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A COMPLETE SET OF HIGHEST-WEIGHT REPRESENTATIONS FOR sl(3,C)

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1. Introduction

Construction of infinite-dimensional irreducible highest-weight representations (HWR's) of complex semisimple Lie algebras represents a problem interesting both mathematically and physically [1,2]. There exists, of course, the classification theorem [3], but the known infinite-dimensional HWR's act on certain factor spaces, and therefore are not very suitable for practical calculations (cf. [1-4] for a more complete discussion and bibliography).

The purpose of this note is to illustrate the method in which the representations are obtained from canonical (boson) or matrix canonical realizations of a given algebra, in particular those described in Ref.5. Using purely canonical realizations [6], we have constructed in Ref.2 the so-called maximal representations of $sl(n+1,\mathbb{C})$, further we have found the sufficient conditions under which they are irreducible HWR's. In this way, complete description of irreducible HWR's is obtained for $n = 1$. Starting from $sl(3,\mathbb{C}) \sim A_2$, a gap opens between irreducible finitedimensional and irreducible maximal representations. As for $sl(3,\mathbb{C})$, we have filled it partially in Ref.7 constructing two classes of the so-called mixed representations.

In the present paper, we resume briefly the results of Refs. 2,7 and construct one more class of mixed representations. They are irreducible HWR's of $sl(3,\mathbb{C})$ corresponding to the remaining weights (the set $\Omega(12)$ - see below); it means that our method yields irreducible HWR's of $sl(3,\mathbb{C})$ for all weights with exception of those to which irreducible finite-dimensional HWR's correspond. Furthermore, explicit form of these representations makes calculation of matrix elements of their generators straightforward. Finally, let us remark that in view of many common features of the mentioned canonical realizations [5] one can hope to perform analogous constructions for A_n, $n \geq 3$, as well as for other complex semisimple Lie algebras and their real forms.

2. Preliminaries

The standard $gl(3,\mathbb{C})$ generators e_{ij}, $i,j = 1,2,3$, obey

$$[e_{ij}, e_{kl}] = \delta_{kj} e_{il} - \delta_{il} e_{kj} \quad . \tag{1}$$

Separating the center, we get the simple subalgebra $sl(3,\mathbb{C})$ with dimension 8 and rank 2 . We shall use the following Cartan-Weyl basis

$$\begin{aligned} h_1 &= e_{22} - e_{11} \ , \quad h_2 = e_{33} - e_{22} \ , \\ e_1 &= e_{21} \ , \quad e_2 = e_{32} \ , \quad e_2 = e_{31} \ , \\ f_1 &= e_{12} \ , \quad f_2 = e_{23} \ , \quad f_3 = e_{13} \ . \end{aligned} \tag{2}$$

A representation $\rho : sl(3,\mathbb{C}) \to \mathcal{L}(V)$ is called representation with the highest weight $\Lambda = (\Lambda_1, \Lambda_2)$ if there exists $x_0 \in V$ such that

(i) $\rho(h_i)x_0 = \Lambda_i x_0$, $i = 1,2$,

(ii) $\rho(e_i)x_0 = 0$, $i = 1,2$,

(iii) x_0 is cyclic for ρ : $\rho(Usl(3,\mathbb{C}))x_0 = V$, where UL is the universal enveloping algebra of L .

To each Λ , there is, up to equivalence, one and only one irreducible representation of $sl(3,\mathbb{C})$ with the highest weight Λ [3]. In particular, an irreducible HWR is finite-dimensional iff both the Λ_1, Λ_2 belong to $\mathbb{N}_0 = \{0,1,2,\dots\}$.

