Derivation of (3.3) does not employ the assumption $v_{00} = 0$.

We see that S' approximates S on $\mathcal{L}$ in the weak operator topology. Strong approximation is also possible but in finite time intervals only ; no essentially stronger inequalities can be expected to hold as discussed in Ref.5 .

4. Some notations

$v_{10} = C_1^*C_0$, $v_{01} = C_0^*C_1$, $v_{00} = B^*A$, where $A, B, C_i \in \mathcal{B}(\mathcal{H})$, $C_1 = P_0C_1P_1$ and the operators A, B, C_0 coincide with their parts in $\mathcal{H}_0$; moreover, $A = \sqrt{|v_{00}|}$, $B = AW^*$, where $W : \overline{\text{Ran } v_{00}} \to \overline{\text{Ran } v_{00}}$ is a partial isometry,

$\varphi_C^t(s) = \chi_{[-t,t]}(s)\, C_0\, e^{-iH_0t}\varphi$, $\varphi_B^t(s) = \chi_{[-t,t]}(s)\, B\, e^{-iH_0t}\varphi$ and similarly $\varphi_A^t(s)$ is defined for each $\varphi \in \mathcal{H}_0$, where χ_M is characteristic function of the set M ,

$x_{11}(t) = C_0\, e^{-iH_0t} C_0^*\, \theta(t)$, $x_{10}(t) = C_0\, e^{-iH_0t} B^*\theta(t)$,

$x_{01}(t) = A\, e^{-iH_0t} C_0^*\, \theta(t)$, $x_{00}(t) = A\, e^{-iH_0t} B^*\, \theta(t)$ and

$z_{11}(t) = C_1\, e^{-ih_1t} C_1^*\theta(t)$, where θ is Heaviside function,

$\|\cdot\|$, $(\cdot,\cdot)$ Hilbert-space norm and the corresponding inner product in $\mathcal{H}, \mathcal{H}_0$,

$\|\cdot\|_2$, $\langle\cdot,\cdot\rangle$ Hilbert-space norm and the corresponding inner product in $L^2(\mathbb{R};\mathcal{H}_0)$,

$\|\cdot\|_u$ operator norm in $\mathcal{B}(\mathcal{H}_0)$,

$\|\cdot\|_{uf}$ operator norm in $\mathcal{B}(L^2(\mathbb{R};\mathcal{H}_0))$

$\|f\|_1 = \int_{\mathbb{R}} \|f(x)\|_u\, dx$ and $\|f\|_\infty = \sup\text{ess}\{\|f(x)\|_u : x \in \mathbb{R}\}$ for a measurable operator-valued function $f : \mathbb{R} \to \mathcal{B}(\mathcal{H}_0)$,

$\hat{f}(.)$ is Fourier transform of f : $\hat{f}(x) = \int_{\mathbb{R}} e^{ixy} f(y)\, dy$,

F is the operator of "left multiplication" by $f : \mathbb{R} \to \mathcal{B}(\mathcal{H}_0)$ on $L^2(\mathbb{R};\mathcal{H}_0)$, $(F\psi)(x) = f(x)\psi(x)$, and similarly for $\hat{F}$; clearly $\|F\|_{uf} = \|f\|_\infty$,

$\tilde{F}$ is convolution of F , i.e., the operator on $L^2(\mathbb{R};\mathcal{H}_0)$ defined by $(\tilde{F}\psi)(x) = \int_{\mathbb{R}} f(y)\psi(x-y)\, dy$.

5. The main result

The central trick of Kato theory consists of factorizing the perturbation into product of a pair of operators which are smooth w.r.t. the unperturbed Hamiltonian. In the considered case, we choose suitable factorization using notation from the previous section and assume the quantities $\|x_{ij}\|_1$, $i,j = 0,1$, and $\|z_{11}\|_1$ to be finite. Since the operators C_0, C_1 are given up to a multiplicative constant, $C_1^* C_0 = (\alpha C_1)^* (\alpha^{-1} C_0)$ with a positive α , and similarly for λv , the last assumption can be in view of further purpose reformulated without loss of generality as follows*)

$$\text{(N)} \qquad \|x_{00}\|_1 = 1 \quad , \quad \max\{\|x_{10}\|_1, \|x_{01}\|_1 , \|x_{11}\|_1\} = 1 \quad ,$$
$$\|z_{11}\|_1 = \eta < \infty \quad .$$

With these prerequisites, we can formulate the above-mentioned assertion which shall be proved in the next three sections :

