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1/N-EXPANSION FOR THE ANHARMONIC OSCILLATOR

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1. In this paper we report the results obtained while investigating the $1/N$ -expansion in the problem of quantum-mechanical anharmonic oscillator. This problem has already been discussed by a number of authors (see, e.g., Ref.[1-3]), who have obtained the first three terms of the expansion (up to the order $1/N$). Here we present the first seven terms of the expansion for the energy ground and first excited levels.

The large-order behaviour of the $1/N$ -expansion coefficients has been studied by Brezin and Hikami[4]. Applying the same method we use such a parametrization of the Hamiltonian that enables us to find the solution in a closed analytical form.

The asymptotic series obtained are quasialternating in sign: they differ from proper alternating ones by an oscillating factor. For arbitrary values of the coupling constant and the sign of the mass squared these series are Borel summable, and to sum them we apply the Pade-Borel method. The comparison of the energies thus obtained with the exact values calculated numerically at computer demonstrates a rather good accordance.

So, let us consider the Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \frac{m^2}{2}\sum_{i=1}^{N} x_i^2 + \frac{g}{N}\left(\sum_{i=1}^{N} x_i^2\right)^2 \tag{1.1}$$

confining ourselves to the case of the quartic anharmonicity, though this method can be easily generalized. In the Hamiltonian (1.1) there are two parameters with the dimension of energy (m and $g^{1/3}$) which define the energy scale for two different limits ($g\to 0$ and $g\to\infty$). It is convenient to introduce such a parameter with the dimension of energy (we denote it by ω), that would fix the energy scale for arbitrary values of g. Then the ratio E/ω will be a function of dimensionless coupling constant $\lambda = g/\omega^3$ only. We define ω by the following relations

$$m^2/\omega^2 = 1-2\lambda, \quad \lambda = g/\omega^3 . \tag{1.2}$$

Similar relations for the effective mass and the coupling constant have been used by Caswell[5] with the only difference in the numerical coefficients.

It seems that the relations (1.2) have been introduced *ad hoc*, but we would like to stress, that they appear quite naturally when the $1/N$ -expansion is constructed with the help of path integrals. As will become clear further ω is the splitting between energies of the ground and the first excited states when N tends to infinity.

From eq. (1.2) it is clear that such a parametrization is really convenient for the investigation of different limits. When m^2 is positive and g varies from 0 to ∞, effective coupling constant λ varies from 0 to 1/2 with ω varying from m to $(2g)^{1/3}$. When m^2 is negative, we have the case of a double-well potential and λ varies from 1/2 to ∞. Introducing the dimensionless energy E/ω and coordinates $x_i/\sqrt{\omega}$, we can transform the radial part of the Hamiltonian (1.1) as follows

$$H = -\frac{1}{2}\frac{d^2}{dr^2} + \frac{(N-1)(N-3)}{8r^2} + \frac{1-2\lambda}{2} r^2 + \frac{\lambda}{N} r^4 . \qquad (1.3)$$

In the limit of large N the ground-state energy is given by the minimum of the asymptotic potential in (1.3):

$$E_0/\omega = V_{as}(r_0), \qquad dV_{as}(r_0)/dr_0 = 0,$$

$$V_{as}(r) = \frac{N^2}{8r^2} + \frac{1-2\lambda}{2} r^2 + \frac{\lambda}{N} r^4$$

and we easily find

$$r_0 = \sqrt{(N/2)}, \qquad E_0/\omega = N\frac{2-\lambda}{4}.$$

Shifting the origin of coordinates to the point of the minimum $(r = r_0 + x)$ and redefining the energy $E_0/\omega = N\epsilon$ we obtain the Schrödinger equation in the form suitable for calculating the subsequent coefficients of the $1/N$ -expansion. In this case the wave function is expanded in powers of $N^{-1/2}$

$$\psi = \psi_0 (1 + \sum_{k=1} \psi_k N^{-k/2}), \quad \psi_0 = [4(1+\lambda)/\pi^2]^{1/8} \exp[-(1+\lambda)^{1/2} x^2]$$

while the energy is expanded in integer powers of $1/N$

$$\epsilon = \epsilon_0 + \sum_{k=1} \epsilon_k / N^k, \qquad \epsilon_0 = 1/2 - \lambda/4 . \qquad (1.4)$$

The next part of this paper contains the formulae for the first ϵ_k of the expansion (1.4), which we have derived in this way.

