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A.Galperin, E.Ivanov, V.Ogievetsky

N=1 SUPERFIELD ANATOMY OF THE FAYET-SOHNIUS MULTIPLET

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1. The simplest representation of N=2 extended supersymmetry with central charges, the hypermultiplet (called here for definiteness the Fayet-Sohnius (FS) multiplet), was studied already in a number of papers /1-3/ . Sohnius /2/, Stelle and West /3/ investigated it in N= 2 superspace $\{x^m, z, \theta^{\alpha i}, \bar{\theta}^{\dot{\alpha}}_i\}$ with a central charge coordinate z . They solved the constraints on the FS multiplet:

$$\mathcal{D}_{\alpha(i}\phi_{j)} = 0 \quad , \quad \bar{\mathcal{D}}_{\dot{\alpha}(i}\phi_{j)} = 0 \tag{1.1}$$

in terms of component fields. Here ϕ_i is an isospinor scalar superfield, $\mathcal{D}_{\alpha i}, \bar{\mathcal{D}}_{\dot{\alpha} i}$ are the spinor derivatives of N=2 supersymmetry and the parentheses mean symmetrization in the SU(2) indices i,j. Fayet /1/ analyzed the FS-multiplet in terms of on-shell component fields and on-shell N=1 superfields.

In this article we present a solution of the constraints (1.1) in terms of N=1 off-shell superfields. The constraints (1.1) are considered below as Grassmann analyticity conditions /4/ with respect to different pairs of the spinor variables. The FS-multiplet is represented by a pair of N=1 chiral superfields. We need no special variable for the central charge which is realized as a bilinear combination of spinor derivatives. The component results are of course identical to those of /2,3/ .

Let us motivate our interest in the FS multiplet. In the real superspace approach to N= 1 supergravity constraints on the torsion components have to be postulated /5/ .As was shown by Gates, Stelle and West /6/ the main meaning of these constraints consists in preserving chiral representations of rigid supersymmetry in curved superspace. The chiral superfield is defined in the complex (4.2) superspace. Indeed, a chiral superfield is a general scalar complex superfield in the real (4,4)-dimensional superspace $\{x^m, \theta^\alpha, \bar{\theta}^{\dot{\alpha}}\}$ constrained by

$$\bar{\mathcal{D}}_{\dot\alpha}\Psi(x,\theta,\bar\theta)=0 \tag{1.2}$$

which is just the Cauchy-Riemann condition in the sense of [4]. The solution of (1.2) in the flat case is given by

$$\Psi(x,\theta,\bar\theta)=\varphi(x_L,\theta)\quad,\quad x_L^m=x^m-i\theta\sigma^m\bar\theta, \tag{1.3}$$

where φ is an arbitrary complex superfield. Defining general analytic transformations in the complex (4.2) superspace $\{x_L^m, \theta^\alpha\}$ and identifying the imaginary part of x_L^m with the axial gravitational superfield one obtains the geometrical formulation of N=1 supergravity suggested by one of the authors (V.O.) and E.Sokatchev [7] (see also [8]). Being expressed in terms of real differential superspace geometry this approach solves automatically the chirality preserving constraints. Analogous analysis of the N=2 supergravity by Stelle and West [9] shows, that in this case the representations preserved are the FS multiplet (1.1) as well as the chiral one. We hope that the solution of constraints (1.1) in the flat case will suggest the choice of an adequate complex superspace for N=2 supergravity and the corresponding complex geometry.

The FS multiplet is solved in terms of a pair of chiral N=1 superfields defined on two complex (4,2) superspaces. Each of these superspaces is not SU(2) invariant, their vector coordinates transform according to a reducible representation $1\oplus 3$ of the SU(2) group. Besides, there is a correlation between the external SU(2) index of the superfield and the superspace on which it is given. This and other exotic properties of the basis discussed need further clarification. However, by analogy with the N=1 case there are the earnest reasons to believe that this basis plays an important role in the N=2 case and may help in searching for the adequate complex geometry of N=2 supergravity.

2. Let us first list some basic definitions and notations. We use two-component formalism. In manifestly SU(2)-covariant form the N=2 superalgebra reads

$$\begin{aligned}
\{Q_{\alpha i},\bar Q_{\dot\beta}^{\ j}\} &= 2\,\delta_i^{\ j}(\sigma_m)_{\alpha\dot\beta}P^m\\
\{Q_{\alpha i},Q_{\beta j}\} &= -2i\,\varepsilon_{ij}\,\varepsilon_{\alpha\beta}\,Z \qquad\qquad (2.1)\\
\{\bar Q_{\dot\alpha}^{\ i},\bar Q_{\dot\beta}^{\ j}\} &= -2i\,\varepsilon^{ij}\,\varepsilon_{\dot\alpha\dot\beta}\,Z^\dagger \qquad (\varepsilon^{12}=-\varepsilon_{12}=1),
\end{aligned}$$

where $\bar Q_{\dot\alpha}^{\ i}=(Q_{\alpha i})^\dagger$. The spinor generators transform with respect to the SU(2) group as follows (T^a are the SU(2) generators):

$$
\begin{aligned}
[T^a, Q_{\alpha i}] &= -\frac{1}{2}(\tau^a)_i{}^k Q_{\alpha k} \\
[T^a, \bar{Q}_{\dot\alpha}{}^i] &= \frac{1}{2}(\tau^a)_k{}^i \bar{Q}_{\dot\alpha}{}^k .
\end{aligned}
\tag{2.2}
$$

We begin with a general situation, when two central charges are present (a scalar central charge $\sim Z+Z^\dagger$, a pseudoscalar one $\sim i(Z-Z^\dagger)$).