Theorem : Assume (S),(B) and (N), then for $|\lambda| < (2\sqrt{\eta_0})^{-1}$, $\eta_0 = \max\{1, \eta\}$ and an arbitrary $\psi \in \mathcal{H}_0$ the estimate

$$\| S^t_{00}\psi - S^t_{opt}(E)\psi\| \leq \tag{5.1}$$
$$\leq \lambda^2 \{\alpha(E) + \sqrt{2}\,(\beta(E) + 2\gamma(E))\|\psi\|\}(\sqrt{2}\,(1-2|\lambda|\sqrt{\eta_0}))^{-1}$$

---

*) Instead of the normalization (N), one can choose some other one. For instance, after fixing λ by $\|x_{00}\|_1 = 1$, one can choose C_0 so that some of the remaining $\|x_{ij}\|_1$'s would equal one. In such a case, however, the following considerations would contain three parameters and the estimation would be much more complicated.

holds where $S^t_{opt}(E)$ refer to the optical potential (1.4) and

$$\alpha(E) = \|(\tilde{Z}_{11} - \hat{Z}_{11}(E))\psi^t_C\|_2 \quad , \tag{5.2a}$$

$$\beta(E) = \|(\tilde{Z}_{11} - \hat{Z}_{11}(E))\tilde{X}_{11}\|_{uf} \quad , \tag{5.2b}$$

$$\gamma(E) = \|(\tilde{Z}_{11} - \hat{Z}_{11}(E))\tilde{X}_{10}\|_{uf} \quad . \tag{5.2c}$$

6. The basic estimates

Proposition 2 : If (N) is valid, then the vectors φ^t_C, φ^t_A, φ^t_B belong to $L^2(\mathbb{R};\mathcal{H}_0)$ and

$$\|\varphi^t_K\|_2^2 \leq 2\,\|\varphi\|^2 \quad , \quad K = C, A, B \tag{6.1}$$

for each $t > 0$ and all $\varphi \in \mathcal{H}_0$.

Proof : Consider first φ^t_C, we have

$$\|\varphi^t_C\|_2^2 \leq \int_{\mathbb{R}} \|C_0\, e^{-iH_0 s}\varphi\|^2\, ds \tag{*}$$

and according to Ref.6, the rhs is $\leq 2\pi\,\|C_0\|^2_{H_0}\,\|\varphi\|^2$; thus one has to show that C_0 is H_0-smooth, $\|C_0\|_{H_0} < \infty$. Kato derived six equivalent expressions of the norm $\|.\|_{H_0}$, among them

$$\|C_0\|^2_{H_0} = a_3 = \frac{1}{2\pi} \sup_{z,\psi} |((R(z)-R(\bar{z}))C_0^*\psi, C_0^*\psi)|$$

(Ref.6, Theorem 5.1), where $R(z) = (H - z)^{-1}$, $z \in \mathbb{C}\setminus\mathbb{R}$ and $\psi \in D(C_0^*)$, $\|\psi\| = 1$. Using

$$R(z) - R(\bar{z}) = i \int_{\mathbb{R}} \exp(i\xi s - \eta|s| - iH_0 s)\, ds \quad ,$$

where $z = \xi + i\eta$, $\eta > 0$, together with simple estimates, we get

$$2\pi\,\|C_0\|^2_{H_0} \leq \int_{\mathbb{R}} \|C_0\, e^{-iH_0 s}\, C_0^*\|_u\, ds \quad ; \tag{6.2}$$

further the integrand in (6.2) is an even function of s so (*) and (N) give (6.1). As for $K = A, B$, the partial isometry W is Hermitean and commutes with A because v_{00} is Hermitean, thus $A^* = B^* W$ so that

$$2\pi \| A \|^2_{H_0} \leq 2 \int_0^\infty \| A e^{-iH_0 s} A^* \|_u \, ds \leq 2 \| x_{00} \|_1 ,$$

and similarly for φ^t_B. ■

<u>Remark</u> : The inequality (6.1) differs from the one used in Ref.5 by factor 2 on the rhs. It cannot be avoided as shown by the following example : let $U_s = \exp(-iH_0 s)$ be translations in $L^2(\mathbb{R})$, $(U_s\varphi)(x) = \varphi(x+s)$ and let C be a one-dimensional projection containing unit vector ψ_0 in its range. Then (6.1) corresponds to the inequality

