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NONLINEAR SCHRÖDINGER EQUATION WITH U(p,q) ISOTOPICAL GROUP.

II. The U(1,1) System

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This work continues the previous one[1],-therefore using formulae of the latter we put the number I in front of them,- nevertheless we shall try to present the results in a way that it may be read independently.

The equation under consideration is

$$i\Psi_t + \Psi_{xx} + 2(\overline{\Psi}\Psi)\Psi = 0, \qquad \text{(I)}$$

where $(\Psi)_a = \Psi^{(a)}(x,t)$ $(a=1,\dots,n)$ and

$$(\overline{\Psi}\Psi) = \sum_{a=1}^{p} |\Psi^{(a)}|^2 - \sum_{a=p+1}^{n} |\Psi^{(a)}|^2 \qquad (p+q=n). \qquad \text{(II)}$$

This system is quite complicated since a large set of boundary conditions are possible for it. We reduce the problem to simpler one but conserving in main features essential properties that stands it out of previously discussed. Let us for that look at the plane wave solutions to system (I) ("condensate" solutions):

$$\Psi^{(a)} = S^{(a)} \exp\left[-i(\omega_a t - \kappa_a x)\right]$$

with $\omega_a = \kappa_a^2 - (\bar{S}S)$. Following to ref.[2] one can study the stability of these solutions under infinitesimal perturbations and obtain as a result the dispersion relation between the perturbation frequency Ω and wave number K

$$\Omega^2 = \kappa^2\left[\kappa^2 - 2(\bar{S}S)\right].$$

It implies the condensate to be stable when $(\bar{S}S) \leqslant 0$, i.e., the isotopic space metric is negative $\sum_{a=1}^{p}|S^{(a)}|^2 \leqslant \sum_{a=p+1}^{n}|S^{(a)}|^2$.
Therefore solutions with <u>nontrivial</u> boundary conditions exist (if at all) only in the case of either noncompact group U(p,q) or compact U(o,q). For example in both models U(2,0) studied by MANAKOV[3] and U(1,1) (see ref.[4]) there exists solution

$$\Psi_1 = s^{(1)} \operatorname{sech} \varkappa x \quad ,$$
$$\Psi_2 = s^{(2)} \tanh \varkappa x \quad , \quad \varkappa^2 = (\bar{s} s) \tag{III}$$

but in the first case it is unstable due to instability of the condensate (vacuum) which this solution was constructed on. In the models U(1,1) and U(0,2) solution (II) should be stable. In general, nontrivial boundary conditions mean that we have a problem of the interaction of infinite number of particles that leads to renormalization of physical quantities and the most adequate language for treating it is the language of noncompact groups all which unitary representations are infinite-dimensional. It is this why we have a rich spectrum of stable particle-like excitations even in the framework of the simplest pseudo-unitary group U(1,1).

The plan of the paper is as follows: after a short review on U(1,0) and U(0,1) NLS equations the properties of soliton solutions of system U(1,1) are discussed and their possible physical interpretation is given *.

In Appendix the set of integral equations of the inverse method for the U(p,q) system is given to conclude the general analysis of part I, and the one soliton solution is obtained using them. It coincides with one derived by isotopic rotation of U(1,0) one soliton solution.

1. U(1,0) AND U(0,1) EQUATIONS

$$i\Psi_t + \Psi_{xx} + 2\varkappa |\Psi|^2 \Psi = 0 \; . \tag{1}$$

We recall briefly the properties of solutions to these equations (see also ref. /4/ and works cited there).

a) In the first case U(1,0) system or $\varkappa > 0$, the Cauchy problem may be completely solved for the functions $\Psi(x)$ rapidly vanishing at both infinities. A number of solitons and waveground occur as a result of an initial wave packet breaking down. The first are related to the discrete spectrum of the linear problem operator L, the second are related to the continuous one. Such solutions may be at the quantum and quasiclassical levels thought as follows: Solitons are the bound states of N_s "particles" of mass equal $\frac{1}{2}$ which energy (spectrum)

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* These results have partly been published in ref. /2/.

assumes the form:

$$E_s = \frac{P_s^2}{N_s} - \frac{\varkappa^2}{12} N_s^3 \quad , \qquad (N_s \gg 1) \tag{2}$$

$$P_s = \frac{1}{2} v_s N_s \quad , \quad M_s = \frac{N_s}{2} \quad .$$

The second term in the first relation is the binding energy. These formulae may be easily obtained via the Bohr-Sommerfeld quantization. Note also that one can rewrite the relation (2) in differential form

$$dE = \mu \, dN_s \tag{3}$$

with $\mu = \frac{1}{4}(v_s^2 - \varkappa^2 N_s^2)$ being the frequency of the soliton solution

$$\Psi_s = \frac{a}{\sqrt{\varkappa}} \operatorname{sech} a(x - v_s t - x_0) \exp\left[i\left(\frac{v_s}{2} x - \omega t\right)\right], \; a = \frac{\varkappa}{2} N_s . \tag{4}$$

Continuous spectrum of the operator L (wave background) corresponds to "particles" of masses $\frac{1}{2}$ and dispersion relation $E=p^2$. Then the integrals of particle number, momentum and energy look as follows

$$\begin{aligned} N &= \int_{-\infty}^{\infty} \psi^* \psi \, dx = \int_{-\infty}^{\infty} n(p) \, dp + \sum_{S=1} N_s \quad , \\ P &= \frac{1}{2i} \int_{-\infty}^{\infty} (\psi^* \psi_x - \psi_x^* \psi) \, dx = \int_{-\infty}^{\infty} p \, n(p) \, dp + \sum_{S=1} \frac{1}{2} v_s N_s \quad , \\ E &= \int_{-\infty}^{\infty} (|\psi_x|^2 - \varkappa |\psi|^4) \, dx = \int_{-\infty}^{\infty} p^2 n(p) dp + \sum_{S=1} \frac{1}{12} (3 v_s^2 N_s - \varkappa^2 N_s^3). \end{aligned} \tag{5}$$

By adding into the right hand side (2) the term $\frac{\varkappa^2 N_s}{12}$ one obtains binding energy

$$E_b = -\frac{\varkappa^2}{12} (N_s^3 - N_s)$$

coinciding exactly with that found in ref./5/ where the quantum N_s Bose-particle problem was solved.

b) In the case of repulsive potential (U(0,1) - system or $\varkappa < 0$) treating becomes considerably complicated. The operator L is now self-conjugate and does not possess a discrete spectrum under trivial boundary conditions at infinities and solutions behave as $\psi \propto t^{-\frac{1}{2}}$ when $/t/ \to \infty$.

Non-vanishing boundary conditions correspond to a Bose-gas of finite density ρ, which is defined by the asymptotic value of field, i.e., $|\psi|^2 \to \rho$ at $|x| \to \infty$. To eliminate time-dependence of the field phase a chemical potential μ used to be inserted into the equation of motion, or proceed to $H' \to H - \mu N$ and $\mu = 2\varkappa\rho$. The interaction term then assumes the form $2\varkappa(|\psi|^2 - \rho)\psi$ *. The integrals of particle number, momentum and energy should be in this case renormalized as follows

$$\bar{N} = \int_{-\infty}^{\infty} (|\psi|^2 - \rho)\, dx ,$$

$$P = \frac{1}{2i} \int_{-\infty}^{\infty} (\psi^* \psi_x - \psi_x^* \psi)\, dx + \rho\alpha , \qquad (6)$$

$$E = \int_{-\infty}^{\infty} \left[|\psi_x|^2 + \varkappa\left(|\psi|^2 - \rho\right)^2 \right] dx$$

with α being the phase shift: $\psi(\infty) = \sqrt{\rho}$, $\psi(-\infty) = \sqrt{\rho}\, e^{i\alpha}$. We notice that at the quantum level such a renormalization appears naturally due to commutation procedure at constructing a physical vacuum state.