2. The calculations give:

$$\epsilon_0 = \frac{2-\lambda}{4},$$

$$\epsilon_1 = (1+\lambda)^{1/2} - 1,$$

$$\epsilon_2 = \frac{\lambda}{(1+\lambda)^2}\left[\frac{\lambda}{4} + 3 - 3(1+\lambda)^{1/2}\right],$$

$$\epsilon_3 = \frac{\lambda^2}{(1+\lambda)^5}\left[-\frac{27}{2}\lambda^2 + 6\lambda + \frac{39}{2} + (1+\lambda)^{1/2}\left(\frac{27}{16}\lambda^2 + \frac{45}{4}\lambda - \frac{39}{2}\right)\right],$$

$$\epsilon_4 = \frac{\lambda^3}{(1+\lambda)^7}\left[\frac{595}{64}\lambda^3 - \frac{269}{16}\lambda^2 - \frac{6025}{16}\lambda + 270 - (1+\lambda)^{1/2}\left(\frac{1155}{16}\lambda^2 - \frac{885}{2}\lambda + 270\right)\right], \qquad (2.1)$$

$$\epsilon_5 = \frac{\lambda^4}{(1+\lambda)^{10}}\left[-\frac{6945}{16}\lambda^4 + \frac{195489}{32}\lambda^3 - \frac{235509}{32}\lambda^2 - \frac{69789}{8}\lambda + \frac{41433}{8} + (1+\lambda)^{1/2}\left(\frac{59931}{1024}\lambda^4 - \frac{67089}{64}\lambda^3 - \frac{82131}{32}\lambda^2 + \frac{194571}{16}\lambda - \frac{41433}{8}\right)\right],$$

$$\epsilon_6 = \frac{\lambda}{(1+\lambda)^{12}}\left[\frac{1733637}{4096}\lambda^5 - \frac{18403621}{1024}\lambda^4 + \frac{1154631}{32}\lambda^3 + 238245\lambda^2 - \frac{13247963}{32}\lambda + 121104 - (1+\lambda)^{1/2}\right.$$

$$\times \left(\frac{2958357}{1024}\lambda^4 - \frac{10955343}{128}\lambda^3 + \frac{6406191}{16}\lambda^2 - \right.$$

$$\left. - \frac{7385751}{16}\lambda + 121104 \right)],$$

and the ground state energy is expanded as follows

$$E_0/\omega = N\epsilon_0 + \sum_{k=0} \epsilon_{k+1}/N^k . \qquad (2.2)$$

Taking into account the connections of the first excited level with the ground-state energy of the oscillators with a different number of components

$$E_1(N,g) = E_0\left(N+2,\ g\frac{N+2}{N}\right), \qquad (2.3)$$

it is possible to obtain for E_1 an expansion of the form (2.2) with ϵ_k' instead of ϵ_k. The formulae (2.1)-(2.3) then lead to the expressions

$$\epsilon_0' = \frac{2-\lambda}{4},$$

$$\epsilon_1' = (1+\lambda)^{1/2},$$

$$\epsilon_2' = \frac{\lambda}{(1+\lambda)^2}\left[-\frac{3}{4}\lambda + 2 + (1+\lambda)^{1/2}(2\lambda-1)\right],$$

$$\epsilon_3' = \frac{\lambda^2}{(1+\lambda)^5}(1+\lambda)^{1/2}\left(-\frac{165}{16}\lambda^2 + \frac{57}{4}\lambda - \frac{9}{2}\right),$$

$$\epsilon_4' = \frac{\lambda^3}{(1+\lambda)^7}\left[-\frac{333}{64}\lambda^3 + \frac{2707}{16}\lambda^2 - \frac{5713}{16}\lambda + 89 - \right.$$

$$\left. -(1+\lambda)^{1/2}\left(4\lambda^3 - 57\lambda^2 - \frac{63}{2}\lambda + \frac{59}{2}\right)\right],$$

$$\epsilon_5' = \frac{\lambda^4}{(1+\lambda)^{10}}(1+\lambda)^{1/2}\left(\frac{47451}{1024}\lambda^4 + \frac{87561}{64}\lambda^3 - \right.$$

$$\left. - \frac{70809}{8}\lambda^2 + \frac{121521}{16}\lambda - \frac{8253}{8}\right),$$

$$\epsilon_6' = \frac{\lambda^5}{(1+\lambda)^{12}}\left[-\frac{916731}{4096}\lambda^5 + \frac{19485563}{1024}\lambda^4 - \frac{2606457}{16}\lambda^3 + \right.$$