The general N=2 superspace is defined in the usual way as a space of left cosets of the N=2 supergroup over the direct product of the Lorentz and SU(2) groups. The central charges may be included either in the coset space (then an additional bosonic coordinate is needed) or in the stability subgroup. We prefer here the second possibility, so the algebra (2.1) will be implemented in the real (4.8) N=2 superspace $\{ x^m, \theta_\alpha{}^i, \bar\theta_{\dot\alpha i} = (\theta_\alpha^i)^\dagger \}$. Usually the manifestly SU(2) covariant symmetric parametrization of coset spaces is used (see Appendix). As we are interested in the N=1 superfield description of the FS multiplet it is more convenient for us to use a nonsymmetric parametrization, in which one of the supersymmetries (e.g., the first one) is realized in the standard N=1 fashion . The following notation is therefore appropriate:

$$
\begin{aligned}
&S_\alpha \equiv Q_{\alpha 1} , \quad Q_\alpha \equiv Q_{\alpha 2} ; \quad \theta_\alpha \equiv \theta_\alpha{}^1 , \; \eta_\alpha \equiv \theta_\alpha{}^2 \\
&\bar{S}_{\dot\alpha} \equiv \bar{Q}_{\dot\alpha}^1 , \quad \bar{Q}_{\dot\alpha} \equiv \bar{Q}_{\dot\alpha}^2 ; \quad \bar\theta_{\dot\alpha} \equiv \bar\theta_{\dot\alpha 1} , \; \bar\eta_{\dot\alpha} \equiv \bar\theta_{\dot\alpha 2}
\end{aligned}
\tag{2.3}
$$

An element of cosets in the nonsymmetric parametrization is defined as

$$
G(x,\theta,\bar\theta,\eta,\bar\eta) = e^{i x^m P_m} e^{i(\theta S + \bar\theta\bar S)} e^{i(\eta Q + \bar\eta\bar Q)} . \tag{2.4}
$$

The corresponding realization of various symmetry generators and spinor covariant derivatives can be found either directly or using the connection with the symmetric parametrization given in Appendix:

$$
\begin{aligned}
&S_\alpha = i\frac{\partial}{\partial\theta^\alpha} - (\not\partial\theta)_\alpha , \quad Q_\alpha = i\frac{\partial}{\partial\eta^\alpha} - (\not\partial\eta)_\alpha - 2\theta_\alpha Z \\
&\bar{S}_{\dot\alpha} = -i\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + (\theta\not\partial)_{\dot\alpha} , \quad \bar{Q}_{\dot\alpha} = -i\frac{\partial}{\partial\bar\eta^{\dot\alpha}} + (\eta\not\partial)_{\dot\alpha} - 2\bar\theta_{\dot\alpha} Z^\dagger
\end{aligned}
\tag{2.5}
$$

$$
\begin{aligned}
T^1 &= \frac{1}{2}\left[\theta\frac{\partial}{\partial\eta} + \eta\frac{\partial}{\partial\theta} - \bar\theta\frac{\partial}{\partial\bar\eta} - \bar\eta\frac{\partial}{\partial\bar\theta}\right] + \frac{i}{2}\left[(\theta\theta+\eta\eta)Z - (\bar\theta\bar\theta+\bar\eta\bar\eta)Z^\dagger\right] + \bar{T}^1 \\
T^2 &= \frac{i}{2}\left[\eta\frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial\eta} + \bar\eta\frac{\partial}{\partial\bar\theta} - \bar\theta\frac{\partial}{\partial\bar\eta}\right] + \frac{1}{2}\left[(\theta\theta-\eta\eta)Z + (\bar\theta\bar\theta-\bar\eta\bar\eta)Z^\dagger\right] + \bar{T}^2 \\
T^3 &= \frac{1}{2}\left[\theta\frac{\partial}{\partial\theta} - \eta\frac{\partial}{\partial\eta} - \bar\theta\frac{\partial}{\partial\bar\theta} + \bar\eta\frac{\partial}{\partial\bar\eta}\right] + \bar{T}^3
\end{aligned}
\tag{2.6}
$$

$$\mathcal{D}^{\theta}_{\alpha} \equiv \mathcal{D}_{1\alpha} = \frac{\partial}{\partial\theta^{\alpha}} - i(\not{\partial}\bar{\theta})_{\alpha} + 2i\eta_{\alpha} Z \;, \quad \mathcal{D}^{\eta}_{\alpha} \equiv \mathcal{D}_{2\alpha} = \frac{\partial}{\partial\eta^{\alpha}} - i(\not{\partial}\eta)_{\alpha}$$
$$\bar{\mathcal{D}}^{\theta}_{\dot\alpha} \equiv \bar{\mathcal{D}}^{1}_{\dot\alpha} = -\frac{\partial}{\partial\bar{\theta}^{\dot\alpha}} + i(\theta\not{\partial})_{\dot\alpha} + 2i\bar{\eta}_{\dot\alpha} Z^{\dagger} \;, \quad \bar{\mathcal{D}}^{\eta}_{\dot\alpha} \equiv \bar{\mathcal{D}}^{2}_{\dot\alpha} = -\frac{\partial}{\partial\bar{\eta}^{\dot\alpha}} + i(\eta\not{\partial})_{\dot\alpha}, \tag{2.7}$$

where T^a are the matrix parts of the SU(2) generators acting on the external SU(2) indices of superfields. A general group variation of the N=2 superfield $\Phi(x, \theta^i, \bar{\theta}_k)$ has the form (external indices of superfield are suppressed):

$$\delta\Phi = \left[-i\varepsilon^i Q_i - i\bar{\varepsilon}_i \bar{Q}^i - i\rho^a T^a + i\lambda Z + i\lambda^{\dagger} Z^{\dagger} - i a^m P_m - \frac{i}{2} a^{mn} L_{mn} \right] \Phi \tag{2.8}$$

$\varepsilon^i, \bar{\varepsilon}_i, \rho^a, \lambda, \lambda^{\dagger}, a^m, a^{mn}$ being the corresponding group parameters.