In this case the operator L possesses again both continuous and discrete spectra. The continuous one corresponding to the Bogolubov spectrum of excitation has the form

$$\omega_1(p) = |p|\left(p^2 + 4\varkappa\rho\right)^{1/2} , \qquad (7)$$

$$p\,(k) = 2\sqrt{k^2 - \varkappa\rho} , \qquad |k| > \sqrt{\varkappa\rho}$$

and may be obtained via the perturbation theory. The discrete spectrum corresponds to the so-called "hole" excitation mode which was first found and examined at the quantum level by Lieb/6/. In the classical case we have (see ref./4/)

$$\bar{N}_h = -\frac{2}{\varkappa}\sqrt{\rho\varkappa - \frac{v^2}{4}} ,$$

$$P_h = \frac{2}{\varkappa}\left(\varkappa\rho \cos^{-1}\frac{v}{2\sqrt{\varkappa\rho}} - \frac{v}{2}\sqrt{\varkappa\rho - \frac{v^2}{4}}\right) , \qquad (8)$$

$$E_h = \frac{8}{3\varkappa}\left(\varkappa\rho - \frac{v^2}{4}\right)^{3/2} = -\frac{1}{3}\varkappa^2 \bar{N}_h^{\,3} ,$$

$$\alpha = 2\cos^{-1}\frac{v}{2\sqrt{\varkappa\rho}} ,$$

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* We should stress here that the transformation $\psi \to \psi\, e^{-i\mu t}$ relates transformed equation and equation (1) as well as their solutions.

where v is the velocity of soliton-bubble. Upon calculating integrals (6) the spectrum (8) can be easily verified to occur for the following soliton solutions of the kink type

$$\Psi_h = \sqrt{\varkappa\rho - \frac{v^2}{4}}\, \tanh\left[\sqrt{\varkappa\rho - \frac{v^2}{4}}\,(x - vt - x_0)\right] + i\,\frac{v}{2} \quad . \tag{9}$$

Classical spectra (7) and (8) go over into corresponding quantum ones at $\varkappa \to 0$ /4/. A characteristic feature of the hole excitations is the velocity dependence of the number of elementary holes $N_h = -\bar{N}$ bounded in the soliton-bubble: the greater v the less number of holes are bounded in the bubble and at $v \to 2\sqrt{\varkappa\rho}$ the soliton mode (bubbles) disappears (we have analogously $dE/_{dv} < 0$). Subsequently we will call dispersion a dependence of inverse soliton size on v and N_h and spectrum a $E(N_h, v)$ dependence.

2. U(1,1) SYSTEM

We have now the following set of equations *

$$\begin{aligned} i\Psi_t^{(1)} + \Psi_{xx}^{(1)} + 2\left(|\Psi^{(1)}|^2 - |\Psi^{(2)}|^2\right)\Psi^{(1)} = 0 \\ i\Psi_t^{(2)} + \Psi_{xx}^{(2)} + 2\left(|\Psi^{(1)}|^2 - |\Psi^{(2)}|^2\right)\Psi^{(2)} = 0 \end{aligned} \tag{10}$$

Hamiltonian density

$$\mathcal{H} = |\Psi_x^{(1)}|^2 - |\Psi_x^{(2)}|^2 - \left(|\Psi^{(1)}|^2 - |\Psi^{(2)}|^2\right) \tag{11}$$

and the integrals of particle number, momentum and energy

$$\begin{aligned} N_i &= \int_{-\infty}^{\infty} |\Psi^{(i)}|^2 dx \quad , \\ P &= \frac{i}{2}\int_{-\infty}^{\infty}\left(\Psi_x^{(1)*}\Psi^{(1)} - \Psi^{(1)*}\Psi_x^{(1)} - \Psi_x^{(2)*}\Psi^{(2)} + \Psi^{(2)*}\Psi_x^{(2)}\right)dx \, , \\ E &\stackrel{def}{=} H = \int_{-\infty}^{\infty} \mathcal{H}(x,t)\,dx \quad . \end{aligned} \tag{12}$$

In the long wave continuous approximation the Hubbard model may be reduced to a system of type (10) /8/.

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* We were awared that an analogous system was obtained in ref. /7/, but no comments on it were given by the authors barring the existence of three types of solutions.

Let us first focus on those certain properties of this system which result form the general analysis of part I.

Transformations $\Psi' = R\Psi$ when conserve the inner product $(\bar{\Psi}\Psi)$ generate four parameter pseudo-unitary group U(1,1). Corresponding Noether currents J^{α}_{ik} (i,k=1,2) form 2x2 matrices with components

$$J^{0}_{ik} = \bar{\Psi}^{(i)}\Psi^{(k)} \quad , \quad J^{1}_{ik} = i\left(\bar{\Psi}^{(i)}_{x}\Psi^{(k)} - \bar{\Psi}^{(i)}\Psi^{(k)}_{x}\right) \tag{13}$$

so that $\partial^{\alpha}J^{ik}_{\alpha} = 0$.

The Poisson brackets of the elements of "charge" matrix $Q_{ik} = \int\limits^{\infty} J^{0}_{ik}\,dx$ and the Hamiltonian H vanishe, i.e., $\{Q_{ik}, H\} = 0$. The charges Q_{ik} may be utilized to construct four hermitian generators of the U(1,1) group:

$$Q_{11} \ , \ Q_{22} \ , \ T_{12} = i\left(Q_{12} + Q_{21}\right), \ K_{12} = Q_{12} - Q_{21} \ .$$

The first two

$$N_1 = \int\limits_{-\infty}^{\infty} |\Psi^{(1)}|^2\,dx \quad \text{and} \quad N_2 = -\int\limits_{-\infty}^{\infty} |\Psi^{(2)}|^2\,dx$$

are the numbers of type "1" or "2" particles.