$$+ \frac{5299749}{16}\lambda^2 - \frac{5869337}{32}\lambda + \frac{88251}{4} + (1+\lambda)^{1/2}$$

$$\times\left(24\lambda^5 - \frac{2073}{2}\lambda^4 + \frac{3945}{2}\lambda^3 + \frac{42879}{8}\lambda^2 + \right.$$

$$\left.\left. + \frac{2145}{4}\lambda - \frac{14325}{8}\right)\right].$$

3. Let us dwell on some properties of the coefficients $\epsilon_k(\lambda)$. To find their large-order behaviour, we have applied the general method of Brezin-Le Guillou-Zinn-Justin /6/. This method has already been used for the $1/N$-expansion /4/, where some of its peculiarities in this specific problem were mentioned. The essence of this method lies in calculating the contribution of the complex saddle point, i.e., the corresponding complex instantons to the path integral. We want to stress once more that the suitable choice of the parametrization enables us to obtain the result in closed form:

$$\epsilon_k(\lambda) \underset{k\to\infty}{\sim} \frac{4}{\pi^{3/2}} \frac{(1+\lambda)^{7/4}}{(\sqrt{1+\lambda}+1)^2}(-1)^{k+1}\frac{\Gamma(k-1/2)}{A^{k-1/2}}\sin(k-1/2)\theta,$$

$$A = (Z^2+\pi^2/4)^{1/2}, \quad \theta = \arccos(Z/A), \qquad (3.1)$$

$$Z = \frac{1-2\lambda}{3\lambda}(1+\lambda)^{1/2} + \ln\frac{1+(1+\lambda)^{1/2}}{\lambda^{1/2}}.$$

The validity of formulae (3.1) was verified by the comparison of the asymptotic coefficients with the exact ones obtained numerically for different values of the coupling constant.

Now consider the case of $m^2 > 0$ and $g \to 0$. Then, reexpanding (2.1) in powers of g, we find

$$E_0/m = \frac{N}{2} + \sum_{k=1} A_k(N)\left(\frac{g}{Nm^3}\right)^k , \qquad (3.2)$$

where $A_k(N)$ are the coefficients of the standard perturbative expansion, known for arbitrary N up to $k=14$ (see, e.g., the paper by Dolgov, Eletzsky, Popov /3/). Formulae (2.1) for the coefficients $\epsilon_0(\lambda),\dots,\epsilon_6(\lambda)$ enable us to reproduce correctly $A_k(N)$ with $k=0, 1, \dots, 6$.

When $\lambda \to 0$, coefficients $\epsilon_k(\lambda)$ behave as

$$\epsilon_k(\lambda) = \alpha_k \lambda^k [1 + O(\lambda)]. \qquad (3.3)$$

Consider the properties of α_k. Taking into account that $\omega \simeq m$ and $\lambda \simeq g/m^3$, we obtain from (3.3) and (2.2)

$$E_0/m = \frac{N}{2} + N \sum_{k=1}[\alpha_k + O(N)]\left(\frac{g}{Nm^3}\right)^k . \qquad (3.4)$$

Comparing (3.4) and (3.2) we see that

$$\alpha_k = \lim_{N\to 0} A_k(N)/N$$

because of the fact that $A_k(N)$ are polynomials in N and $A_k(0)=0$.

Since

$$A_k(N) \underset{k\to\infty}{\sim} (-1)^{k+1} \frac{6^{N/2}}{\pi\Gamma(N/2)} \Gamma(k+N/2) 3^k ,$$

we have when $N \to 0$

$$\alpha_k \underset{k\to\infty}{\sim} \frac{(-1)^{k+1}}{2\pi} 3^k \Gamma(k) . \qquad (3.5)$$

Using the exact expression for the ground-state energy when $N=0$, found in Ref. /3/, we can derive the recurrent equations for α_k

$$\alpha_{k+1} = -\alpha_k(3k-1) - \sum_{m=1}^{k} \alpha_m \alpha_{k+1-m}, \quad \alpha_1 = 1/2. \qquad (3.6)$$

These equations can be easily solved numerically, so it was possible to verify the asymptotic formula (3.5). It is worth to note that when deriving it, we first took the limit $\lambda \to 0$, and only then the limit $k \to \infty$. These two limits are noncommutative. Really, when the limit $\lambda \to 0$ in the asymptotic formula (3.1) is taken, we have

$$\epsilon_k(\lambda) \underset{\lambda\to 0}{\sim} \sqrt{3\pi\lambda}\, \left[\frac{(-1)^{k+1}}{2\pi} 3^k \Gamma(k+1/2)\right] \lambda^k \sim$$

$$\sim \sqrt{3\pi\lambda k}\, \left[\frac{(-1)^{k+1}}{2\pi} 3^k \Gamma(k)\right] \lambda^k .$$