We divide the set of all weights $\Lambda = (\Lambda_1, \Lambda_2)$ into five mutually disjoint subsets :

$$\Omega = \Omega_{fin} \cup \Omega(1) \cup \Omega(2) \cup \Omega(12) \cup \Omega_{max} \quad , \tag{3}$$

where

$$\Omega_{fin} = \Omega(1,2,12) = \{ \Lambda : \Lambda_i \in N_0 , i = 1,2 \} ,$$
$$\Omega(1) = \{ \Lambda : \Lambda_1 \in N_0 , \Lambda_2 \notin N_0 \} ,$$
$$\Omega(2) = \{ \Lambda : \Lambda_1 \notin N_0 , \Lambda_2 \in N_0 \} , \quad (4)$$
$$\Omega(12) = \{ \Lambda : \Lambda_i \notin N_0 , i = 1,2 , 1 + \Lambda_1 + \Lambda_2 \in N_0 \} ,$$
$$\Omega_{max} = \Omega(\emptyset) = \{ \Lambda : \Lambda_i \notin N_0 , i = 1,2 , 1 + \Lambda_1 + \Lambda_2 \notin N_0 \} .$$

As noticed above, the irreducible HWR´s corresponding to $\Lambda \in \Omega_{fin}$ are finite-dimensional; they are well-known (cf.,e.g.,[8], sec.10.1).

3. The sets Ω_{max} , $\Omega(1)$ and $\Omega(2)$

The canonical realizations of $L = sl(3,C)$ are homomorphisms of L into the Weyl algebra W_{2N} , which is the associative algebra with unity 1 generated by q_i , p_j , $i,j = 1,\dots,N$, obeying

$$[p_i,p_j] = [q_i,q_j] = 0 , \quad [p_i,q_j] = \delta_{ij} 1 , \quad (5)$$

or to the matrix Weyl algebra $W_{2N,M}$, which is the tensor product of W_{2N} with the associative algebra of $N \times N$ matrices. The realizations used below stem from the formulae [6] :

$$\tau(h_1) = -q_1p_1 + q_2p_2 + H_1 ,$$
$$\tau(h_2) = -q_1p_1 - 2q_2p_2 - \frac{1}{2} H_1 + \alpha 1 ,$$
$$\tau(e_{21}) = q_2p_1 + M_{21} , \quad \tau(e_{32}) = -p_2 , \quad \tau(e_{31}) = -p_1 ,$$
$$\tau(e_{12}) = q_1p_2 + M_{12} , \quad (6)$$
$$\tau(e_{23}) = q_2(q_1p_1 + q_2p_2 + \frac{1}{2} H_1 - \alpha) + q_1M_{21} ,$$
$$\tau(e_{13}) = q_1(q_1p_1 + q_2p_2 - \frac{1}{2} H_1 - \alpha) + q_2M_{12} ,$$

where $H_1 = 2M_{22}$, M_{12} , M_{21} are generators of a realization of $sl(2,C)$ commuting with q_1,p_1,q_2,p_2 and α is a complex parameter. Then we have either the purely canonical realizations $\tau^{\Lambda}_{max} : L \to W_6$ given by (6) with $\alpha = \Lambda_2 + \frac{1}{2}\Lambda_1$ and

$$H_1 = -2q_3p_3 + \Lambda_1 \mathbb{1} \ ,$$
$$M_{21} = -p_3 \ , \qquad (7)$$
$$M_{12} = q_3(q_3p_3 - \Lambda_1)$$

or the matrix canonical realizations $\tau^{\Lambda}_{mix} : L \to W_{4,2k+1}$, $k = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$, with (7) replaced by (2k+1)-dimensional irreducible matrix representations of $sl(2,\mathbb{C})$ and $\Lambda = (\Lambda_1, \Lambda_2) = (2k, \alpha - k)$.

Further we introduce the vector spaces V and U_k , $k = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$, spanned by

$$\left|\begin{smallmatrix} m & n \\ s & \end{smallmatrix}\right| \ , \quad m,n,s \in N_0 \ , \text{ and}$$
$$\left\|\begin{smallmatrix} m & n \\ s & \end{smallmatrix}\right\| \ , \quad m,n \in N_0 \ , \ s = -k, -k+1, \dots, k \ , \qquad (8)$$

respectively, and the annihilation and creation operators a_i , $\bar{a}_i$, $i = 1,2,3$, b_j , $\bar{b}_j$, $j = 1,2$, on them :

$$a_1 \left|\begin{smallmatrix} m & n \\ s & \end{smallmatrix}\right| = m^{1/2} \left|\begin{smallmatrix} m-1, & n \\ s & \end{smallmatrix}\right| \ , \quad \bar{a}_1 \left|\begin{smallmatrix} m & n \\ s & \end{smallmatrix}\right| = (m+1)^{1/2} \left|\begin{smallmatrix} m+1, & n \\ s & \end{smallmatrix}\right| \ ; \qquad (9)$$