The remaining

$$T_{12} = i\int\limits_{-\infty}^{\infty}\left(\bar{\Psi}^{(1)}\Psi^{(2)} + \bar{\Psi}^{(2)}\Psi^{(1)}\right)dx$$

and

$$K_{12} = \int\limits_{-\infty}^{\infty}\left(\bar{\Psi}^{(1)}\Psi^{(2)} - \bar{\Psi}^{(2)}\Psi^{(1)}\right)dx \tag{14}$$

are elements of the subalgebra SU(1,1) and generate transformations that mix components of the vector $\Psi^{T} = (\Psi^{(1)}\Psi^{(2)})$. The transformation matrix $R_1 \in SU(1,1)$ is of the form

$$R_1 = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix} \quad , \quad |\alpha|^2 - |\beta|^2 = 1$$

and may be parametrized as follows

$$\alpha = \cosh\theta\, e^{i\varphi_\alpha} \quad , \quad \beta = \sinh\theta\, e^{i\varphi_\beta}$$

Using these transformations one can construct the whole class of solutions to system (10) on the basis of its particular solutions, e.g.,

$$\begin{cases}\psi^{(1)}=\widetilde{\psi}^{(1)}\\ \psi^{(2)}=0\end{cases} \quad \text{or} \quad \begin{cases}\psi^{(1)}=0\\ \psi^{(2)}=\widetilde{\psi}^{(2)}\end{cases}. \tag{15}$$

But (15) are the solutions to the one-component NLS with positive, U(1,0) or negative, U(0,1), coupling constant respectively. It means that the overall set of its solutions can be used to construct solutions to system (10). Consider two particular solutions of type (15) with

$$\widetilde{\psi}^{(1)}=a\,e^{i\theta_1}\operatorname{sech} ax \quad \text{or} \quad \widetilde{\psi}^{(2)}=b\,e^{i\theta_2}\tanh bx$$

making an isotopic rotation R_1 we get two solutions of system (10)

$$\begin{cases}\widetilde{\psi}^{(1)}(x)=\alpha a\,e^{i\theta_1}\operatorname{sech} ax\\ \widetilde{\psi}^{(2)}(x)=\beta^* a\,e^{i\theta_1}\operatorname{sech} ax\end{cases} \quad \text{or} \quad \begin{cases}\widetilde{\psi}^{(1)}(x)=\beta b\,e^{i\theta_2}\tanh bx\\ \widetilde{\psi}^{(2)}(x)=\alpha^* b\,e^{i\theta_2}\tanh bx\end{cases}. \tag{16}$$

The first solution differs from one obtained by Manakov in ref. /3/ by only the definition of polarization vector. In our case its norm is pseudo-Euclidean, $|\alpha|^2-|\beta|^2=1$ (according to subgroup SU(1,1)) in the Manakov case $|\alpha|^2+|\beta|^2=1$ (subgroup SU(2)) it is Euclidean. This unessential at a first glance difference leads to the principally different structure of the system under consideration. It manifests through the difference in physically natural boundary conditions: in the case U(2,0) natural boundary conditions are fields vanishing at both infinites, in our case, U(1,1) variant, there are at any rate four various types /2/.*

The simplest type is vanishing boundary conditions

$$\psi^{(a)}(\pm\infty)=0 \quad , a=1,2\,. \tag{17}$$

The general analysis of Part I works in this case from which it follows that elements $S_{ii}(\xi)$ i=1,2,3, $S_{23}(\xi)$ and $S_{32}(\xi)$ of transition matrix $\hat{S}(\xi)$ (see formula I(37) for definition) are independent of time.

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* Their sense becomes clear through the physical interpretation of solutions obtained.

Only $S_{11}(\xi)$ of them generates an infinite set of local conservation laws. To construct them we consider eq. I(A.11) of Part I at n=2

$$e^{\Phi_1(x)} = 1 - \int_{-\infty}^{x} dy \int_{-\infty}^{y} dz\, e^{i\frac{3}{2}\xi(y-z)} \left(\overline{\Psi}^{(1)}(y)\Psi^{(1)}(z) + \overline{\Psi}^{(2)}(y)\Psi^{(2)}(z)\right) e^{\Phi_1(z)} .$$

Upon differentiating it thrice and eliminating quantities

$$A_{1,2}(x) = \int_{-\infty}^{x} dy \exp\left[i\frac{3}{2}\xi(x-y) + \Phi_1(y)\right]\Psi^{(1,2)}(y)$$

we come to the equation for functions $\Phi_{1x}(x)$

$$\Phi_{1x}\left[\left(\frac{3}{2}i\xi\right)^2 + \frac{\Delta_x}{\Delta}\left(\frac{3}{2}i\xi\right) + \frac{1}{\Delta}\left(\overline{\Psi}^{(2)}_{xx}\Psi^{(1)}_x - \overline{\Psi}^{(1)}_{xx}\Psi^{(2)}_x\right) - (\overline{\Psi}\Psi)\right] =$$

$$-D_x^2\Phi_{1x} + \left[\frac{\Delta_x}{\Delta} + 2\left(\frac{3}{2}i\xi\right)\right]D_x\Phi_{1x} + \left[\frac{\Delta_x}{\Delta} + \left(\frac{3}{2}i\xi\right)\right](\overline{\Psi}\Psi) -$$

$$-(\overline{\Psi}\Psi)_x - (\overline{\Psi}_x\Psi) ,$$

where $D_x = \frac{d}{dx} + \Phi_{1x}$, $\Delta = \overline{\Psi}^{(2)}_x\overline{\Psi}^{(1)} - \overline{\Psi}^{(2)}\overline{\Psi}^{(1)}_x$.
Expanding $\Phi_{1x}(x)$ in the following series

$$\Phi_{1x}(x) = \sum_{k=1}^{\infty} \frac{\varphi_{11}^{(k)}(x)}{\left(\frac{3}{2}i\xi\right)^k}$$

we obtain the recurrense formulae for coefficients $f_k \overset{def}{=} \varphi_{11}^{(k)}(x)$:

$$f_{k+2} = \left(2\frac{d}{dx} - \frac{\Delta_x}{\Delta}\right)f_{k+1} + \left[-\frac{d^2}{dx^2} + \frac{\Delta_x}{\Delta}\frac{d}{dx} + \frac{1}{\Delta}\left(\overline{\Psi}^{(1)}_{xx}\overline{\Psi}^{(2)}_x - \overline{\Psi}^{(2)}_{xx}\overline{\Psi}^{(1)}_x\right)\right.$$

$$\left.-(\overline{\Psi}\Psi)\right]f_k + 2\sum_{k_1+k_2=k+1} f_{k_1}f_{k_2} + \sum_{k_1+k_2=k} f_{k_2}\left(2-3\frac{d}{dx}\right)f_{k_1} - \sum_{k_1+k_2+k_3=k} f_{k_1}f_{k_2}f_{k_3}$$

and k=1,2,...; $f_1 = (\overline{\Psi}\Psi)$, $f_2 = (\overline{\Psi}\Psi_x)$.
First four integrals of motion can be got in the form

$$I_{11}^{(1)} = \int_{-\infty}^{\infty} dx\,(\overline{\Psi}\Psi)(x,t) ,$$
$$I_{11}^{(2)} = \int_{-\infty}^{\infty} dx\,(\overline{\Psi}\Psi_x)(x,t) ,$$
$$I_{11}^{(3)} = \int_{-\infty}^{\infty} dx\left[(\overline{\Psi}\Psi_{xx}) + (\overline{\Psi}\Psi)^2\right](x,t) ,$$
$$I_{11}^{(4)} = \int_{-\infty}^{\infty} dx\left[(\overline{\Psi}\Psi_{xxx}) + 3(\overline{\Psi}\Psi)(\overline{\Psi}_x\Psi)\right](x,t)$$

and coincide with those described by I(49) at n=2.

One soliton solution to system (10) is found via isorotation (formula (16a), see also Appendix).

It reads

$$\psi^{(j)}(x,t)=\exp\left\{i\left(\frac{v}{2}x-\omega_j t\right)\right\}a_j \operatorname{sech} \varkappa\zeta \ ,(j=1,2). \tag{18}$$

where $\omega_1=\omega_2=\frac{v^2}{4}-(\bar{a}a)$, $\varkappa^2=(\bar{a}a)>0$ and $\zeta=x-vt-x_0$.

We call hereafter this solution double drop (2D).