It is clear from (2.5) that the coordinates $x^m, \theta^{\alpha}, \bar{\theta}^{\dot\alpha}$ constitute the standard real N=1 superspace with respect to S-supersymmetry. The generators T^a get additional Z-dependent terms. The nonsymmetric parametrization was used already in /4/ .

3. Now we proceed to solving the constraints (1.1). For this purpose it is useful to rewrite (1.1) in the nonmanifestly SU(2) - covariant notation (2.7):

$$\mathcal{D}^{\theta}_{\alpha}\phi_1 = 0\,, \quad \mathcal{D}^{\eta}_{\alpha}\phi_2 = 0\,, \quad \mathcal{D}^{\theta}_{\alpha}\phi_2 + \mathcal{D}^{\eta}_{\alpha}\phi_1 = 0\,, \tag{3.1a}$$

$$\bar{\mathcal{D}}^{\theta}_{\dot\alpha}\phi_2 = 0\,, \quad \bar{\mathcal{D}}^{\eta}_{\dot\alpha}\phi_1 = 0\,, \quad \bar{\mathcal{D}}^{\theta}_{\dot\alpha}\phi_1 - \bar{\mathcal{D}}^{\eta}_{\dot\alpha}\phi_2 = 0\,. \tag{3.1b}$$

The next step is to perform an appropriate SU(2) rotation in the external indices of the superfields:

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \rightarrow \begin{pmatrix} \check{\phi}_1 \\ \check{\phi}_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_1 - i\phi_2 \end{pmatrix}. \tag{3.2}$$

In terms of $\check{\phi}_i$ equations (3.1 a,b) take the form

$$\left(\mathcal{D}^{\theta}_{\alpha} + i\mathcal{D}^{\eta}_{\alpha}\right)\check{\phi}_1 = 0 \tag{3.3a}$$

$$\left(\mathcal{D}^{\theta}_{\alpha} - i\mathcal{D}^{\eta}_{\alpha}\right)\check{\phi}_2 = 0 \tag{3.3b}$$

$$\mathcal{D}^{\theta}_{\alpha}\left(\check{\phi}_1 + \check{\phi}_2\right) = 0 \tag{3.3c}$$

$$\left(\bar{\mathcal{D}}^{\theta}_{\dot\alpha} + i\bar{\mathcal{D}}^{\eta}_{\dot\alpha}\right)\check{\phi}_1 = 0 \tag{3.4a}$$

$$\left(\bar{\mathcal{D}}^{\theta}_{\dot\alpha} - i\bar{\mathcal{D}}^{\eta}_{\dot\alpha}\right)\check{\phi}_2 = 0 \qquad (3.4b)$$

$$\bar{\mathcal{D}}^{\theta}_{\dot\alpha}\,(\check{\phi}_1 - \check{\phi}_2) = 0 . \qquad (3.4c)$$

Note that in the representation (3.2) the SU(2) generators are given by

$$\check{T}^a = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ 1 & -i\end{pmatrix}\frac{\tau^a}{2}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i\end{pmatrix}$$

or

$$\check{T}^1 = -\frac{\tau^2}{2}\,,\quad \check{T}^2 = -\frac{\tau^3}{2}\,,\quad \check{T}^3 = \frac{\tau^1}{2}\,. \qquad (3.5)$$

Let us analyze constraints (3.3), (3.4). Equations (3.3a,3.4a) and (3.3b, 3.4b) are just Grasmann Cauchy-Riemann conditions /4/, which express the analyticity with respect to $\{\theta - i\eta, \bar\theta - i\bar\eta\}$ and $\{\theta + i\eta, \bar\theta + i\bar\eta\}$, correspondingly. In other words, the superfield $\check\phi_1$ is analytical, $\check\phi_2$ is antianalytical, that is, they are reduced to some complex scalar N=1 superfields. Equations (3.3a,b), (3.4a,b) can be easily solved /4/ :

$$\check\phi_1(x,\theta,\bar\theta,\eta,\bar\eta) = e^{-\eta\eta Z - \bar\eta\bar\eta Z^{\dagger}}\, e^{i(\eta\mathcal{D}+\bar\eta\bar{\mathcal{D}})}\,\check\varphi_1(x,\theta,\bar\theta) \qquad (3.6)$$

$$\check\phi_2(x,\theta,\bar\theta,\eta,\bar\eta) = e^{\eta\eta Z + \bar\eta\bar\eta Z^{\dagger}}\, e^{-i(\eta\mathcal{D}+\bar\eta\bar{\mathcal{D}})}\,\check\varphi_2(x,\theta,\bar\theta),$$

where $\mathcal{D}_\alpha$, $\bar{\mathcal{D}}_{\dot\alpha}$ are the ordinary spinor derivatives of N=1 supersymmetry

$$\mathcal{D}_\alpha = \mathcal{D}^{\theta}_{\alpha}\big|_{\eta=0} = \frac{\partial}{\partial\theta^\alpha} - i(\partial\!\!\!/\bar\theta)_\alpha \qquad (3.7)$$

$$\bar{\mathcal{D}}_{\dot\alpha} = \bar{\mathcal{D}}^{\theta}_{\dot\alpha}\big|_{\eta=0} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + i(\theta\partial\!\!\!/)_{\dot\alpha} .$$

Note that the O(2) group from /4/ coincides with the O(2) subgroup of SU(2) generated by T^2 : the Cauchy-Riemann conditions are separately covariant under transformations from this subgroup.