$a_2 , \bar{a}_2$ and $a_3 , \bar{a}_3$ act analogously on the upper right and the lower indices of $\left|\begin{smallmatrix} m & n \\ s & \end{smallmatrix}\right|$, respectively, and similarly for b_j , $\bar{b}_j$ and the corresponding indices of $\left\|\begin{smallmatrix} m & n \\ s & \end{smallmatrix}\right\|$.

In order to get representations from the canonical realizations, one has to represent elements of the used Weyl algebra. Let us denote as π the representation $W_6 \to \mathcal{L}(V)$ generated by

$$\pi(q_i) = \bar{a}_i \ , \quad \pi(p_i) = a_i \ , \quad i = 1,2,3 \qquad (10)$$

and as ν_k the representation $W_{4,2k+1} \to \mathcal{L}(U_k)$ in which

$$\nu_k(q_j) = \bar{b}_j \ , \quad \nu_k(p_j) = b_j \ , \quad j = 1,2 \qquad (11)$$

and a matrix A is represented by the operator $\nu_k(A)$ the matrix representation of which on each subspace spanned by $\left\|\begin{matrix} m & n \\ s & \end{matrix}\right\|$, m,n fixed, $s=-k,-k+1,\dots,k$, is given just by A. In particular, we may write

$$\nu_k(M_{12})\left\|\begin{matrix} m & n \\ s & \end{matrix}\right\| = (k+s)\left\|\begin{matrix} m & n \\ s-1 & \end{matrix}\right\| ,$$
$$\nu_k(M_{21})\left\|\begin{matrix} m & n \\ s & \end{matrix}\right\| = (k-s)\left\|\begin{matrix} m & n \\ s+1 & \end{matrix}\right\| , \qquad (12)$$
$$\nu_k(H_1)\left\|\begin{matrix} m & n \\ s & \end{matrix}\right\| = 2s\left\|\begin{matrix} m & n \\ s & \end{matrix}\right\| .$$

The last mapping we need here is the automorphism φ of $sl(3,\mathbb{C})$ generated by

$$\varphi(h_1) = h_2 \ , \quad \varphi(h_2) = h_1 \ ,$$
$$\varphi(e_{21}) = e_{32} \ , \quad \varphi(e_{32}) = e_{21} \ , \quad \varphi(e_{31}) = -e_{13} \ , \qquad (13)$$
$$\varphi(e_{12}) = e_{23} \ , \quad \varphi(e_{23}) = e_{12} \ , \quad \varphi(e_{13}) = -e_{13} \ .$$

The following assertions were proved in Refs 2,7 :

Proposition 1 : (a) $\varrho_\Lambda = \pi \circ \tau^\Lambda_{max}$ given by (6),(7) and (10) is the irreducible representation of $sl(3,\mathbb{C})$ with the highest weight Λ iff $\Lambda \in \Omega_{max}$; in this case $x_0 = \left|\begin{matrix} 0 & 0 \\ 0 & \end{matrix}\right|$.

(b) If $\Lambda \in \Omega(1)$ with $\Lambda_1 = 2k$, then $\mu^{(1)}_\Lambda = \nu_k \circ \tau^\Lambda_{mix}$ given by (6),(11) and (12) is the irreducible representation of $sl(3,\mathbb{C})$ with the highest weight Λ and the highest-weight vector $x_0 = \left\|\begin{matrix} 0 & 0 \\ k & \end{matrix}\right\|$.