Thus, the permutation of the limits leads to the appearance of an additional factor $(3\pi\lambda k)^{1/2}$. It means that in the region $1 \ll k \lesssim 1/3\pi\lambda$ it is necessary to use formulae (3.3, 3.5), and the asymptotic formula (3.1) is satisfactory only for $k \gg 1/3\pi\lambda$.

4. Consider now the double-well potential, i.e., the case $\lambda \to \infty$. The asymptotic formula (3.1) gives then

$$\epsilon_k(\lambda) \underset{\lambda\to\infty}{\sim} \beta_k \lambda^{1-k/2},$$
$$\beta_k \underset{k\to\infty}{\sim} \frac{4}{\pi} \sqrt{\frac{2}{3\pi}} \left(\frac{3}{2}\right)^k \Gamma(k-1/2). \qquad (4.1)$$

This formula seems to reveal the constant sign nature of the corresponding series. The reason for this is that θ of eqs. (3.1) tends to π and $\sin(k-1/2)\theta \to (-1)^{k+1}$. But it is not so for all $k \gg 1$. Since $\theta \sim \pi(1-3/4\sqrt{\lambda})$ when $\lambda\to\infty$, the angle θ cannot be considered equal to π for $k \sim \sqrt{\lambda}$ and the change of sign will take place. As a result, the series in powers of $1/N$ is Borel-summable for arbitrary values of λ.

The exact expressions for $\epsilon_k(\lambda)$ can be used for obtaining the first terms of the expansion in powers of g for the energy levels of an N-component oscillator with negative m^2 (double-well potential). Changing m^2 to $(-m^2)$ we obtain

$$E_0/m = -N\frac{m^3}{16g} + \frac{1}{\sqrt{2}} + \sum_{k=1} B_k(N)\left(\frac{g}{Nm^3}\right)^k \qquad (4.2)$$

with the coefficients $B_k(N)$ given by the expressions:

$$B_1 = N^2/2 - 2N + 1/2,$$

$$B_2 = \sqrt{2}\,(3N^2 - 12N + 27/4),$$

$$B_3 = -N^4 + 8N^3 + 38N^3 - 216N + 595/4, \qquad (4.3)$$

$$B_4 = \sqrt{2}\,(-33N^4 + 264N^3 + 99/2 \cdot N^2 - 2310N + 59931/32),$$

$$B_5 = 8N^6 - 96N^5 - 1288N^4 + 12864N^3 - 12862N^2 -$$

$$-55560N + 1733637/32\,,$$

$$B_6 = \sqrt{2}\,(534N^6 - 6408N^5 - 28335/2\cdot N^4 +$$

$$+ 284220N^3 - 7230315/16\cdot N^2 -$$

$$- 2958357/2\cdot N + 115032927/128).$$

Note that the coefficients β_k, introduced above, are equal to $4B_{k-1}(N=0)\ /\ (2\sqrt{2})^k$ (compare with relations between a_k and A_k).

Using (2.3) we can find a similar expansion for the first excited level, where $B'_k(N)$ appear instead of $B_k(N)$:

$$B'_1 = N^2/2 - 3/2\,,$$

$$B'_2 = \sqrt{2}\,(3N^2 - 21/4),$$

$$B'_3 = -N^4 + 62N^2 - 333/4\,,$$

$$B'_4 = \sqrt{2}\,(-33N^4 + 1683/2\cdot N^2 - 30885/32)\,, \qquad (4.4)$$

$$B'_5 = 8N^6 - 1768N^4 + 27650N^2 - 916731/32,$$

$$B'_6 = \sqrt{2}\,(534N^6 - 92415/2\cdot N^4 + 8462805/16\cdot N^2 -$$

$$- 65518401/128).$$

When $N=1$ all the coefficients B_k are equal to B'_k, as the exponentially small splitting of the levels due to quantum-mechanical tunnelling cannot be reproduced by the perturbation theory.

It is worth to mention that the double-well potential is not investigated so completely as the ordinary anharmonic one. In particular, we do not know formulae describing the asymptotics of $B_k(N)$ when $k\to\infty$ except for the case $N=1$ which has been

studied in Ref. /7/. Formulae (4.3) enable us to guess some of the relations.