The second type of boundary conditions is a "constant" fiels type

$$\psi^{(j)}(x,t)e^{i\omega_j t}=\begin{cases} a_0^{(j)} , & x\to\infty \\ a_0^{(j)}e^{i\delta_j}, & x\to-\infty \end{cases} \quad j=1,2. \tag{19}$$

Eqs.(2) are convenient to rewrite in the form with chemical potentials μ_1 and μ_2 :

$$\begin{gathered} i\psi_t^{(1)}+\psi_{xx}^{(1)}+2\left(|\psi^{(1)}|^2-|\psi^{(2)}|^2+\mu_1\right)\psi^{(1)}=0 \\ i\psi_t^{(2)}+\psi_{xx}^{(2)}+2\left(|\psi^{(1)}|^2-|\psi^{(2)}|^2+\mu_2\right)\psi^{(2)}=0 \end{gathered} \tag{20}$$

then Hamiltonian density will be

$$\tilde{\mathcal{H}}=-\mathcal{H}+2\mu_1|\psi^{(1)}|^2-2\mu_2|\psi^{(2)}|^2 . \tag{21}$$

In addition to plane wave solution (condensate)

$$\psi_c^{(1)}=a_0^{(1)} \ , \psi_c^{(2)}=a_0^{(2)}\exp\left\{2i(\mu_2-\mu_1)t\right\} \ , -(\bar{a}_0a_0)=\mu_1 \tag{22}$$

eqs.(20) possess one soliton solution of the kink type*

$$\begin{gathered} \psi^{(1)}=a^{(1)}\left(\tanh\varkappa\zeta+i\frac{v}{2\varkappa}\right) \\ \psi^{(2)}=a^{(2)}\exp\left\{2i(\mu_2-\mu_1)t\right\}\left(\tanh\varkappa\zeta+i\frac{v}{2\varkappa}\right). \end{gathered} \tag{23a}$$

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* There are other kink type solutions but since their properties are analogous to these considered we did not discuss them here.

where $\varkappa^2 = -(\bar{a}a) = > 0$, $(\bar{a}a) = -\mu_1$, $\zeta = x - vt - x_0$, $a_0^{(2)} > a_0^{(1)}$. Using formulae (19) one can easily find a dispersion relation

$$\varkappa^2 = \mu_1 - \frac{v^2}{4} . \tag{23b}$$

We call this solution double bubble (2B).

The third type of boundary conditions is "quasiconstant" one

$$\psi^{(1)}(x,t) = \begin{cases} a_0^{(1)} & , x \to \infty \\ a_0^{(1)} e^{i\delta_1} & , x \to -\infty \end{cases} , \psi^{(2)}(x,t) e^{i(\omega_2 t - \frac{v_0}{2}x)} = \begin{cases} a_0^{(2)}, & x \to \infty \\ a_0^{(2)} e^{i\delta_2}, & x \to -\infty . \end{cases} \tag{24}$$

The condensate solution to eqs. (2) is now (with accuracy of a constant phase)

$$\begin{gathered} \psi_c^{(1)} = a_0^{(1)} , \ \psi_c^{(2)} = a_0^{(2)} \exp\{-i(\omega_2 t - \tfrac{v_0}{2} x)\} , \\ \omega_2 = 2(\mu_1 - \mu_2) + \frac{v_0^2}{4} , (\bar{a}a) = -\mu_1 < 0 \ \rightarrow \ a_0^{(2)} > a_0^{(1)} \end{gathered} \tag{25}$$

and one soliton solution is

$$\begin{gathered} \psi^{(1)} = a^{(1)}\left(\tanh \varkappa\zeta + i \frac{v}{2\varkappa}\right) , \\ \psi^{(2)} = a^{(2)} \exp\{-i(\omega_2 t - \tfrac{v_0}{2} x)\}\left(\tanh \varkappa\zeta + \frac{i}{2\varkappa}(v - v_0)\right) \end{gathered} \tag{26a}$$

here again $\varkappa^2 = -(\bar{a}a) > 0$, $(\bar{a}a) = -\mu_1$.

It follows from (25)

$$\left(a^{(1)}\right)^2\left(1 + \frac{v^2}{4\varkappa^2}\right) = \left(a_0^{(1)}\right)^2 , \ \left(a^{(2)}\right)^2\left(1 + \frac{(v - v_0)^2}{4\varkappa^2}\right) = \left(a_0^{(2)}\right)^2 .$$

Solving these equation with respect to x^2 we get the dispersion relation

$$\begin{gathered} 8\varkappa^2 = \left[4\mu_1 - v^2 - (v - v_0)^2\right] \pm \Big\{\left[4\mu_1 - v^2 - (v - v_0)^2\right] + \\ + 4(4\mu_1 - v^2)(v - v_0)^2 + 16\left(a_0^{(2)}\right)^2 v_0 (2v - v_0)\Big\}^{1/2} . \end{gathered} \tag{26b}$$

We call this solution double bubble in a system of interpenetrating gases ($2B_0$).

Finilly, the fourth mixed type of boundary conditions

$$\psi^{(1)}(\pm\infty) = 0 , \ \psi^{(2)}(x,t) = \begin{cases} a_0^{(2)} & , x \to \infty \\ a_0^{(2)} e^{i\delta_2} & , x \to -\infty \end{cases} \tag{27}$$

defines the condensate

$$\psi_c^{(1)} = 0 \quad , \quad \psi_c^{(2)} = a_0^{(2)} = \sqrt{\mu_2} \tag{28}$$

and one soliton solution

$$\psi^{(1)} = a^{(1)} \exp\{-i(\omega_1 t - v\tfrac{x}{2})\} \operatorname{sech} \varkappa\zeta \ , \tag{29a}$$

$$\psi^{(2)} = a^{(2)} \tanh\left(\varkappa\zeta + i\frac{v}{2\varkappa}\right)$$

and $\omega_1 = \frac{v^2}{4} - \varkappa^2 + 2(\mu_2 - \mu_1)$, $\varkappa^2 = (a^{(1)})^2 + (a^{(2)})^2$, $(a_0^{(2)})^2 = \mu_2$.
From formulae (27) and (29a) we have also

$$(a^{(2)})^2\left(1 + \frac{v^2}{4\varkappa^2}\right) = (a_0^{(2)})^2 = \mu_2 \tag{29b}$$

or the dispersion relation

$$2(a^{(2)})^2 = \mu_2 - (a^{(1)})^2 + \frac{v^2}{4} + \left[\left(\mu_2 - (a^{(1)})^2 + \frac{v^2}{4}\right)^2 + 4(a^{(1)})^2\mu_2\right]^{1/2} . \tag{29c}$$

This solution can be naturally called a drop in a bubble (BD)

Solutions (2B) and (2D), i.e., (18) and (23) respectively are simply two field modifications of corresponding solutions to one component equations U(1,0) and U(0,1) -drop (4) and bubble (9). These can be easily obtained by use of the transformations (14) and $\psi \to e^{i\omega t}\psi$. Supposing, for example, in (18) $\varkappa = a$ we get

$$(a^{(1)})^2 = a^2 + (a^{(2)})^2 \overset{def}{=} |\alpha|^2 a^2 \ , (a^{(2)})^2 = (a^{(1)})^2 - a^2 \overset{def}{=} |\beta|^2 a^2$$

so that $|\alpha|^2 - |\beta|^2 = 1$, i.e., solutions (16) (analogously for (23)).