(c) Let $\Lambda \in \Omega(2)$ with $\Lambda_2 = 2k$ and denote $\Lambda' = (\Lambda_2,\Lambda_1)$ then $\mu^{(2)}_\Lambda = \mu^{(1)}_{\Lambda'} \circ \varphi = \nu_k \circ \tau^{\Lambda'}_{mix} \circ \varphi$ is the irreducible representation of $sl(3,\mathbb{C})$ with the highest weight Λ and the highest-weight vector $x_0 = \left|\begin{matrix} 0 & 0 \\ k & \end{matrix}\right|$.

4. The set $\Omega(12)$

In order to get the HWR's corresponding to $\Lambda \in \Omega(12)$, we change first the representation of the canonical pairs. We introduce $\sigma_k : W_{4,2k+1} \to \mathcal{L}(U_k)$ by

$$\sigma_k(q_1) = b_1 \quad , \quad \sigma_k(p_1) = -\bar{b}_1 \ ,$$
$$\sigma_k(q_2) = \bar{b}_2 \quad , \quad \sigma_k(p_2) = b_2 \quad ; \tag{14}$$

the "matrix part" of σ_k coincides with that of ν_k, i.e., $\sigma_k(A) = \nu_k(A)$. Further we denote by χ the following automorphism of $sl(3,\mathbb{C})$:

$$\chi(h_1) = -h_1 - h_2 \quad , \quad \chi(h_2) = h_2 \quad ,$$
$$\chi(e_{21}) = e_{13} \quad , \quad \chi(e_{32}) = e_{32} \quad , \quad \chi(e_{31}) = -e_{12} \quad , \tag{15}$$
$$\chi(e_{12}) = e_{31} \quad , \quad \chi(e_{23}) = e_{23} \quad , \quad \chi(e_{13}) = -e_{21} \quad .$$

If $\Lambda \in \Omega(12)$, then $\Lambda'' = (1 + \Lambda_1 + \Lambda_2, -\Lambda_1 - 2) \in \Omega(1)$ so that $\tau^{\Lambda''}_{mix}$ exists. For $1 + \Lambda_1 + \Lambda_2 = 2k \in \mathbb{N}_0$, we can define the representation $\mu^{(12)}_\Lambda = \sigma_k \circ \tau^{\Lambda''}_{mix} \circ \chi$ of $sl(3,\mathbb{C})$ the explicit form of which follows from (6),(12),(14) and (15) :

$$\mu^{(12)}_\Lambda(h_1) = -2\bar{b}_1 b_1 + \bar{b}_2 b_2 - \tfrac{1}{2} H^k_1 - \tfrac{1}{2}(\Lambda_2 - \Lambda_1 + 1)I \quad ,$$
$$\mu^{(12)}_\Lambda(h_2) = \bar{b}_1 b_1 - 2\bar{b}_2 b_2 - \tfrac{1}{2} H^k_1 + \tfrac{1}{2}(\Lambda_2 - \Lambda_1 - 1)I \quad ,$$
$$\mu^{(12)}_\Lambda(e_{21}) = (-\bar{b}_1 b_1 + \bar{b}_2 b_2 - \tfrac{1}{2} H^k_1 - \tfrac{1}{2}(\Lambda_2 - \Lambda_1 + 1))b_1 + b_2 M^k_{12} \ ,$$
$$\mu^{(12)}_\Lambda(e_{32}) = -b_2 \quad ,$$
$$\mu^{(12)}_\Lambda(e_{31}) = -b_1 b_2 - M^k_{12} \quad , \tag{16}$$
$$\mu^{(12)}_\Lambda(e_{12}) = \bar{b}_1 \quad ,$$
$$\mu^{(12)}_\Lambda(e_{23}) = \bar{b}_2(-\bar{b}_1 b_1 + \bar{b}_2 b_2 + \tfrac{1}{2} H^k_1 - \tfrac{1}{2}(\Lambda_2 - \Lambda_1 - 1)) + b_1 M^k_{21} \ ,$$
$$\mu^{(12)}_\Lambda(e_{13}) = \bar{b}_1 \bar{b}_2 - M^k_{21} \quad ,$$

where $H^k_1 = \sigma_k(H_1)$, $M^k_{ij} = \sigma_k(M_{ij})$. The following assertion holds :

<u>Proposition 2</u> : Let $\Lambda \in \Omega(12)$, then $\mu^{(12)}_\Lambda$ given by (16) is the irreducible representation of $sl(3,\mathbb{C})$ with the highest weight Λ and the highest-weight vector $x_0 = \left\| \begin{matrix} 0 & 0 \\ -k & \end{matrix} \right\|$.