$$\int_{\mathbb{R}} |(\psi_0, U_s\varphi)|^2 \, ds \leq K \|\varphi\|^2 \int_{\mathbb{R}} |(\psi_0, U_s\psi_0)| \, ds$$

with $K = 1$. Choosing $\varphi = \varphi_\alpha = \alpha^{-1}\chi_{[0,\alpha^2]}$ for $\alpha \geq 1$ and $\psi_0 = \varphi_1$, we get

$$K \geq 1 - (3\alpha^2)^{-1}$$

so that $K = \frac{1}{2}$ is not possible and $K = 1$ is saturated for $\alpha \to \infty$. Consequently, one must add the factor $\sqrt{2}$ on appropriate places, especially in Theorem 3.1 of Ref.5 .

<u>Proposition 3</u> : If (N) is valid, then the operators $\widetilde{X}_{ij}$, $\widetilde{Z}_{11}$ and $\hat{Z}_{11}(E)$ are bounded :

$$\|\widetilde{X}_{ij}\|_{uf} \leq 1 \quad , \quad i,j = 0,1 \quad , \tag{6.3a}$$

$$\|\widetilde{Z}_{11}\|_{uf} \leq \eta \tag{6.3b}$$

and

$$\|\hat{Z}_{11}(E)\|_{uf} \leq \eta \quad , \quad \forall E \in \mathbb{R} \quad . \tag{6.3c}$$

<u>Proof</u> : Let us take $f : \mathbb{R} \to \mathcal{B}(\mathcal{H}_0)$ with $\|f\|_1 < \infty$ and $g \in L^2(\mathbb{R};\mathcal{H}_0)$. It holds

$$\|\widetilde{F}g\|^2_2 = \int_{\mathbb{R}} \Big\| \int_{\mathbb{R}} f(y)g(x-y) \, dy \Big\|^2 dx \leq$$
$$\leq \int_{\mathbb{R}} dy \, \|f(y)\|_u \int_{\mathbb{R}} dz \, \|f(z)\|_u \int_{\mathbb{R}} dx \, \|g(x-y)\| \, \|g(x-z)\|$$

so that Hölder inequality applied to the last integral gives $\|\widetilde{F}g\|^2_2 \leq \|f\|^2_1 \, \|g\|^2_2$, i.e.,

$$\|\widetilde{F}\|_{uf} \leq \|f\|_1 \quad , \tag{6.4}$$

which proves (6.3a,b). Further $\|\hat{Z}_{11}(E)\|_{uf} = \|\hat{z}_{11}(E)\|_u$ because $\hat{Z}_{11}(E)$ acts as a "constant operator" on $L^2(\mathbb{R};\mathcal{H}_0)$. Consequently, $\|\hat{Z}_{11}(E)\|_{uf} \leq \|z_{11}\|_1$, which proves (6.3c). ∎

7. Dyson expansions of S^t_{00} and S^t_{opt}

Using the notation introduced above, the optical potential (1.4) can be rewritten as follows

$$V_{opt}(E) = \lambda B^* A - i\lambda^2\, C_0 \hat{Z}_{11}(E) C_0^* \tag{7.1}$$

(one can perform the limit $\varepsilon \to 0+$ because $\|z_{11}\|_1 < \infty$). Both v and $V_{opt}(E)$ are bounded so S^t_{00} and S^t_{opt} may be expressed via Dyson expansion*). Consider first S^t_{00} and $\varphi, \psi \in \mathcal{H}_0$: it holds

$$(\varphi, S^t_{00}\psi) = (\varphi,\psi) + \sum_{k=1}^{\infty} (-i\lambda)^k \int_{-t}^{t} ds_1 \int_{-t}^{s_1} ds_2 \ldots \int_{-t}^{s_{k-1}} ds_k (\varphi, e^{iH_0 s_1} v$$

$$e^{-i(H_0+h_1)(s_1-s_2)} v \ldots v\, e^{-iH_0 s_k} \psi) \quad .$$

We substitute $v = v_{00} + v_{10} + v_{01}$ and use the factorization from Sec.4, then the last expression can be rewritten as follows

$$(\varphi, S^t_{00}\psi) = (\varphi,\psi) + \sum_{k=1}^{\infty} (-i\lambda)^k s^t_k \quad , \tag{7.2}$$