Solitons (26) and (29) appear for the first time namely in the system U(1,1), therefore we discuss their properties in more detail. We also give them the simplest possible physical interpretation. First dispersion formulae will be considered:

1) Soliton (29) is a bubble (rare faction) in the second component condensate $\psi_c^{(2)}$ moving at a velocity v together with a concentric drop of the first component $\psi^{(1)}$. The size of this aggregate $\ell \simeq 1/\varkappa$ depends on the velocity v , however the form of this dependence differs essentially from the convential hole-like one (see, e.g., (23b)).

In the formula $\varkappa^2 = (a^{(1)})^2 + (a^{(2)})^2$ only $(a^{(2)})^2$ is a function of v -which results from (29b) -so that solution (29) depends on two free parameters $a^{(1)}$ and v (abstracting from constant phase φ_i and initial position x_0 unessential for us). Function $a^{(2)}$ determined by formula (29c) has no zeros at finite $a^{(1)}$ and v and behaves asymptotically as $(a^{(2)})^2 \sim \mu_2 \frac{(a^{(1)})^2}{v^2}$ when $v \to \infty$.

It means that the soliton turns gradually from a drop in a bubble into a drop which size is defined by its amplitude $a^{(1)}$ *. The intensity of rarefaction in a bubble (the bubble amplitude) tends to zero at increasing v.

2) Soliton (26) is a double bubble in such two component condensate in which one component (e.g., the second one) moves with respect to another at a velocity v_0 related with boundary condition**. The additional parameter of velocity dimension appear in this case to change qualitatively properties of the system (and, of course, solutions) remind, e.g., a beam in a plasma. Note that we have now two types of bubbles at rest ($v=0$) for the same values $\mu_1, a_0^{(2)}$ (or $a_0^{(1)}$) and v_0 :

$$2\bar{\varkappa}_{\pm}^{o2} = 1 - \beta_0^2 \pm \sqrt{(\beta_0^2 - \varepsilon^2)(\beta_0^2 - \varepsilon^{-2})} \tag{30}$$

if

$$\beta_0^2 \leq \frac{a_0^{(2)} - a_0^{(1)}}{a_0^{(2)} + a_0^{(1)}} \overset{def}{=} \varepsilon^2 < 1 \quad ,$$

where

$$(\alpha_0^{(i)})^2 = \frac{(a_0^{(i)})^2}{\mu_1}, (i=1,2), \beta = \frac{v}{2\sqrt{\mu_1}}, \beta_0 = \frac{v_0}{2\sqrt{\mu_1}}, \bar{\varkappa}^2 = \frac{\varkappa^2}{\mu_1} \tag{31}$$

therefore $(\alpha_0^{(2)})^2 - (\alpha_0^{(1)})^2 = 1$ and $\varepsilon^2 = (1 + 2(\alpha_0^{(1)})^2)^{-1}$.
They are narrow bubble $l_n \sim 1/\varkappa_+$ and a broad one $l_b \sim 1/\varkappa_-$.
The difference between their widths $\Delta = l_b - l_n$ vanishes when β_0 tends to ε and $\varkappa^2$ turns to be complex (soliton disappear) at $\beta_0 > \varepsilon$ (see Fig.1). How does this picture change at $v \neq 0$?
In this case

$$2\bar{\varkappa}_{\pm}^2 = 1 - \beta_0^2 - 2\beta^2 + 2\beta\beta_0 \pm \sqrt{(\beta_0^2 - \varepsilon^2)(\beta_0^2 - \varepsilon^{-2}) + 4\beta\beta_0\left[\beta\beta_0 - (\beta_0^2 - \varepsilon^{-2})\right]} \tag{32}$$

and there are real solutions for $\varkappa$ in the region $\varepsilon^2 \leq \beta_0^2 \leq \varepsilon^{-2}$ as well. Velocities of such solitons have, however, both upper and lower limits. The plots in Fig.2 display dependence $\bar{\varkappa}^2$ as function of β , at various values ε^2 .The existence of region of narrow solitons diminishes and that of broad ones

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* The amplitude $a^{(1)}$ is in turn determined by an initial state.

** We should underline that the "density" of the first component condensate $n_1 \sim |\psi_c^{(1)}|^2 = (a_0^{(1)})^2$ is less than the second one $n_2 \sim |\psi_c^{(2)}|^2 = (a_0^{(2)})^2$.

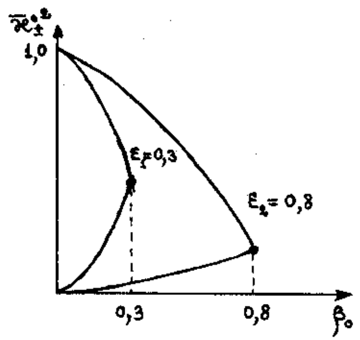

Fig.1

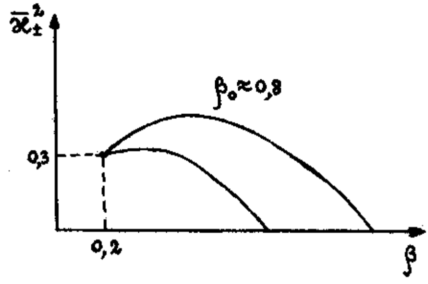

Fig.2

disappears at all when $\beta_0 > \sqrt{8/5}$ and $\varepsilon \ll 1$. When $\beta_0^2 \geqslant \frac{1}{2}(\varepsilon^2 + \varepsilon^{-2})$ region of narrow soliton existence enlarges again and is

$$\beta_{cr}^{(1)} \leq \beta \leq \beta_{cr}^{(2)},$$

where $\beta_{cr}^{(1)} = \frac{1}{\varepsilon}\left(1 - \sqrt{\frac{1+\varepsilon^2}{2}}\right)$, $\beta_{cr}^{(2)} = \frac{1}{\varepsilon}\left(1 + \sqrt{\frac{1+\varepsilon^2}{2}}\right)$

at $\beta_0 = 1/\varepsilon$.

It decreases at further growth of β_0 and vanishes at $\beta_c \approx O(1/\varepsilon)$.

Thus in the above "beam-gas" system there appear novel hole-like solutions with the following features: a) two branches at the same values of parameters ε, β_0 and β, b) bubbles are forced by the beam to follow it at $\beta_0^2 > \varepsilon^2$ (i.e., there are no bubble at rest) and c) they disappear at $\beta_0 \simeq O(1/\varepsilon)$.

Let us now discuss the spectra of the solutions obtained. Calculating functionals N and E on the solution (18) one gets

$$E^{(I)} = \frac{1}{4} N_S^{(I)}\left[v^2 - \frac{1}{3}(N_S^{(I)})^2\right], \tag{33}$$

$$N_S^{(I)} = N_1 - N_2 = 2(a_1^2 - a_2^2)^{1/2} = 2\varkappa, \quad \omega = \frac{1}{4}\left[v^2 - (N^{(I)})^2\right],$$

we see that the soliton energy is a function of two parameters v and $N_S^{(I)}$. We can also check the formula

$$dE^{(I)} = \sum_i \mu_i \, dN_i = \omega \, dN_S^{(I)} \qquad \left(\mu_i = (-1)^{i-1}\omega\right) \tag{34}$$

to be valid like in the case U(1,0).