At first, $B_k(N)$ can be represented as polynomials in even powers of $(N-2)$, from which the relation

$$B_k(N) = B_k(4-N) \qquad (4.5)$$

follows and, for instance, $B_k(0)=B_k(4)$ and $B_k(1)=B_k(3)$. Then a straightforward substitution leads to the connection between coefficients of the perturbative series in the cases of the double-well potential and ordinary anharmonic oscillators:

$$B_k(1) = (-1)^k\, 2^{-(k+1)/2} A_k(2)\,,$$

$$B_k(2) = (-1)^k\, 2^{(k+1)/2} A_k(1). \qquad (4.6)$$

Formulae (4.5, 4.6) are derived by considering the first few terms of the perturbative series, but the accidental coincidence is almost incredible and we can believe in their validity for all k. We know the asymptotics for $A_k(N)$ and that $B_k(0)$ is connected with β_k of eqs. (4.1), so we can obtain the large-order behaviour of $B_k(N)$ when $N=0,\dots,4$:

$$B_k(0) = B_k(4) \underset{k\to\infty}{\sim} \frac{2\sqrt{3}}{\pi^{3/2}}\,(3\sqrt{2})^k\,\Gamma(k+1/2),$$

$$B_k(1) = B_k(3) \underset{k\to\infty}{\sim} -\frac{3\sqrt{2}}{\pi}\left(\frac{3}{\sqrt{2}}\right)^k \Gamma(k+1), \qquad (4.7)$$

$$B_k(2) \underset{k\to\infty}{\sim} -\frac{2\sqrt{3}}{\pi^{3/2}}\,(3\sqrt{2})^k\,\Gamma(k+1/2).$$

Note that our $B_k(1)$ are related to E_k of Ref. /7/ as follows

$$E_k = 2^{(k-1)/2} B_k(1)\,,$$

so the results of Ref. /7/ are reproduced correctly. As is known, the large-order behaviour of $B_k(1)$ is caused by the contribution of the instanton-anti-instanton pair. It would be interesting to find the classic solutions which would lead to the expressions (4.7).

5. Consider now the summation of the $1/N$ -series taking as an example the ground-state energy. The problem is to sum the series

$$f(1/N) = \sum_{k=0} \epsilon_{k+1} / N^k . \quad (5.1)$$

Using the Borel summation method we get

$$f(1/N) = 2 \int_0^\infty dt \ \exp(-t^2) \, F(t^2/N), \quad (5.2)$$

where

$$F(x) = \sum_{k=0} \epsilon_{k+1} x^k / \Gamma(k+1/2). \quad (5.3)$$

The asymptotic formulae (3.1) enable us to reveal the singularity structure of the function F in the plane of the Borel variable x. It is easy to see that the nearest singularities are located at the points $x = \pm A \exp(\pm i\theta)$ which are out of the integration contour. Then, the integral (5.2) converges and the series (5.1) is Borel summable.

To do the numerical calculations one has to continue the Borel transform F(x) to the whole real axis. To make such a continuation we use the asymmetric Pade-approximation. Keeping in mind the exact expressions (2.1) for six coefficients ϵ_k, our representation takes the form

$$F(x) \sim \sum_{k=0}^{5} \epsilon_{k+1} x^k / \Gamma(k+1/2) = C \frac{1 + d_1 x + d_2 x^2}{1 + b_1 x + b_2 x^2 + b_3 x^3}$$

The results thus obtained are placed in the Figure which represents the relative accuracy $\Delta = |E_0 - E_{exact}| / |E_{exact}|$ as a function of the potential parameters.

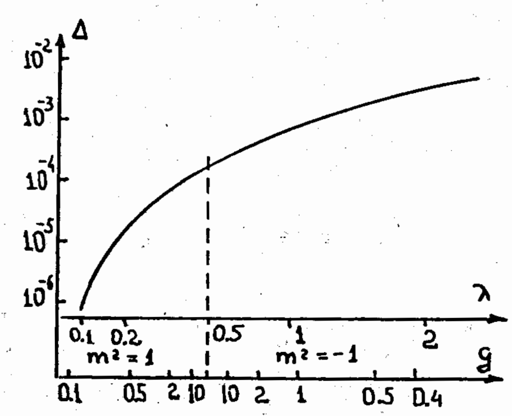

Fig. The relative accuracy of the Pade-Borel reconstructed ground-state energy when seven terms of the 1/N -expansion are taken into account.

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