where

$$\begin{aligned}
s^t_1 &= \langle \varphi^t_B, \psi^t_A \rangle \quad , \\
s^t_2 &= \langle \varphi^t_B, \tilde{X}_{00}\psi^t_A \rangle + \langle \varphi^t_C, \tilde{Z}_{11}\psi^t_C \rangle \quad , \\
s^t_3 &= \langle \varphi^t_B, \tilde{X}_{00}\tilde{X}_{00}\psi^t_A \rangle + \langle \varphi^t_B, \tilde{X}_{01}\tilde{Z}_{11}\psi^t_C \rangle + \langle \varphi^t_C, \tilde{Z}_{11}\tilde{X}_{10}\psi^t_A \rangle \quad , \qquad (7.3) \\
s^t_4 &= \langle \varphi^t_B, \tilde{X}_{00}\tilde{X}_{00}\tilde{X}_{00}\psi^t_A \rangle + \langle \varphi^t_B, \tilde{X}_{00}\tilde{X}_{01}\tilde{Z}_{11}\psi^t_C \rangle + \langle \varphi^t_B, \tilde{X}_{01}\tilde{Z}_{11}\tilde{X}_{10}\psi^t_A \rangle + \\
&\quad + \langle \varphi^t_C, \tilde{Z}_{11}\tilde{X}_{10}\tilde{X}_{00}\psi^t_A \rangle + \langle \varphi^t_C, \tilde{Z}_{11}\tilde{X}_{11}\tilde{Z}_{11}\psi^t_C \rangle \quad ,
\end{aligned}$$

---

*) Cf.Ref.8 ; notice that in the following we deal with matrix elements of S^t_{00} and S^t_{opt} , i.e., with "weak" solutions to the evolution equations only. Notice further that the use of Dyson expansion is not conditioned by hermiticity of the interaction.

etc. In order to illustrate the structure of a general s_k^t , let us exhibit the allowed pairings :

$$\begin{aligned}
\text{containing}\quad & \tilde{Z}_{11} : \tilde{X}_{11}\tilde{Z}_{11} , \tilde{X}_{01}\tilde{Z}_{11} \; ; \; \tilde{Z}_{11}\tilde{X}_{11} , \tilde{Z}_{11}\tilde{X}_{10} , \\
& \tilde{X}_{11} : \tilde{Z}_{11}\tilde{X}_{11} \; ; \; \tilde{X}_{11}\tilde{Z}_{11} , \\
& \tilde{X}_{10} : \tilde{Z}_{11}\tilde{X}_{10} \; ; \; \tilde{X}_{10}\tilde{X}_{01} , \tilde{X}_{10}\tilde{X}_{00} , \\
& \tilde{X}_{01} : \tilde{X}_{10}\tilde{X}_{01} , \tilde{X}_{00}\tilde{X}_{01} \; ; \; \tilde{X}_{01}\tilde{Z}_{11} , \\
& \tilde{X}_{00} : \tilde{X}_{00}\tilde{X}_{00} , \tilde{X}_{10}\tilde{X}_{00} \; ; \; \tilde{X}_{00}\tilde{X}_{00} , \tilde{X}_{00}\tilde{X}_{01} .
\end{aligned} \tag{7.4}$$

We need to know the number n_k of terms contained in s_k^t . Notice that each operator product in s_k^t is due to (7.4) uniquely determined by positions of the operators $\tilde{Z}_{11}$; let c_{k-1}^m denote in how many ways m operators $\tilde{Z}_{11}$ can be distributed over $k-1$ places. According to (7.4), we have $c_{k-1}^m = c_{k-3}^{m-1} + c_{k-4}^{m-1} + \ldots$ so that $c_{k-1}^m = \binom{k-m}{m}$ for $0 \leq m \leq [k/2]$, where $[\cdot]$ denotes entire part. Consequently,

$$n_k = \sum_{m=0}^{[k/2]} \binom{k-m}{m} \quad . \tag{7.5}$$

Let us now pass to S_{opt}^t . Dyson expansion gives

$$(\varphi, S_{opt}^t\psi) = (\varphi,\psi) + \sum_{k=1}^{\infty} b_k^t \quad , \tag{7.6}$$