Using Bohr-Sommerfeld quantization formula

$$\frac{1}{2}\int_0^T dt \int_{-\infty}^{\infty} dx \left(\frac{\delta \mathcal{L}}{\delta \psi_t^{(i)}} \psi_t^{(i)} + c.c.\right) = 2\pi n_i$$

we get

$$\int_0^T dt\int_{-\infty}^{\infty} dx\,\frac{i}{2}\left(\psi_t^{(i)}\psi^{(i)*}-\psi_t^{(i)*}\psi^{(i)}\right)=\omega\int_0^T dt\int_{-\infty}^{\infty} dx\,|\psi^{(i)}|^2=\omega\int_0^T dt\,N_i=\omega T N_i$$

or $N_i=n_i$ with n_i being large integer number. 2D solution is therefore a bound state of Bose-quanta of the fields $\psi^{(1)}$ and $\psi^{(2)}$ and the latters "2" give the negative contribution in total energy, see formula (11)*.

Equation (34) may be thought of as thermodynamical relation for the two component mixture which chemical potentials are μ_i i.e., $\mu_1=\omega$, $\mu_2=-\omega$. We note that such drops can arise only at those places of a system where density of gravitating Bose-gas ($\rho_1=|\psi^{(1)}|^2$) is greater than that of anti-gravitating ($\rho_2=|\psi^{(2)}|^2$). All the properties of drops are governed by the difference between numbers of bosons of the first and the second components, i.e., $N^{(I)}=N_1-N_2$.

Then an initial bunch will disperse as far as the inequality $\rho_2>\rho_1$ holds.

At constant boundary conditions energy of the system is defined mainly by the second component, so we rewrite Hamiltonian in the form (21) (i.e., with negative sign compared with (11)) and add terms with the chemical potentials $2\mu_1$ and $2\mu_2$.

To make expressions for integrals of particle number and energy finite their normalization should be done as in the case of U(0,1) variant

$$\overline{N_i}=\int_{-\infty}^{\infty}\left(|\psi^{(i)}(x,t)|^2-|\psi^{(i)}(\infty)|^2\right)dx$$

$$E=\int_{-\infty}^{\infty}\left(\mathcal{H}(\psi^{(i)}(x,t))-\mathcal{H}(\psi^{(i)}(\infty)\right)dx\,.$$

2B soliton (21)

$$\begin{aligned} E^{(II)}&=\frac{1}{3}N^3+2\,\frac{(\mu_2-\mu_1)}{\mu_1}\,(a_0^{(2)})^2N\,,\\ N&=N_2-N_1=2\varkappa=2\sqrt{\mu_1-\frac{v^2}{4}}\,,\\ N_i&=-\overline{N_i}=2\,(a^{(i)})^2/\varkappa=2\,(a_0^{(i)})^2\left(\varkappa+\frac{v^2}{4\varkappa}\right)^{-1} \end{aligned}\qquad(35)$$

* Hamiltonian of a quantum system for which (10) may be considered as quasi-classical approximation is $H=-\sum_{n=1}^{N}\frac{\partial^2}{\partial x_n^2}+\sum_{m=1}^{M}\frac{\partial^2}{\partial y_m^2}-2\sum_{n<n'}^{N}\delta(x_n-x_{n'})-2\sum_{m<m'}^{M}\delta(y_m-y_{m'})+2\sum_{n=1}^{N}\sum_{m=1}^{M}\delta(x_n-y_m)$.

is a simplest solution in this case and its energy, as one can see from (35), depends on one parameter v (or N). Equation (35) may be rewritten in the differential form

$$dE^{(II)} = \mu\, dN = \sum_i \gamma_i\, dN_i \,, \qquad (\gamma_2 = -\gamma_1 = \mu),$$

where γ_i are now the chemical potentials of holes coupling in the bubble

$$\mu = N^2 + 2\,\frac{\mu_2 - \mu_1}{\mu_1}\,(a_0^{(2)})^2 \,. \tag{36}$$

Such bubbles may occur in the infinite system of two gases if density of the second (antigravitating) one is greater than that of the first gas, $\rho_2 > \rho_1$. Note that the condensate (ground state) is stable only under this condition and becomes unstable at $\rho_2 \le \rho_1$ even for infinitesimal perturbations of density /2/. Since the energy of bubbles (23), (34) is positive they arise when a finite energy perturbation is brought in the system.

Condensate (25) corresponds to the analogous system in which, however, the second gas moves at a constant velocity v_0 in the first one. We have again a double bubble but of a more complex constraction. Its energy

$$E^{(III)} = \frac{1}{3}(N_2 - N_1)^3 + 2\left[\mu_2 - \mu_1 + \frac{v_0}{4}\left(\frac{v_0}{2} - v\right)\right] N_2 \,,$$
$$N_2 - N_1 = 2\varkappa \,, \quad N_1 = \frac{2\,(a_0^{(1)})^2}{\varkappa + \frac{v^2}{4\varkappa^2}} \,, \quad N_2 = \frac{2\,(a_0^{(2)})^2}{\varkappa + \frac{(v - v_0)^2}{4\varkappa}} \tag{37}$$

is also a function of one free parameter v and the dependence $\varkappa(v)$ is given by (30). We can see from (37) that broad bubbles carry, generally speaking, smaller energy than narrow bubbles, therefore the former may be excited at an easier rate than in latter.

Consider a situation where a small amount of gravitating gas $\psi^{(1)}$ is injected into the condensate $\rho_2 \ne 0$, $\rho_1 = 0$. The former gas condensating in a drop will serve as a center of bubble formation and we have the solution of BD type (a drop in a bubble) with spectrum

$$E^{(IV)} = \frac{1}{3}(N_1 + N_2)^3 - (3\mu_2 - 2\mu_1)\, N_1 - \mu_2\,\frac{N_1^2}{N_2} \,,$$
$$N_1 = \frac{2\,(a^{(1)})^2}{\varkappa} \,, \quad N_2 = 2\,\frac{(a^{(2)})^2}{\varkappa + \frac{v^2}{4\varkappa^2}} \,, \quad N_1 + N_2 = 2\varkappa \tag{38}$$

that depends on two free parameters v and $a^{(1)}$ or N_1 and N_2. The differential form of (38) is now

$$dE^{(IV)} = \sum_i \bar{\mu}_i dN_i ,$$

where

$$\bar{\mu}_1 = (N_1+N_2)^2 - 2\mu_2 \frac{N_1}{N_2} - 3\mu_2 + 2\mu_1$$
$$\bar{\mu}_2 = (N_1+N_2)^2 + \mu_2 \frac{N_1^2}{N_2^2}$$

play roles of the chemical potentials of the first component bosons $\bar{\mu}_1$ and of holes in the second component condensate $\bar{\mu}_2$ coupled in the BD soliton. Note that the chemical potential of bosons in the soliton differs from their frequency

$$\omega_1 = \frac{1}{2}\left[(N_1+N_2)^2 - 2\mu_2 \frac{N_1}{N_2}\right] - 3\mu_2 + 2\mu_1 , \text{ i.e. } \bar{\mu}_1 = 2\omega_1 - (3\mu_2 - 2\mu_1) .$$

This is due to the numbers of bosons and holes coupled in the soliton depend on its velocity, - the characteristic feature of the hole mode. When $v = dv = 0$ we come to those usual relations for drops $dE^{(IV)} = \sum \omega_i \; dN_i$ and $\omega_1 = \frac{1}{4}(N_1+N_2)^2 - \frac{1}{2}N_2(N_1+N_2) + \mu_1$, $\omega_2 = 0$ which have been found in ref./2/.