The additional equations (3.3c), (3.4c) applied to the N=1 superfields $\check\varphi_1$, $\check\varphi_2$ are reduced to the usual chirality constraints:

$$\mathcal{D}_\alpha(\check\varphi_1 + \check\varphi_2) = 0 \qquad (3.3c)$$

$$\bar{\mathcal{D}}_{\dot\alpha}(\check\varphi_1 - \check\varphi_2) = 0\,, \qquad (3.4c)$$

that is, $\check\varphi_1$ and $\check\varphi_2$ are simply the sum and difference of two chiral N=1 superfields

$$\check{\varphi}_1(x,\theta,\bar\theta)=\frac{1}{\sqrt2}\left[\varphi_1(x^m+i\theta\sigma^m\bar\theta,\bar\theta)+i\,\varphi_2(x^m-i\theta\sigma^m\bar\theta,\theta)\right]$$
$$\check{\varphi}_2(x,\theta,\bar\theta)=\frac{1}{\sqrt2}\left[\varphi_1(x^m+i\theta\sigma^m\bar\theta)-i\,\varphi_2(x^m-i\theta\sigma^m\bar\theta,\theta)\right]. \tag{3.8}$$

Normalization in (3.8) is chosen in such a way that

$$\varphi_i(x,\theta,\bar\theta)=\phi_i(x,\theta,\bar\theta,\eta,\bar\eta)\,\big|_{\eta=\bar\eta=0}. \tag{3.9}$$

So, with the FS multiplet two different kinds of Grassmann analyticity are associated: N=2 analyticity and chirality, which is the N=1 analyticity /4/ . The combined action of these analyticities is so restrictive that the highly reducible N=2 supermultiplet contained in the N=2 superfields ϕ_i comes down to a pair of chiral N=1 multiplets with highest spin 1/2 x) .

4. The transformation rules of the superfields φ_1 , φ_2 under the central charge variations are fixed by the initial constraints (1.1). It is not hard to show using the algebra of spinor derivatives (which coincides with (2.1)) that:

$$Z\phi_1=\frac{i}{4}\mathcal{D}^\theta\mathcal{D}^\theta\phi_2\ ,\quad Z\phi_2=-\frac{i}{4}\mathcal{D}^\eta\mathcal{D}^\eta\phi_1 \tag{4.1a}$$

$$Z^\dagger\phi_2=-\frac{i}{4}\bar{\mathcal{D}}^\theta\bar{\mathcal{D}}^\theta\phi_1\ ,\quad Z^\dagger\phi_1=\frac{i}{4}\bar{\mathcal{D}}^\eta\bar{\mathcal{D}}^\eta\phi_2 \tag{4.1b}$$

and

$$(ZZ^\dagger+\Box)\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}=0 \tag{4.2}$$

Combining (4.1), (4.2) with (3.9) yields

$$Z\varphi_1=\frac{i}{4}\mathcal{D}\mathcal{D}\varphi_2\ ,\quad Z^\dagger\varphi_2=-\frac{i}{4}\bar{\mathcal{D}}\bar{\mathcal{D}}\varphi_1 \tag{4.3}$$

and

$$(ZZ^\dagger+\Box)\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}=0\ , \tag{4.4}$$

i.e., Z on φ_1 and $Z^\dagger$ on φ_2 are realized bilinearly in ordinary N=1 spinor derivatives, while $Z\varphi_2$ and $Z^\dagger\varphi_1$ are some

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x) It is interesting to note that the whole set of constraints (3.3), (3.4) could be reproduced starting from some O(2)-analytical superfield $\check{\phi}_1$ and extending then O(2) to SU(2). The simplest nontrivial possibility is to allow $\check{\phi}_1$ to be a component of an isodoublet of the same sort as $\mathcal{D}^\theta_\alpha+i\mathcal{D}^\eta_\alpha$ (with the opposite choice the resulting constraints reduce the superfields to constants). Varying (3.3a) (3.4a) by the SU(2) / O(2) transformations one obtains all the other equations.

new independent chiral superfields. If $Z \neq e^{i\alpha} Z^\dagger$, then acting successively by powers of $Z, Z^\dagger$ on φ_1 and φ_2 we shall get an infinite multiplet of chiral superfields. This procedure cannot be interrupted at any finite step. For instance, by setting $Z^n \varphi_2 = 0$ we would get a meaningless constraint $\Box^n \varphi_2 = 0$ (in virtue of (4.4)). Let us emphasize that we want to have finite multiplet. The only possibility to escape the proliferation of FS multiplets is to restrict the supersymmetry algebra (2.1) to one central charge only, i.e., to put x) :

$$Z = Z^\dagger . \tag{4.5}$$

It is clear from (4.3) that the multiplet $\{\varphi_1, \varphi_2\}$ is now closed under the action of central charge:

$$Z \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = -\frac{1}{4}\left(\mathcal{D}\mathcal{D} + \bar{\mathcal{D}}\bar{\mathcal{D}}\right) \tau^2 \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \tag{4.6}$$

or in terms of $\check{\varphi}_1, \check{\varphi}_2$:

$$Z \begin{pmatrix} \check{\varphi}_1 \\ \check{\varphi}_2 \end{pmatrix} = \frac{1}{4}\left(\mathcal{D}\mathcal{D} + \bar{\mathcal{D}}\bar{\mathcal{D}}\right) \tau^3 \begin{pmatrix} \check{\varphi}_1 \\ \check{\varphi}_2 \end{pmatrix} . \tag{4.7}$$

It should be emphasized once more that there is no need in introducing of additional bosonic coordinate to realize the central charge as it is usually done /2,3/ . Instead, Z is expressed through spinor derivatives while the condition (4.4) is fulfilled identically. To prove the last statement we rewrite equation (4.7) as

$$Z_- \check{\varphi}_1 = 0, \quad Z_+ \check{\varphi}_2 = 0 , \tag{4.8}$$

where $$Z_\pm = Z \pm \frac{1}{4}\left(\mathcal{D}\mathcal{D} + \bar{\mathcal{D}}\bar{\mathcal{D}}\right) .$$

One can easily see that

$$Z_+ Z_- = Z_- Z_+ = Z^2 + \Box - \Box\, \Pi_{1/2} , \tag{4.9}$$

where $\Pi_{1/2} = 1 + \frac{1}{16\Box}\left(\mathcal{D}\mathcal{D}\bar{\mathcal{D}}\bar{\mathcal{D}} + \bar{\mathcal{D}}\bar{\mathcal{D}}\mathcal{D}\mathcal{D}\right)$ is the projector singling out superspin 1/2 /10/ . Since superfields $\check{\varphi}_1$ and $\check{\varphi}_2$ contain superspin 0 only, equation (4.4) follows immediately from (4.8), (4.9).

