сообщения
объединенного
института
ядерных
исследований
дубна

E2-81-419

T.D.Palev

NONCANONICAL QUANTIZATION OF TWO PARTICLES INTERACTING VIA A HARMONIC POTENTIAL

1981

I. INTRODUCTION

In the ordinary, canonical quantum mechanics the operators of the Cartesian coordinates $\hat{q}_1,\dots,\hat{q}_n$ and momenta $\hat{p}_1,\dots,\hat{p}_n$, corresponding to a classical system with a Hamiltonian

$$H = \sum_{i=1}^{n} p_i^2/2m_i + U(q_1,\dots,q_n) \tag{1}$$

satisfy the canonical commutation relations (CCR)

$$\begin{aligned} &[\hat{q}_j, \hat{p}_k] = i\hbar\delta_{jk} \\ &[\hat{q}_j, \hat{q}_k] = [\hat{p}_j, \hat{p}_k] = 0. \end{aligned} \tag{2}$$

The quantum canonical variables $\hat{q}_1,\dots,\hat{q}_n, \hat{p}_1,\dots,\hat{p}_n$ possess some interesting algebraical features. First of all, the linear envelope of these operators and the unity is a Lie algebra, the Heisenberg algebra. On the other hand, the position and momentum operators have also well defined Lie superalgebraical properties namely they constitute a basis in the odd part of the orthosymplectic Lie Superalgebra (LS) $B(0,n) = osp(1,2n)$ and generate it [2] (in a Lie-superalgebraical sence). More precisely, $\hat{p}_i$ and $\hat{q}_i$, $i=1,\dots,n$, generate one particular infinite-dimensional irreducible representation of $B(0,n)$, which may be called a Heisenberg representation. Thus, the canonical quantization may be viewed as a mapping

$$q_i \to \hat{q}_i, \quad p_i \to \hat{p}_i, \quad i=1,\dots,n \tag{3}$$

of the classical canonical variables onto a set of operators $\hat{q}_i, \hat{p}_i$ that generate the Heisenberg representation of the LS $B(0,n)$. In view of this property it is natural to aks whether there exists a logically admissible scheme of quantization, based on other representations of the position and momentum operators. The answer is negative for $\hat{p}_i, \hat{q}_i$ being generators of the Heisenberg Lie algebra (2), since the latter has no other (non-equivalent) representations. If, however, a representation means a representation of $\hat{p}_i$ and $\hat{q}_i$ considered as generators of the LS $B(0,n)$, then one can write down several non-equivalent representations. The latter stems from the observation that to every irreducible representation of $B(0,n)$ there corresponds an irreducible representation of the generating algebra operators $\hat{p}_i$ and $\hat{q}_i$ and vice versa. The representations of $B(0,n)$ were recently classified [3]. This LS (and hence $\hat{p}_i$ and $\hat{q}_i$) has infinitely many nonequivalent representations. The more important question is whether all or some of these representations can lead to a selfconsistent quantization.

Before studying this possibility, however, one has to give first of all an independent definition of the concept of (non-canonical) quantization, since, moreover, the word "quantization" is used at present with different meaning [4]. The definition of a quantization we follow is actually due to Wigner [5], who has shown that within a more general frame of this concept the one-dimensional harmonic oscillator can be quantized in several non-canonical ways. In Sec. II we review the Wigner approach and show that the different possibilities for quantizing he has found are in fact different representations of $\hat{p}$ and $\hat{q}$ considered as generators of one and the same Lie superalgebra, the algebra $B(0,1)$. The Wigner quantization of the one-dimensional oscillator was studied in Ref. 6 and recently in great details in Refs. 7,8 and 9.

The Lie-superalgebraical generalization of the results of Wigner to the case of the n-dimensional oscillator is obvious and one easely concludes that the quantization can be performed according to different representations of the LS $B(0, n)$.

A particular feature of the Wigner quantization is its dependence on the dynamics, on the form of the Hamiltonian. The results for the harmonic oscillator cannot be applied, for instance, to quantize a particle in a Coulomb field. Because of the close connection of the approach of Wigner with the simple Lie superalgebra $B(0,n)$, one may call the quantization, adopted in Ref. 5, a B-quantization.