where

$$\begin{aligned}
b_1^t &= -i\lambda \langle \varphi_B^t, \psi_A^t \rangle + (-i\lambda)^2 \langle \varphi_C^t, \hat{Z}_{11}(E)\psi_C^t \rangle \; , \\
b_2^t &= (-i\lambda)^2 \langle \varphi_B^t, \tilde{X}_{00}\psi_A^t \rangle + (-i\lambda)^3 \langle \varphi_B^t, \tilde{X}_{01}\hat{Z}_{11}(E)\psi_C^t \rangle + \\
&\quad + (-i\lambda)^3 \langle \varphi_C^t, \hat{Z}_{11}(E)\tilde{X}_{10}\psi_A^t \rangle + (-i\lambda)^4 \langle \varphi_C^t, \hat{Z}_{11}(E)\tilde{X}_{11}\hat{Z}_{11}(E)\psi_C^t \rangle \; ,
\end{aligned} \tag{7.7}$$

etc. In order to make comparison of the expansions (7.2) and (7.6) possible, one has to rearrange the latter w.r.t. the powers of $-i\lambda$. It may be done if the series converges absolutely. Applying Propositions 2,3 , we get

$$|b_1^t| \leq |\lambda| \|\varphi_B^t\|_2 \|\psi_A^t\|_2 + |\lambda|^2 \|\varphi_C^t\|_2 \|\psi_C^t\|_2 \|\hat{Z}_{11}(E)\|_{uf} \leq 2\|\varphi\|\|\psi\|(|\lambda| + 2\eta\lambda^2),$$

$$|b_2^t| \leq 2\|\varphi\|\|\psi\|(|\lambda|^2 + 2|\lambda|^3\eta + |\lambda|^4\eta^2) \quad ,$$

....

$$|b_k^t| \leq 2\|\varphi\|\|\psi\||\lambda|^k \sum_{s=0}^{k} \binom{k}{s} (\eta|\lambda|)^s$$

so

$$\sum_{k=1}^{\infty} |b_k^t| \leq 2\|\varphi\|\|\psi\| \sum_{k=1}^{\infty} |\lambda|^k (1+\eta|\lambda|^k) \quad ,$$

and therefore the series converges absolutely for $|\lambda|(1+\eta|\lambda|) < 1$, i.e.,

$$|\lambda| < \frac{1}{2\eta}(\sqrt{1+4\eta} - 1) \quad . \tag{7.8}$$

It is easy to see that after the rearrangement, (7.6) differs from (7.2) just by replacement of all $\tilde{Z}_{11}$ by $\hat{Z}_{11}(E)$.

8. Comparison of the expansions

Now we can complete the proof comparing term-by-term the expansions (7.2) and (7.6). We denote

$$(\varphi,(S_{00}^t - S_{opt}^t)\psi) = \sum_{k=2}^{\infty}(-i\lambda)^k d_k^t \quad , \tag{8.1}$$

then Propositions 2,3 togethet with (5.2) give

$$\begin{aligned}
|d_2^t| &\leq \|\varphi_C^t\|_2\, \alpha(E) \leq \sqrt{2}\|\varphi\|\alpha(E) \quad , \\
|d_3^t| &\leq \|\varphi_B^t\|_2\|\tilde{X}_{01}\|_{uf}\alpha(E) + \|\varphi_C^t\|_2\|\psi_A^t\|_2\, \gamma(E) \leq \sqrt{2}\|\varphi\|\{\alpha(E) + \sqrt{2}\|\psi\|\gamma(E)\} \quad , \\
|d_4^t| &\leq \sqrt{2}\|\varphi\|\alpha(E) + 2\sqrt{2}\|\varphi\|\gamma(E)\sqrt{2}\|\psi\| + |\langle\varphi_C^t,(\tilde{Z}_{11}-\hat{Z}_{11}(E))\tilde{X}_{11}\tilde{Z}_{11}\psi_C^t\rangle| + \\
&\quad + |\langle\varphi_C^t,\hat{Z}_{11}(E)\tilde{X}_{11}(\tilde{Z}_{11}-\hat{Z}_{11}(E))\psi_C^t\rangle| \leq \\
&\quad \leq \sqrt{2}\|\varphi\|\{(1+\eta)\alpha(E) + \sqrt{2}\|\psi\|(2\gamma(E)+\eta\beta(E))\} ,
\end{aligned} \tag{8.2}$$

etc. In order to estimate general d_k^t , let us rewrite it as

$$d_k^t = \sum_{m=1}^{[k/2]} d_k^t(m) \quad ,$$

where $d_k^t(m)$ contains terms with just m operators $\tilde{Z}_{11}$ or $\hat{Z}_{11}(E)$. The number of terms in $d_k^t(m)$ is $mc_{k-1}^m = m\binom{k-m}{m}$ because similarly as above one must add and subtract $(m-1)c_{k-1}^m$ terms in order to single out the differences $\tilde{Z}_{11} - \hat{Z}_{11}(E) \equiv D_{11}$. They can be divided into three groups containing terms in which D_{11} is followed by ψ_C^t , $\tilde{X}_{11}$ and $\tilde{X}_{10}$; numbers of their elements are denoted $c_{k-1}^m(\alpha)$, $c_{k-1}^m(\beta)$ and $c_{k-1}^m(\gamma)$, respectively. It holds