Proof : Let us denote $E_{ij} = \mu_\Lambda^{(12)}(e_{ij})$ and $H_i = \mu_\Lambda^{(12)}(h_i)$. It is easy to check that x_0 obeys the conditions (i) and (ii). Let further Z be a non-trivial invariant subspace of $\mu_\Lambda^{(12)}$, then there exists at least one non-zero vector $y = \sum_{mns} a_{mns} \left\| \begin{smallmatrix} m & n \\ s & \end{smallmatrix} \right\|$ which belongs to Z . We abbreviate

$$\bar{n} = \max \{ n : a_{mns} \neq 0 \} \ ,$$

$$\bar{s} = \max \{ s : a_{m\bar{n}s} \neq 0 \} \ ,$$

$$\bar{m} = \max \{ m : a_{m\bar{n}\bar{s}} \neq 0 \} \ .$$

Since $(E_{21})^{\bar{m}}(E_{31})^{\bar{s}+k}(E_{32})^{\bar{n}} y = c\, a_{\bar{m}\bar{n}\bar{s}} (\Lambda_1 - \bar{m} + 1)(\Lambda_1 - \bar{m} + 2) \ldots$ $\ldots (\Lambda_1 - 1)\Lambda_1 x_0$ with a non-zero c and $a_{\bar{m}\bar{n}\bar{s}}$, we see that the assumption $\Lambda_1 \notin N_0$ implies $x_0 \in Z$. Applying further $(E_{12})^m (E_{23})^n$ to x_0 and using $\Lambda_2 \notin N_0$ we obtain $\left\| \begin{smallmatrix} m & n \\ -k & \end{smallmatrix} \right\| \in Z$ for all $m,n \in N_0$. Finally, the relation

$$E_{13} \left\| \begin{smallmatrix} m & n \\ n & \end{smallmatrix} \right\| = ((m+1)(n+1))^{1/2} \left\| \begin{smallmatrix} m+1, & n+1 \\ s & \end{smallmatrix} \right\| - (k-s) \left\| \begin{smallmatrix} m & n \\ s+1 & \end{smallmatrix} \right\|$$

shows that each $\left\| \begin{smallmatrix} m & n \\ s & \end{smallmatrix} \right\|$ belongs to Z . Thus $Z = U_k$ and $\mu_\Lambda^{(12)}$ is irreducible ; at the same time we have verified the condition (iii) of the definition. ∎

5. Concluding remarks

If $\Lambda \in \Omega(1)$, then the mixed representations are obtained directly from the matrix canonical realizations of $sl(3,\mathbb{C})$. For $\Lambda \in \Omega(2)$, we have used the fact that if μ is a representation and $\varphi \in \mathrm{Aut}\, sl(3,\mathbb{C})$, then $\mu \circ \varphi$ is again a representation, in general non-equivalent to μ . We believe that in the same way one can obtain mixed representations of $sl(n+1,\mathbb{C})$, for which up to now the maximal representations were constructed only.

In the case of $\Lambda \in \Omega(12)$, we have used another automorphism of $sl(3,\mathbb{C})$ in combination with an alternative representation of the canonical pairs ; the latter is itself composition of ν_k with an automorphism of $W_{4,2k+1}$. Some generalizations to higher n seem to be possible too, but the problem is much less transparent here.

Let us finally notice that our method yields irreducible HWR's of $sl(3,\mathbb{C})$ for $\Lambda \in \Omega(1), \Omega(2)$ and $\Omega(12)$, while the standard HWR's (elementary representations or Verma modules - cf. [1]) are reducible for these weights.

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