In the present paper we give another example of a non-canonical quantization, which is not, however, B-quantization. We consider a system of two non-relativistic point particles, interacting via harmonic potential. Assuming that the centre of mass variables are quantized in a canonical way and commute with the internal variables, we reduce in the usual way the problem to a three-dimensional harmonic oscillator for the internal degrees of freedom (Sec. III A). In Sec. III B we study a non-canonical quantization for a more general, n-dimensional oscillator. The main algebraical background of the approach considered in this paper is of Lie-superalgebraical nature. The quantization is performed via a mapping of the classical variables $q_1, \ldots, q_n, p_1, \ldots, p_n$ onto position and momentum operators $\hat{q}_1, \ldots, \hat{q}_n, \hat{p}_1, \ldots, \hat{p}_n$ that generate (different irreducible representations of) another simple LS, namely the algebra $A(0, n-1) \equiv sl(1,n)$. Therefore, this quantization may be called A-quantization. It turns out that the energy of the oscillator one can have at most $n+1$ different values. In the case of the two-particle system (Sec. III C) the Hamiltonian commutes with the relative distance operator, which has no more than 4 different eigenvalues. Therefore the particles are confined. On the other hand, the relative coordinates do not commute and therefore the particles cannot be localized on the allowed spheres. For the representations we consider the orbital momentum of the system can be only 0 or 1.

II. WIGNER QUANTIZATION OF A HARMONIC OSCILLATOR

In 1927 Ehrenfest [10] has shown (on an example of one-particle quantum system with Hamiltonian $\hat{H} = \hat{p}^2/2 + \hat{U}(\hat{q})$) that the equations of motion of the classical mechanics, the Hamiltonian equations, considered as operator equations in the Heisenberg picture, namely

$$\dot{\hat{p}}_k = -\frac{\partial \hat{H}}{\partial \hat{q}_k}, \quad \dot{\hat{q}}_k = \frac{\partial \hat{H}}{\partial \hat{p}_k}, \tag{4}$$

are consequence of the quantum equations, e.g., the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi$$

or, equivalently, of the Heisenberg equations (in the Heisenberg representation)

$$\dot{\hat{p}}_k = -\frac{i}{\hbar}[\hat{p}_k, \hat{H}], \quad \dot{\hat{q}}_k = -\frac{i}{\hbar}[\hat{q}_k, \hat{H}]. \tag{5}$$

The inverse also holds, (5) is a consequence of (4). To establish the equivalence of (4) and (5) it is enough to use the equal time canonical commutation relations,

$$[\hat{p}_j, \hat{q}_k] = -i$$
$$[\hat{q}_i, \hat{q}_j] = [\hat{p}_i, \hat{p}_j] = 0. \tag{6}$$

In 1950 Wigner inverted the problem asking whether the condition (6) is also a necessary one, i.e., whether the eqs. (4) and (5) are equivalent <u>if and only if</u> the CCR (6) hold. The equations for the possible $\hat{p}_i$ and $\hat{q}_i$ are obtained from (4) and (5)

$$\frac{\partial \hat{H}}{\partial \hat{q}_k} = \frac{i}{\hbar}[\hat{p}_k, \hat{H}], \quad \frac{\partial \hat{H}}{\partial \hat{p}_k} = -\frac{i}{\hbar}[\hat{q}_k, \hat{H}]. \tag{7}$$

For any solution of eqs. (7) with respect to $\hat{p}_i$ and $\hat{q}_i$ the eqs. (4) and (5) are equivalent. The question of Wigner was whether the only solution was given with the canonical operators (6). On an example of one-dimensional harmonic oscillator with Hamiltonian

$$\hat{H} = \frac{1}{2}(\hat{p}^2 + \hat{q}^2) \tag{8}$$

he has shown that this is not the case, so that the "classical" eqs. (4) and the quantum eqs. (5) are consequence of each other for several non-canonical operators $\hat{p}$ and $\hat{q}$. In view of this it is logically justified to ascribe physical meaning to any solution $\hat{p}_i, \hat{q}_i$ of (7) as to position and momentum operators and to study the properties of the corresponding generalized quantum mechanics. Thus, by a quantization of a classical system we shall understand a mapping

$$q_i \to \hat{q}_i, \quad p_i \to \hat{p}_i, \quad i = 1,\dots,n, \tag{9}$$

which replaces the classical canonical variables q_i, p_i by position and momentum operators $\hat{q}_i, \hat{p}_i$ in the above generalized sence. The properties of $\hat{q}_i$ and $\hat{p}_i$ will depend in general on the Hamiltonian (1), i.e., on the potential $U(q_1,\dots,q_n)$. Hence, the quantization under consideration is not of a geometrical origin, but rather of a dynamical one. Therefore, we refer to it as to a dynamical quantization.