$$c_{k-1}^m(\alpha) = \binom{k-m-1}{m-1} \quad , \quad c_{k-1}^m(\beta) = c_k^{m+1}(\gamma) = m\binom{k-m-1}{m} \tag{8.3}$$

as is shown in Appendix, and therefore (6.1), (6.3) together with (5.2) imply

$$|d_k^t(m)| \leqslant \sqrt{2}\|\varphi\|\eta^{m-1}\Big\{\binom{k-m-1}{m-1}\alpha(E) + + \sqrt{2}\|\psi\|\Big[(m-1)\binom{k-1-m}{m-1}\beta(E) + m\binom{k-1-m}{m}\gamma(E)\Big]\Big\} . \tag{8.4}$$

In order to make use of these inequality, one has to estimate

$$n_k(i) = \sum_{n=1}^{[k/2]} \eta^{m-1} c_{k-1}^m(i) \quad , \quad i = \alpha, \beta, \gamma. \tag{8.5}$$

Assume first $\eta \leqslant 1$. One obtains easily the relations

$$n_k(\alpha) = n_{k-1}(\alpha) + n_{k-2}(\alpha) \quad ,$$

$$n_k(\beta) = n_{k-2}(\alpha) + n_{k-2}(\beta) + n_{k-1}(\beta) \quad ,$$

$$n_k(\gamma) = n_{k+1}(\beta) \quad ,$$

which imply by induction $n_k(\alpha) \leqslant 2^{k-3}$ and $n_k(\beta) = n_{k-1}(\gamma) \leqslant \leqslant 2^{k-3}$. On the other hand, if $\eta \geqslant 1$, then

$$n_k(\alpha) = \eta^{[k/2]-1}\sum_{m=1}^{[k/2]}\binom{k-m-1}{m-1} \leqslant \eta^{-1}(\sqrt{\eta})^k\, 2^{k-3} \quad ,$$

and similarly for $n_k(\beta)$, $n_k(\gamma)$. Together we obtain

$$n_k(\alpha) \leqslant \eta_0^{-1}(\sqrt{\eta_0})^k\, 2^{k-3} \quad ,$$

$$n_k(\beta) \leqslant \eta_0^{-1}(\sqrt{\eta_0})^k\, 2^{k-3} \quad , \tag{8.6}$$

$$n_k(\gamma) \leqslant \eta_0^{-1}(\sqrt{\eta_0})^k\, 2^{k-2} \quad ,$$

where $\eta_0 = \max\{1, \eta\}$. The relations (8.1)-(8.6) imply

$$|(\varphi, (S_{00}^t - S_{opt}^t)\psi)| \leqslant \sum_{k=2}^{\infty} |\lambda|^k \sum_{m=1}^{[k/2]} |d_k^t(m)| \leqslant$$

$$\leqslant \sqrt{2}\|\varphi\|\sum_{k=2}^{\infty} |\lambda|^k \Big\{n_k(\alpha)\alpha(E) + \sqrt{2}\|\psi\|\sum_{i=\beta,\gamma} n_k(i)\, i(E)\Big\} \leqslant$$

$$\leqslant (\sqrt{2})^{-1}\|\varphi\|\,|\lambda|^2 \sum_{l=0}^{\infty} (2|\lambda|\sqrt{\eta_0})^l \Big\{\alpha(E) + \sqrt{2}\|\psi\|(\beta(E) + 2\gamma(E))\Big\} ,$$

i.e., the following inequality

$$|(\varphi,(S^t_{00}-S^t_{opt})\psi)| \leq$$
$$\leq \lambda^2 \|\varphi\| \{\alpha(E) + \sqrt{2}\|\psi\|(\beta(E)+2\gamma(E))\} \, (\sqrt{2}(1-2|\lambda|\sqrt{\eta_0}))^{-1} \quad (8.7)$$

for each $\varphi \in \mathcal{H}_0$, which is equivalent to (5.1). Finally, $(2\sqrt{\eta_0})^{-1} < (2\eta)^{-1}(\sqrt{1+4\eta}-1)$ so the condition (7.8) is fulfilled ; thus the proof is finished.