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x) It is sufficient to restrict oneself by equation (4.5) since the more general condition $Z = e^{i\alpha} Z^\dagger$ is reduced to (4.5) by chiral $U(1)$ transformation $Q \to e^{-i\alpha/4} Q$

Now we can represent the transition formulae (3.6) in their final form:

$$\check{\Phi}_1(x,\theta,\bar\theta,\eta,\bar\eta)=e^{i(\eta\mathcal{D}+\bar\eta\bar{\mathcal{D}})}e^{-\frac{1}{4}(\eta\eta+\bar\eta\bar\eta)(\mathcal{D}\mathcal{D}+\bar{\mathcal{D}}\bar{\mathcal{D}})}\frac{1}{\sqrt{2}}\left[\varphi_1(x_R^m,\bar\theta)+i\,\varphi_2(x_L^m,\theta)\right] \quad (4.10)$$

$$\check{\Phi}_2(x,\theta,\bar\theta,\eta,\bar\eta)=e^{-i(\eta\mathcal{D}+\bar\eta\bar{\mathcal{D}})}e^{\frac{1}{4}(\eta\eta+\bar\eta\bar\eta)(\mathcal{D}\mathcal{D}+\bar{\mathcal{D}}\bar{\mathcal{D}})}\frac{1}{\sqrt{2}}\left[\varphi_1(x_R^m,\bar\theta)-i\varphi_2(x_L^m,\theta)\right] \quad (4.11)$$

$$x_R^m\equiv(x_L^m)^\dagger=x^m+i\theta\sigma^m\bar\theta .$$

5. In this section we give the explicit form of Q supersymmetry and SU(2) transformations (S - supersymmetry is realized in a standard fashion). The action of the generators on chiral superfields can be found as follows: at first one has to act on the superfields $\check{\Phi}_1$, $\check{\Phi}_2$ by the generators in realization (2.5), (2.6) and then put $\eta=\bar\eta=0$:

$$Q_\alpha=-2\theta_\alpha Z\mp\mathcal{D}_\alpha \quad (5.1)$$

$$\bar Q_{\dot\alpha}=-2\bar\theta_{\dot\alpha}Z\mp\bar{\mathcal{D}}_{\dot\alpha}$$

$$T^1=\check{T}^1\pm\frac{i}{2}(\theta\mathcal{D}-\bar\theta\bar{\mathcal{D}})+\frac{i}{2}(\theta\theta-\bar\theta\bar\theta)Z$$

$$T^2=\check{T}^2\pm\frac{1}{2}(\theta\mathcal{D}+\bar\theta\bar{\mathcal{D}})+\frac{1}{2}(\theta\theta+\bar\theta\bar\theta)Z \quad (5.2)$$

$$T^3=\check{T}^3+\frac{1}{2}\left(\theta\frac{\partial}{\partial\theta}-\bar\theta\frac{\partial}{\partial\bar\theta}\right)$$

with the upper sign corresponding to $\check{\varphi}_1$ and the lower one to $\check{\varphi}_2$. The differential part of T^3 is ordinary γ_5 transformation. Substituting in these formulae the concrete realization of Z (4.7) and passing to superfields φ_1,φ_2, in terms of which the formulae are more compact, we get

$$Q_\alpha\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}=-\frac{i}{2}\begin{pmatrix}0 & \mathcal{D}\mathcal{D}\\ -\bar{\mathcal{D}}\bar{\mathcal{D}} & 0\end{pmatrix}\theta_\alpha\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}$$

$$\bar Q_{\dot\alpha}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}=-\frac{i}{2}\begin{pmatrix}0 & \mathcal{D}\mathcal{D}\\ -\bar{\mathcal{D}}\bar{\mathcal{D}} & 0\end{pmatrix}\bar\theta_{\dot\alpha}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} \quad (5.3)$$

$$T^1\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}=-\frac{1}{8}\begin{pmatrix}0 & \mathcal{D}\mathcal{D}\\ -\bar{\mathcal{D}}\bar{\mathcal{D}} & 0\end{pmatrix}(\theta\theta-\bar\theta\bar\theta)\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}$$

$$T^2\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}=\frac{i}{8}\begin{pmatrix}0 & \mathcal{D}\mathcal{D}\\ -\bar{\mathcal{D}}\bar{\mathcal{D}} & 0\end{pmatrix}(\theta\theta+\bar\theta\bar\theta)\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} \quad (5.4)$$

$$T^3\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}=\frac{1}{2}\left(\tau^3+\theta\frac{\partial}{\partial\theta}-\bar\theta\frac{\partial}{\partial\bar\theta}\right)\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}$$

The transformations generated by $Z,Q_\alpha,\bar Q_{\dot\alpha},T^1,T^2$ can be compactly represented by the single formula:

$$\delta \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = -\frac{1}{4} \begin{pmatrix} 0 & \mathcal{D}\mathcal{D} \\ -\bar{\mathcal{D}}\bar{\mathcal{D}} & 0 \end{pmatrix} \left[\lambda(\theta) + \bar{\lambda}(\bar{\theta}) \right] \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \tag{5.5}$$

where $\lambda(\theta)$, $\bar{\lambda}(\bar{\theta}) = \lambda(\theta)^{\dagger}$ are constant chiral "superfields" with the group parameters as components:

$$\lambda(\theta) = \frac{\lambda}{2} + 2\varepsilon^2\theta - \frac{i}{2}(\beta^1 - i\beta^2)\,\theta\theta \tag{5.6}$$
$$\bar{\lambda}(\bar{\theta}) = \frac{\lambda}{2} + 2\bar{\varepsilon}_2\bar{\theta} + \frac{i}{2}(\beta^1 + i\beta^2)\,\bar{\theta}\bar{\theta}.$$