It should be also stressed that isorotation (14) applied to a solution under non-trivial, e.g., mixed boundary conditions gives rise to a solution of a new boundary problem. Solution (16b) of the 2B type may be for example obtained from solution (15b), i.e. that of the mixed boundary-value problem BD (see (29a) at $a^{(1)} = 0$). Moreover boundary values of both components change due to this transformation:

$$\psi^{(1)}(\infty) : \qquad 0 \to \beta\, a_c^{(2)}$$

$$\psi^{(2)}(\infty) : \qquad a_c^{(2)} \to \alpha^* a_c^{(2)}$$

herewith the energy remains constant that follows from (33) and (39) since neither $N_S^{(r)} = 2\sqrt{(a^{(1)})^2 - (a^{(2)})^2}$ nor $N = 2\sqrt{(a^{(2)})^2 - (a^{(1)})^2}$ change.

Ultimately it should be underlined that though NLS solitons with positive coupling constant are bound states of the certain number of constituents-bosons, see (3), upon formation they assume principally novel solitonic properties which differ them substantially from conventional bound states (such as, e.g., that of nucleons in nuclei). The integrability of the fiels equation supplies solitons with additional integrals of motion (for example, integral of total number of solitons like the baryon or lepton charge conservation laws) that leads to their elastic interactions. As a result processes of soliton fusion creating havier solitons although being exoergic are forbidden, so that even at the classical level solitons possess an analog of quantum properties that suppresses collapse type instabilities. An arbitrary initial state breaks up into a number of solitons (and "noise"); if large number,a distribution function over amplitudes or N may be found.

Applying results obtained to examine magnetic excitations ($\propto |\psi^{(2)}|^2$) in the framework of the continuum Hubbard model (system U(1,1)) one needs to bear in mind that the ground state of a chain is antiferromagnetic, $\psi^{(2)}(\infty) = 0$ and may pretend to describe magnetic excitations over antiferromagnetic vacuum. In this connection studying 2D soliton collisions becomes interesting that will in particular be the subject of the third part if the present work.

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APPENDIX

Inverse method for the U(p,q) NLS

Recall briefly the main formulae of Part I we will use in what follows. Matrix Jost solutions are governed by equations

$$\hat{\varphi}(x,\xi) = e^{-i\xi\hat{\Sigma}x} + \int_{-\infty}^{x} e^{-i\xi\hat{\Sigma}(x-y)}\hat{Q}(y)\hat{\varphi}(y,\xi)\,dy \qquad (A.1)$$

$$\hat{\psi}(x,\xi) = e^{-i\xi\hat{\Sigma}x} - \int_{x}^{\infty} e^{-i\xi\hat{\Sigma}(x-y)}\hat{Q}(y)\hat{\psi}(y,\xi)\,dy\,, \qquad (A.2)$$

where

$$\hat{Q}(y) = \begin{bmatrix} 0 & i\bar{q}(y) \\ iq(y) & 0 \end{bmatrix}, \quad \hat{\Sigma} = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{n}I_n \end{bmatrix}.$$

From the relation

$$\frac{d}{dx}\left(\bar{\varphi}(x,\eta)\varphi(x,\xi)\right) = i(\eta^* - \xi)\left(\hat{\bar{\varphi}}(x,\eta)\hat{\Sigma}\,\hat{\varphi}(x,\xi)\right)$$

it follows that

$$\hat{\bar{\varphi}}(x,\xi)\hat{\varphi}(x,\xi) = \hat{I} = \hat{\bar{\psi}}(x,\xi)\hat{\psi}(x,\xi)\,. \qquad (A.3)$$

Given by the relation

$$\hat{\varphi}(x,\xi) = \hat{\Psi}(x,\xi)\hat{S}(\xi) \qquad (A.4)$$

transition matrix is subject to the conditions of unimodularity

$$\det \hat{S}(\xi) = 1 \qquad (A.5)$$

and of pseudounitarity

$$\hat{\bar{S}}(\xi)\hat{S}(\xi) = \hat{I} \quad . \qquad (A.6)$$

Its elements are

$$S_{ik}(\xi) = W(\Psi_1 \ldots \Psi_{i-1}\varphi_k \Psi_{i+1} \ldots \Psi_{n+1})$$

or

$$S_{ik}(\xi) = \bar{\Psi}_i(x,\xi)\varphi_k(x,\xi) . \qquad (A.7)$$

The Jost solutions $\varphi_1(x,\xi), \Psi_2(x,\xi), \ldots, \Psi_{n+1}(x,\xi)$ can be continued in the upper half plane of ξ and $\varphi_2(x,\xi), \varphi_3(x,\xi), \ldots, \varphi_{n+1}(x,\xi), \Psi_1(x,\xi)$ in the lower one. From (A.7) the element $S_{11}(\xi)$ follows to be analytical in the region $\mathrm{Im}\,\xi \geqslant 0$, wheareas $S_{\alpha\beta}(\xi)$ (hereafter Greek indices vary from 2 to n+1) is analytical at $\mathrm{Im}\,\xi \leqslant 0$. Following ref.[3] consider a problem of reconstruction of the potential $\hat{Q}(y)$ on the basis of scattering data $S_{ik}(\xi)$ and $C_{1\alpha}$ whereat

$$\varphi_1(x,\xi) = C_{12}\Psi_2(x,\xi) + \ldots + C_{1n+1}\Psi_{n+1}(x,\xi) \quad . \qquad (A.8)$$

Zeros of the function $S_{11}(\xi)$ are supposed to be simple and located at points $\xi_1, \ldots, \xi_N$. From (A.5) and (A.6) we have

$$\det S_{\alpha\beta}(\xi) = S_{11}^*(\xi^*) . \qquad (A.9)$$

Define a matrix $\Omega_{\alpha\beta}$ inverse to $S_{\alpha\beta}$ as usual

$$S_{\alpha\beta}(\xi)\Omega_{\beta\gamma}(\xi) = \delta_{\alpha\gamma} .$$

It is clear that elements of matrix Ω are analytical in the lower half plane due to (A.9) barring the points $\xi_1^*, \ldots, \xi_N^*$ where they have simple poles

$$\Omega_{\beta\gamma}(\xi) = \frac{a_{\gamma\beta}(\xi)}{\det S_{\alpha\beta}(\xi)} = \frac{a_{\gamma\beta}(\xi)}{S_{11}^*(\xi^*)}$$

with $a_{\gamma\beta}(\xi)$ being algebraic complement of the element $S_{\gamma\beta}(\xi)$.