9. Concluding remarks

Comparing to the results of Ref.5, the estimate (5.2) contains the term proportional to $\gamma(E)$ and the parameter η_0 which are present due to non-zero v_{00} . The right-hand-sides of the two estimates depend on the coupling constant in a slightly different way because of different processes of estimation. What is more surprising, the substantially more complicated structure of Dyson expansions in the case of non-zero v_{00} (we have $\sum_{i=\alpha,\beta,\gamma} n_k(i) \leq 2^{k-1}$ terms in the k-th order of (8.1), in contrast to mere $\frac{1}{2}k$ (for even k only) if $v_{00}=0$) reduces to the combinatorial factor only which in effect multiplies the coupling constant by two in (5.1).

The physical interpretation mentioned in the introduction does not change with non-zero v_{00} . Assume for simplicity*) $E=0$, then the optical approximation can be intuitively expected from (5.1),(5.2) to be good if $z_{11}(.)$ is sharply peaked around zero (in a time scale appropriate for $\psi^t_0(.)$, $x_{11}(.)$ and $x_{10}(.)$). This assertion can be formulated rigorously as a direct generalization to Theorem 3.2 of Ref.5 . Let us notice that the appearance of three natural time scales is proper not only to the optical model : remember the well-known problem of deviations from the exponential decay law due to semiboundedness of the energy spectrum (cf. Refs.9,10 and references given in these papers). In the most models, there are two regions in which these deviations affect the decay law significantly : for times much larger/11-13/ and much smaller/14-18/ than the mean life of the considered system.

---

*) Cf. the footnote on page 3 .

The most problematic part of the discussed approximation lies in replacement of h_1 by h_1' with continuous spectrum*). As is shown in Sec.3, it needs the initial wavepackets φ to be such that $t \mapsto \| v \exp(-iH_0 t)\varphi \|$ decreases sufficiently rapidly with $|t|$; such a behaviour could be expected for φ with slowly varying energy density. On the other hand, considerations of Sec.2 show that the use of the energy-dependent optical potential (1.4) can be justified if the energy density is sharply peaked. The arguments leading to this contradiction are, of course, only roughly qualitative ; a more careful analysis and model examples are needed in order to decide whether and under which conditions the described scheme works.

Let us finally mention, that though (1.4) is "the best" optical potential in view of Sec.2, in practical calculations it is often replaced by some local one/1,4,7/, i.e., by operator of multiplication by a complex function. Such potentials can be handled more easily because one can adapt for them methods elaborated for real local potentials/19,20/ ; nevertheless, there is a lot of open problems related to them, especially in scattering theory.

Acknowledgement

The authors are grateful to Drs.J.Blank and M.Havlíček for useful discussions.

Appendix

Consider first $c^m_{k-1}(\alpha)$. In the corresponding terms, D_{11} must stand first from the right and its left neighbour is $\tilde{X}_{11}$ or $\tilde{X}_{01}$ due to (7.4). Thus we have $k-3$ places for arranging of $m-1$ operators Z_{11} (i.e., $\tilde{Z}_{11}$ or $\hat{Z}_{11}(E)$) , symbolically

$$\underset{k-3}{\overset{m-1}{\longmapsto}} \tilde{X} D_{11} \quad ,$$

which gives $c^m_{k-1}(\alpha) = c^{m-1}_{k-3} = \binom{k-m-1}{m-1}$. As for $c^m_{k-1}(\beta)$, we

*) The analogous problem concerns H_0 the spectrum of which must be continuous too if v_{00} is non-zero.

have the following diagrams

$$D_{11}\tilde{X}_{11}Z_{11}\tilde{X}\underset{k-5}{\overset{m-2}{\longmapsto}} ,$$

$$\underset{k-l-6}{\overset{m-s-2}{\longmapsto}}\tilde{X}D_{11}\tilde{X}_{11}Z_{11}\tilde{X}\underset{l}{\overset{s}{\longmapsto}} , \quad l=0,1,\ldots,k-6 ,$$

$$\underset{k-5}{\overset{m-2}{\longmapsto}}\tilde{X}D_{11}\tilde{X}_{11}Z_{11} ,$$

which give

$$c_{k-1}^{m}(\beta) = 2\binom{k-m-2}{m-2} + \sum_{l=0}^{k-6}\sum_{s=0}^{\left[\frac{l+1}{2}\right]}\binom{k-l-m+s-3}{m-s-2}\binom{l+1-s}{s} ,$$

i.e.,

$$c_{k-1}^{m}(\beta) = \sum_{r=0}^{k-4}\sum_{s=0}^{[r/2]}\binom{k-r-m+s-2}{m-s-2}\binom{r-s}{s} .$$