For the sake of completeness we also write down the transfor – mation laws of the components fields. It is not hard to deduce from (5.4) the well-known SU(2) structure of this multiplet /2/ : two scalar isodoublets $A_i(x)$, $F_i(x)$ and two isoscalar spinors $\psi(x)$, $\bar{\varkappa}(x)$. Supersymmetry and central charge are realized on these fields by /2/ :

$$\begin{aligned} \delta A_i &= \varepsilon_i\psi + \bar{\varepsilon}_i\bar{\varkappa} - \lambda F_i \\ \delta\psi &= -2i(\partial\!\!\!/\bar{\varepsilon}_i)A^i - 2\varepsilon^i F_i + i\lambda(\partial\!\!\!/\bar{\varkappa}) \\ \delta\bar{\varkappa} &= 2i(\varepsilon^k\partial\!\!\!/)A_k + 2\bar{\varepsilon}_k F^k - i\lambda(\bar{\partial\!\!\!/}\psi) \\ \delta F_k &= -i(\varepsilon_k\partial\!\!\!/\bar{\varkappa}) + i(\bar{\varepsilon}_k\bar{\partial\!\!\!/}\psi) - \lambda\Box A_k . \end{aligned} \tag{5.7}$$

The fields $A_i, \psi, \bar{\varkappa}, F_i$ are contained in the N=1 chiral superfields φ_1, φ_2 as follows

$$\begin{aligned} \varphi_1(x^m_R, \bar{\theta}) &= A_1(x_R) + \bar{\theta}\bar{\varkappa}(x_R) + \bar{\theta}\bar{\theta}\, F_2(x_R) \\ \varphi_2(x^m_L, \theta) &= A_2(x_L) + \theta\psi(x_L) - \theta\theta\, F_1(x_L). \end{aligned} \tag{5.8}$$

6. In this section we establish an invariant N=1 superfield free action formula for the FS multiplet and its connection with the N=2 action formula /2,3/ .

Taking into account the complex conjugation rules

$$\begin{aligned} (\varphi_i)^* &= \varphi^{*i} \equiv \varepsilon^{ij}\varphi^*_j \\ (\varphi^i)^* &= -\varphi^*_i \end{aligned} \tag{6.1}$$

one can easily build Hermitean expressions for the kinetic and mass terms for the FS-multiplet which are manifestly invariant under S -supersymmetry

$$S_{kin} = -\frac{1}{4}\int d^4x\, d^4\theta\ \left[\varphi_1\varphi_2^* - \varphi_2\varphi_1^*\right] =$$
$$= \frac{1}{8}\int d^4x d^4\theta \left[\delta(\theta)\varphi_1 \mathcal{D}\mathcal{D}\varphi_2^* + \delta(\bar\theta)\varphi_2\bar{\mathcal{D}}\bar{\mathcal{D}}\varphi_1^*\right] \tag{6.2}$$

$$S_{mass} = -\frac{im}{2}\int d^4x d^4\theta \left[\delta(\theta)\varphi_1\varphi_1^* - \delta(\bar\theta)\varphi_2\varphi_2^*\right]. \tag{6.3}$$

The total action can be cast, with the help of (4.6) , in the compact form:

$$S = S_{kin} + S_{mass} = -\frac{i}{4}\int d^4x d^4\theta \left[\delta(\theta)\varphi_1(\overleftrightarrow{Z} + 2m)\varphi_1^* - \delta(\bar\theta)\varphi_2\left(\overleftrightarrow{Z} + 2m\right)\varphi_2^*\right]. \tag{6.4}$$

The superfield equations of motion are just

$$\left(Z + m\right)\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} = 0. \tag{6.5}$$

After going to fields and integrating over $d\theta$ the action (6.4) takes the familiar form [2] :

$$S = \int d^4x \Big\{ \partial_n A_i \partial^n A^{*i} + \frac{i}{2}\left(\varkappa \partial\!\!\!/ \bar\varkappa + \psi \partial\!\!\!/ \bar\psi\right) - F_i F^{*i} -$$
$$- im\left[A_i F^{*i} + F_i A^{*i} + \frac{1}{2}\left(\varkappa\psi - \bar\varkappa\bar\psi\right)\right]\Big\} \tag{6.6}$$

Sohnius [2] , Sohnius, Stelle and West [3] derived an action formula for FS multiplet in terms of constrained N=2 superfields:

$$S = \frac{1}{48}\int d^4x \left(\tilde{\mathcal{D}}^i\tilde{\mathcal{D}}^j + \bar{\tilde{\mathcal{D}}}^i\bar{\tilde{\mathcal{D}}}^j\right)\tilde{K}_{ij}\ , \tag{6.7}$$

where $\tilde{K}_{ij}$ is a linear selfconjugated N=2 supermultiplet built out of N=2 superfields [3]

$$\tilde{K}_{ij} = i\left[\tilde\Phi_i(\overleftrightarrow{Z}+2m)\tilde\Phi_j^* + \tilde\Phi_j(\overleftrightarrow{Z}+2m)\tilde\Phi_i^*\right], (\tilde{K}_{ij})^\dagger = \tilde{K}^{ij} = \varepsilon^{il}\varepsilon^{jn}K_{ln} \tag{6.8}$$
$$\tilde{K}_{ij} = \tilde{K}_{ji}$$

$$\tilde{\mathcal{D}}_{\alpha(k}\tilde{K}_{ij)} = 0\ ,\quad \bar{\tilde{\mathcal{D}}}_{\dot\alpha(k}\tilde{K}_{ij)} = 0. \tag{6.9}$$

Using (6.9) one can check that the expressions

$$\tilde{\mathcal{D}}_{\beta l}\left(\tilde{\mathcal{D}}^i\tilde{\mathcal{D}}^j - \bar{\tilde{\mathcal{D}}}^i\bar{\tilde{\mathcal{D}}}^j\right)\tilde{K}_{ij}\ ,\quad \bar{\tilde{\mathcal{D}}}_{\dot\beta l}\left(\tilde{\mathcal{D}}^i\tilde{\mathcal{D}}^j - \bar{\tilde{\mathcal{D}}}^i\bar{\tilde{\mathcal{D}}}^j\right)\tilde{K}_{ij}$$

are total X^m derivatives. The proof of supersymmetry and central charge invariances of (6.7) is based on this fact.