Then statement of the inverse problem is reduced to reconstructing following piecewise analytical functions

$$\Phi_1(x,\xi) = \begin{cases} \dfrac{\varphi_1(x,\xi)\, e^{i\xi x}}{S_{11}(\xi)} & , \ Im\,\xi > 0 \\ \psi_1(x,\xi) e^{i\xi x} & , \ Im\,\xi < 0 \end{cases} \tag{A.10}$$

$$\Phi_\alpha(x,\xi) = \begin{cases} \psi_\alpha(x,\xi) e^{-i\frac{\xi}{n}x} & , \ Im\,\xi > 0 \\ \varphi_\sigma(x,\xi)\Omega_{\sigma\alpha}(\xi) e^{-i\frac{\xi}{n}x} & , \ Im\,\xi < 0 \end{cases} \tag{A.11}$$

having along the real axis following jumps

$$\tilde{\Phi}_1(x,\xi) = \psi_\alpha(x,\xi)\ \frac{S_{\alpha 1}(\xi)}{S_{11}(\xi)}\, e^{i\xi x} , \tag{A.12}$$

$$\tilde{\Phi}_\alpha(x,\xi) = \psi_1(x,\xi)\frac{\overline{S}_{1\alpha}(\xi)}{S^*_{11}(\xi)} e^{-i\frac{\xi}{n}x} = \psi_1(x,\xi)\frac{S^*_{\alpha 1}(\xi)}{S^*_{11}(\xi)}(\Gamma_0)_{\alpha\alpha} e^{-i\frac{\xi}{n}x} . \tag{A.13}$$

In getting these formulae we used the equality

$$S_{1\sigma}(\xi)\Omega_{\sigma\alpha}(\xi) = -\frac{\overline{S}_{1\alpha}(\xi^*)}{S^*_{11}(\xi^*)} = -\frac{S^*_{\alpha 1}(\xi^*)}{S^*_{11}(\xi^*)}(\Gamma_0)_{\alpha\alpha} , \tag{A.14}$$

$$\hat{\Gamma}_0 = diag(\underbrace{+1,\dots,+1}_{p},\underbrace{-1,\dots,-1}_{q})$$

(there is no summing over α).

Function $\Phi_1(x,\xi)$ has simple poles at points $\zeta_1,\dots,\zeta_N$ with residues

$$\Gamma_1^{(m)} = \frac{c_{1\alpha}^{(m)}\,\psi_\alpha(x,\zeta_m)\exp(i\zeta_m x)}{S'_{11}(\zeta_m)} \tag{A.15}$$

and $\Phi_\alpha(x,\xi)$ at points $\zeta_1^*,\zeta_2^*,\dots,\zeta_N^*$ with residues

$$\Gamma_\alpha^{(m)} = -\frac{\overline{c}_{1\alpha}^{(m)}\,\psi_1(x,\zeta_m^*)\exp(-i\frac{\zeta_m^*}{n}x)}{(S'_{11}(\zeta_m))^*} , \tag{A.16}$$

where $\overline{c}_{1\alpha}^{(m)} = \overline{S}_{1\alpha}(\zeta_m^*)$. Matrix function $\hat{\Phi}(x,\xi)$ is reconstructed using bundary values, residues and jumps according to the equations:

$$\hat{\Phi}(x,\xi) = \hat{I} + \sum_{m=1}^{N}\frac{\hat{\Gamma}^{(m)}}{\xi - \zeta_m^{(i)}} + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\hat{\tilde{\Phi}}(x,\xi')\,d\xi'}{\xi' - \xi} ,$$

where

$$\zeta_m^{(i)} = \begin{cases} \zeta_m & \text{when } i=1 \\ \zeta_m^* & \text{when } i=2,3,\dots,n+1 \end{cases}$$

(boundary conditions for the functions $\hat{\Phi}$ are due to those for the Jost functions φ_1 and Ψ_α).

Therefore we have

$$\Psi_1(x,\zeta_m^*)\exp(i\zeta_m^* x)=e_1+\sum_{k=1}^{N}\frac{\Gamma_1^{(k)}}{\zeta_m^*-\zeta_k}+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\widetilde{\Phi}_1(x,\xi')\,d\xi'}{\xi'-\zeta_m^*} \quad (A.17)$$

$$\Psi_\alpha(x,\zeta_m)\exp(-i\frac{\zeta_m}{n}x)=e_\alpha+\sum_{k=1}^{N}\frac{\Gamma_\alpha^{(k)}}{\zeta_m-\zeta_k^*}+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\widetilde{\Phi}_\alpha(x,\xi')\,d\xi'}{\xi'-\zeta_m} \quad (A.18)$$

where e_i is the "i" column of unit-matrix. We have then the following limiting values $\Phi_i(x,\xi)$ at the real axis

$$\Psi_1(x,\xi)\exp(i\xi x)=e_1+\sum_{k=1}^{N}\frac{\Gamma_1^{(k)}}{\xi-\zeta_k}+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\widetilde{\Phi}_1(x,\xi')\,d\xi'}{\xi'-\xi+i0} \quad (A.19)$$

$$\Psi_\alpha(x,\xi)\exp(-i\frac{\xi}{n}x)=e_\alpha+\sum_{k=1}^{N}\frac{\Gamma_\alpha^{(k)}}{\xi-\zeta_k^*}+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\widetilde{\Phi}_\alpha(x,\xi')\,d\xi'}{\xi'-\xi-i0} \quad (A.20)$$

Comparing asymptotic expansion of $\Phi_i(x,\xi)$ obtained from (A.2) and (A.19) at $\xi\to\infty$ we get "potentials"

$$q^{(a)}(x)=-\frac{n+1}{n}\left(\sum_{m=1}^{N}\Gamma_1^{(m)}+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\widetilde{\Phi}_1(x,\xi)\,d\xi\right)_{(a+1)} \quad (A.21)$$

The set of equations (A.12)-(A.21) solves in principle the inverse problem completely.

It is interesting to note that the potentials $q^{(a)}(x)$ are reconstructed on the basis of only S_{11} and $S_{1\alpha}$.

Consider the simplest case of S_{11} having only zero ζ and all the matrix elements $S_{1\alpha}$ (ξ) vanishing. Then we have a system of algebraic equations

$$S_{11}(\xi,\zeta)=\frac{\xi-\zeta}{\xi-\zeta^*},$$

$$\Psi_1(x,\xi)e^{i\xi x}=e_1+\frac{\Gamma_1}{\xi-\zeta},$$

$$\Psi_\alpha(x,\xi)e^{-i\frac{\xi}{n}x}=e_\alpha+\frac{\Gamma_\alpha}{\xi-\zeta^*},$$

$$\Gamma_1=\frac{c_{1\beta}\Psi_\beta(x,\zeta)\exp(i\zeta x)}{S_{11}'(\zeta)},$$

$$\Gamma_\alpha = - \frac{\bar{c}_{1\alpha}\,\Psi_1(x,\zeta^*)\exp(-i\frac{\zeta^*}{n}x)}{(S_{11}'(\zeta))^*} .$$

Using variables

$$\zeta = \frac{n}{n+1}(\xi + i\eta) \quad , \quad x_0 = \frac{1}{2\eta}\ln(\bar{S}S),$$

$$c_a = \frac{S_a}{(\bar{S}S)^{1/2}} \quad , \quad S_a = C_{1,a+1}(0),$$

$$(\bar{S}S) = \sum_{a=1}^{p}|S_a|^2 - \sum_{a=p+1}^{n}|S_a|^2$$

one gets one soliton solution

$$q^{(a)}(x,t) = -i\eta\,\frac{c^{(a)}\exp\{-i\xi x - i(\xi^2-\eta^2)t\}}{\cosh\{\eta(x-x_0)+2\xi\eta t\}} \tag{A.22}$$

and $v = -2\xi$ is a soliton velocity, η is its amplitude, x_0 is its position at t=0. The components of "polarization" vector $C_{(a)}$, ($\bar{C}C$) =1, may be arbitrary large. This solution is seen to coincide with (18), i.e.,obtained earlier via the isorotation of the U(1,0) soliton.

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