Analogously, the diagrams

$$D_{11}\tilde{X}_{10}\tilde{X}\underset{k-4}{\overset{m-1}{\longmapsto}} ,$$

$$\underset{k-l-5}{\overset{m-s-1}{\longmapsto}}\tilde{X}D_{11}\tilde{X}_{10}\tilde{X}\underset{l}{\overset{s}{\longmapsto}} , \quad l=0,1,\ldots,k-5$$

$$\underset{k-4}{\overset{m-1}{\longmapsto}}\tilde{X}D_{11}\tilde{X}_{10}$$

yield

$$c_{k-1}^{m}(\gamma) = \sum_{r=0}^{k-3}\sum_{s=0}^{[r/2]}\binom{k-r-m+s-2}{m-s-1}\binom{r-s}{s} = c_{k}^{m+1}(\beta) .$$

In view of the last relation, it is sufficient to evaluate for instance $c_{k-1}^{m}(\gamma)$. We change summation indices to $p=r-s$ and s ; it holds $0 \leq s \leq p \leq s+k-2m-1 \leq k-m-2$ and $p+s \leq$ $\leq k+2s-2m-1 \leq k-3$ so*)

$$c_{k-1}^{m}(\gamma) = \sum_{s=0}^{m-1}\sum_{p=s}^{s+k-2m-1}\binom{k-m-2-p}{m-1-s}\binom{p}{s} = \sum_{s=0}^{m-1}\binom{k-m-1}{m} ,$$

and therefore (8.3) is valid.

References

1 Feshbach,H., Porter,C.E. and Weisskopf,V.F., Phys.Rev.1954, v.96, pp.448-464.

2 Feshbach,H., Ann.Phys.1958, v.5, pp.357-390.

---

*) Cf. 4.2.5.39 in Ref.21 ; one has to correct an evident typographical error in this formula.

3 Feshbach,H., Ann.Phys.1962, v.19, pp.287-313.
4 Úlehla,I., Gomolčák,L. and Pluhař,Z.: Optical Model of the Atomic Nucleus, Czech.Acad.Sci.Publ., Prague 1964.
5 Davies,E.B., Ann.Inst.H.Poincaré 1978, v.A29, pp.395-413.
6 Kato,T., Math.Annalen 1966, v.162, pp.258-279.
7 Taylor,J.R.: Scattering Theory - The Quantum Theory of Non-relativistic Collisions, Wiley, New York 1972 ; § 19.4.
8 Reed,M. and Simon,B.: Methods of Modern Mathematical Physics, II.Fourier Analysis. Self-Adjointness, Academic Press, New York 1975 ; sec.X.11 .
9 Fonda,L., Ghirardi,G.C. and Rimini,A., Rep.Progr.Phys.1978, v.41, pp.587-631.
10 Exner,P., Rep.Math.Phys.1980, v.17, pp.87-97.
11 Khalfin,L.A., JETP 1957, v.33, pp.1371-1382.
12 Matthews,P.T. and Salam,A., Phys.Rev.1958, v.115, pp.1079-1084.
13 Höhler,G., Zs.Phys.1958, v.152, pp.542-565.
14 Beskow,A. and Nilsson,J., Arkiv for Fysik 1967, v.34, pp.561-569.
15 Havlíček,M. and Exner,P., Czech.J.Phys.1973, v.B23, pp.594-600.
16 Degasperis,A., Fonda,L. and Ghirardi,G.C., N.Cim.1974, v.21A, pp.471-484.
17 Misra,B. and Sinha,K., Helv.Phys.Acta 1977, v.50, pp.99-104.
18 Misra,B. and Sudarshan,E.C.G., J.Math.Phys.1977, v.18, pp.756-763.
19 Blank,J., Exner,P. and Havlíček,M., Czech.J.Phys.1979, v.B29, pp.1325-1341.
20 Exner,P. and Kolerov,G.I., Phys.Lett.1981, v.83A, pp.203-206.
21 Prudnikov,A.P., Brychkov,Yu.A. and Marichev,O.I.: Integrals and Series, Nauka, Moscow 1981.

Received by Publishing Department
on September 21 1981.