The action (6.7) coincides exactly with (6.4). Their identity is proved using the equivalent form of (6.7) involving integration over the whole N=2 superspace:

$$S=\frac{1}{48}\int d^4x\,d^4\theta\,d^4\bar\theta\left[\delta^4(\bar\theta)\,\theta^\rho\theta^\kappa+\delta^4(\theta)\,\bar\theta^\rho\bar\theta^\kappa\right]\widetilde{K}_{\rho\kappa}\,, \tag{6.7'}$$

where

$$d^4\theta\equiv\frac{1}{12}\left(d\theta_i\,d\theta_j\right)\left(d\theta^j\,d\theta^i\right)=d^2\theta\,d^2\eta$$

$$\delta^4(\theta)\equiv\frac{1}{12}\left(\theta_i\theta_j\right)\left(\theta^j\theta^i\right)=\delta(\theta)\,\delta(\eta)\,.$$

The next steps are to pass to the nonsymmetric parametrized ϕ_i,ϕ_i^* (see A.7), use (4.10) and then integrate over $d^2\eta$, $d^2\bar\eta$

7. Finally we discuss the possibility of interpretation of the basic equations (4.10) in analogy with the chiral N =1 superfield as shifts of superspace coordinates, i.e., as a transition from the general real basis of N=2 superspace to some complex basis of lower dimensionality, which is adequate to the FS-multiplet.

Exchanging the exponentials in (4.10) and using the identity

$$e^{i(\eta\mathcal{D}+\bar\eta\bar{\mathcal{D}})}=e^{(\eta\not{\partial}\bar\theta-\theta\not{\partial}\bar\eta)}\,e^{i(\eta\frac{\partial}{\partial\theta}+\bar\eta\frac{\partial}{\partial\bar\theta})} \tag{7.1}$$

we get

$$\check\phi_1=e^{-(\eta\eta+\bar\eta\bar\eta)Z}\,\frac{1}{\sqrt2}\left[\varphi_1(x^m_{\mathrm{I}},\bar\theta_{\mathrm{I}})+i\,\varphi_2(x^m_{\mathrm{II}},\theta_{\mathrm{II}})\right] \tag{7.2}$$

$$\check\phi_2=e^{(\eta\eta+\bar\eta\bar\eta)Z}\,\frac{1}{\sqrt2}\left[\varphi_1(x^m_{\mathrm{III}},\bar\theta_{\mathrm{III}})+i\,\varphi_2(x^m_{\mathrm{IV}},\theta_{\mathrm{IV}})\right].$$

Here

$$x^m_{\mathrm{I}}=x^m+i\theta\sigma^m\bar\theta-i\eta\sigma^m\bar\eta-2\theta\sigma^m\bar\eta=x^m+ig^{m\,1}{}_1-g^{m1}{}_2\;;\;\bar\theta_{\mathrm{I}}=\bar\theta+i\bar\eta \tag{7.3a}$$

$$x^m_{\mathrm{II}}=x^m-i\theta\sigma^m\bar\theta+i\eta\sigma^m\bar\eta+2\eta\sigma^m\bar\theta=x^m+ig^{m2}{}_2+g^{m2}{}_1\;;\;\theta_{\mathrm{II}}=\theta+i\eta \tag{7.3b}$$

$$x^m_{\mathrm{III}}=(x^m_{\mathrm{II}})^\dagger,\quad\bar\theta_{\mathrm{III}}=(\theta_{\mathrm{II}})^\dagger \tag{7.3c}$$

$$x^m_{\mathrm{IV}}=(x^m_{\mathrm{I}})^\dagger,\quad\theta_{\mathrm{IV}}=(\bar\theta_{\mathrm{I}})^\dagger \tag{7.3d}$$

and

$$g^{m\,i}{}_k\equiv\theta^i\sigma^m\bar\theta_k+\theta_k\sigma^m\bar\theta^i\quad,\quad g^{m\,i}{}_i=0\,. \tag{7.4}$$

From (7.3), (7.4) one concludes that the minimal superfield structure of the FS multiplet is naturally associated with four chiral N=1 superspaces with the crossing-type complex conjugation rule: the first chiral superspace is the conjugate of the fourth, the second of the third. These superspaces are closed only under S - supersymmetry, however, they are correlated with the external index i of the superfield φ_i in such a way that all the other transformations (Q -supersymmetry, SU(2), central charge) do not take a superfield out of the domain of its definition. In other words, due to (5.3), (5.4) the variation of φ_i is defined on the same superspace, as φ_i itself. This unusual property of basis (7.3) is still to be understood. It would be important to be able to construct this basis by purely geometrical reasons. Recall, that standard chiral N=1 superspaces appear when including one of the spinor generators as well as the Lorentz generators in the little group /11/ . Besides, the central charge role requires a deeper understanding, the central charge operator is realized as a second order differential operator and its action does not reduce to shifts of the $x, \theta, \bar{\theta}$ coordinates. However, one important circumstance is already clear, which seems to us to be crucial for a future pure geometrical minimal formulation of N=2 supergravity. The minimal formulation of N=1 supergravity /7/ is based on the complexification of x^m as suggested by the existence of chiral N=1 superspaces: x^m- shift in (1.3) is pure imaginary. x^m - shifts in (7.3) are isotriplet components (7.4) indicating thus the fundamental role for the N=2 case of the object $Z^{m\,ij}$ which is transformed as $1 \oplus 3$ of SU(2) and is reduced in the flat limit to :

$$Z_0^{m\,ij} = \frac{1}{2}\varepsilon^{ij}x^m - \frac{i}{2}g^{m\,ij} . \tag{7.5}$$

The bosonic coordinates of basis (7.3) are just components of (7.5), $Z_0^{m\,ij}$ transforms under supersymmetry as follows

$$\delta Z_0^{m\,ij} = -i\left(\varepsilon^i\sigma^m\bar{\theta}^j + \theta^j\sigma^m\bar{\varepsilon}^i\right). \tag{7.6}$$

A detailed discussion of all these problems will be given elsewhere.

As was mentioned above, in N=2 supergravity the FS-multiplet and the chiral multiplet are preserved. Sokatchev /12/ has achieved an essential progress with chirality preservation. The N=1 superfield anatomy of the FS multiplet exposed in the present paper will be useful together with chirality preservation for construction of adequate geometry of N=2 supergravity.

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Appendix. Basic relations of the N=2 superfield formalism in the symmetric parametrization

An element of left cosets is defined as follows:

$$\widetilde{G}(x,\theta^i,\bar\theta_k) = e^{ix^m P_m} e^{i(\theta^i Q_i + \bar\theta_i \bar Q^i)} \equiv$$

$$\equiv e^{ix^m P_m} \cdot e^{i(\theta S + \bar\theta \bar S + \eta Q + \bar\eta \bar Q)} \tag{A.1}$$

Generators of various symmetries and covariant derivatives in the basis (A.1) are calculated by standard technique. They have manifestly SU(2) - covariant form:

$$\widetilde{Q}_{\alpha i} = i\frac{\partial}{\partial\theta^{\alpha i}} - (\not\partial\bar\theta_i)_\alpha - \theta_{\alpha i} Z \tag{A.2}$$

$$\widetilde{\bar Q}_{\dot\alpha}^{i} = -i\frac{\partial}{\partial\bar\theta^{\dot\alpha}_{i}} + (\theta^i\not\partial)_{\dot\alpha} + \bar\theta^i_{\dot\alpha} Z^\dagger$$

$$\widetilde{T}^a = T^a + \frac{1}{2}\theta^k(\tau^a)_k{}^l\frac{\partial}{\partial\theta^l} + \frac{1}{2}\bar\theta^k(\tau^a)_k{}^l\frac{\partial}{\partial\bar\theta^l} \tag{A.3}$$

$$\widetilde{\mathcal{D}}_{\alpha i} = \frac{\partial}{\partial\theta^{\alpha i}} - i(\not\partial\bar\theta_i)_\alpha - i\theta_{\alpha i} Z$$

$$\widetilde{\bar{\mathcal{D}}}_{\dot\alpha}^{i} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}_{i}} + i(\theta^i\not\partial)_{\dot\alpha} + i\bar\theta^i_{\dot\alpha} Z^\dagger . \tag{A.4}$$

The element (A.1) is related to the corresponding element in the parametrization (2.4) in the following way

$$\widetilde{G} = G \cdot K \tag{A.5}$$

$$K = e^{-i(\eta\theta Z + \bar\eta\bar\theta Z^\dagger)} = \exp\frac{i}{2}\left[\theta^k(\tau^3)_k{}^l\theta_l Z - \bar\theta^k(\tau^3)_k{}^l\bar\theta_l Z^\dagger\right] \tag{A.6}$$

as can easily be seen from Campbell-Hausdorff formula and algebra (2.1). The unitary operator K relates the N=2 superfields and realized on them operators in both parametrizations:

$$\phi(x,\theta^i,\bar\theta_k) = K\,\widetilde{\phi}(x,\theta^i,\bar\theta_k) \tag{A.7}$$

$$\hat{O} = K\,\hat{\widetilde{O}}\,K^\dagger \tag{A.8}$$

$$Q_{\alpha i} = K \tilde{Q}_{\alpha i} K^{\dagger} = \tilde{Q}_{\alpha i} + (\tau^3)_i{}^k \theta_{\alpha k} Z$$

$$\bar{Q}_{\dot{\alpha}}{}^i = K \bar{\tilde{Q}}_{\dot{\alpha}}{}^i K^{\dagger} = \bar{\tilde{Q}}_{\dot{\alpha}}{}^i - \bar{\theta}^k{}_{\dot{\alpha}} (\tau^3)_k{}^i Z^{\dagger} \qquad (A.9)$$

$$T^a = K \tilde{T}^a K^{\dagger} = \tilde{T}^a + \frac{1}{2} \varepsilon^{a3b} \left[\theta^k (\tau^b)_k{}^l \theta_l Z - \bar{\theta}^k (\tau^b)_k{}^l \bar{\theta}_l Z^{\dagger} \right] \qquad (A.10)$$

$$\mathcal{D}_{\alpha i} = K \tilde{\mathcal{D}}_{\alpha i} K^{\dagger} = \tilde{\mathcal{D}}_{\alpha i} - i (\tau^3)_i{}^k \theta_{\alpha k} Z$$

$$\bar{\mathcal{D}}_{\dot{\alpha}}{}^i = K \bar{\tilde{\mathcal{D}}}_{\dot{\alpha}}{}^i K^{\dagger} = \bar{\tilde{\mathcal{D}}}_{\dot{\alpha}}{}^i + i \bar{\theta}^k{}_{\dot{\alpha}} (\tau^3)_k{}^i Z^{\dagger} \qquad (A.11)$$

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