The irreducible inequivalent representations of $\hat{p}$ and $\hat{q}$ found in Ref.5 are labelled by one continuous parameter E_0; the corresponding representation spaces $W(E_0)$ are infinite-dimensional. If $|E_0;n\rangle$, $n=1,2,\dots$ is a basis in $W(E_0)$, then

$$\begin{aligned} \hat{q}|E_0;n\rangle &= x_{n-1,n}|E_0;n-1\rangle + x_{n,n+1}|E_0,n+1\rangle, \\ \hat{p}|E_0;n\rangle &= -ix_{n-1,n}|E_0;n-1\rangle + ix_{n,n+1}|E_0,n+1\rangle, \end{aligned} \tag{10}$$

where

$$x_{n,n+1} = (E_0 + n/2)^{1/2} \quad \text{for even } n$$

$$x_{n,n+1} = (n/2 + 1/2)^{1/2} \quad \text{for odd } n.$$

Only in the case $E_0 = 1/2$ the operators $\hat{p}$ and $\hat{q}$ satisfy the CCR (6). Otherwise they do not generate any finite-dimensiobal Lie algebra. It turns out, however, that $\hat{p}$ and $\hat{q}$ generate a finite-dimensional Lie superalgebra. To establish this it

is convenient to introduce the creation ($\xi=+$) and annihilation ($\xi=-$) operators (CAO's),

$$a^\xi = \frac{1}{\sqrt{2}}(\hat{q} - i\xi\hat{p}). \tag{11}$$

Then

$$\hat{H} = \frac{1}{2}\{a^+, a^-\} \tag{12}$$

and the CAO's transform the basis vectors as follows

$$\begin{aligned} a^-|E_0;2n\rangle &= (2n)^{1/2}|E_0;2n-1\rangle, \\ a^-|E_0;2n+1\rangle &= (2n+2E_0)^{1/2}|E_0;2n\rangle, \\ a^+|E_0;2n\rangle &= (2n+2E_0)^{1/2}|E_0;2n+1\rangle, \\ a^+|E_0;2n+1\rangle &= (2n+2)^{1/2}|E_0;2n+2\rangle. \end{aligned} \tag{13}$$

By a straightforward computation one shows that in any representation space $W(E_0)$ the operators a^+ and a^- satisfy one and the same relations

$$[\{a^\xi, a^\eta\}, a^\epsilon] = (\epsilon - \xi)a^\eta + (\epsilon - \eta)a^\xi. \tag{14}$$

Here and throughout the paper $\xi, \eta, \epsilon, \delta = \pm$ or ± 1; $[x,y] = xy - yx$ and $\{x,y\} = xy + yx$. From (14) one also derives

$$\begin{aligned} [\{a^\xi, a^\eta\}, \{a^\epsilon, a^\delta\}] &= (\epsilon - \xi)\{a^\eta, a^\delta\} + \\ &+ (\epsilon - \eta)\{a^\xi, a^\delta\} + (\delta - \xi)\{a^\eta, a^\epsilon\} + (\delta - \eta)\{a^\xi, a^\epsilon\}. \end{aligned} \tag{15}$$

Therefore, the linear envelope B of the operators $a^\pm$, $(a^\pm)^2$ and $\{a^+, a^-\}$ is a Lie superalgebra with a^+, a^- as a basis in the odd part B_1 and a three-dimensional even subalgebra

$$B_0 = \text{lin. env.}\{(a^+)^2, (a^-)^2, \{a^+, a^-\}\}, \tag{16}$$

which is isomorphic to the Lie algebra $sp(2) \simeq sl(2)$. One can check that B is isomorphic to the 5-dimensional orthosymplectic LS $osp(1,2) \equiv B(0,1)$. Thus, the representations (10) of $\hat{p}$ and $\hat{q}$, found by Wigner, define a class of infinite-dimensional irreducible representations of the LS $B(0,1)$. To every such representation there corresponds a selfconsistent generalization of the ordinary quantization of the oscillator with momentum and position operators, that are not unitarily equivalent to the canonical $\hat{p}$ an $\hat{q}$.

Consider now n non-interacting oscillators with a Hamiltonian

$$\hat{H} = \sum_{j=2}^{n} \left(\frac{1}{2m_j}\hat{p}_j^2 + \frac{m_j\,\omega_j^2}{2}\hat{q}_j^2\right). \tag{17}$$

In terms of the CAO's

$$a_k^{\xi} = \left(\frac{m_k\omega_k}{2\hbar}\right)^{1/2}\hat{q}_k - i\xi(2m_k\omega_k\hbar)^{-1/2}\,\hat{p}_k \tag{18}$$

the eqs. (7) read

$$\sum_{i=1}^{n}\frac{\omega_i\hbar}{2}[\{a_i^+, a_i^-\}, a_k^{\xi}] = \xi\frac{\omega_k\hbar}{2}a_k^{\xi}, \quad \xi = \pm. \tag{19}$$

One (among several other) solution of eqs. (19) is given with operators, which are straightforward generalization of the relation (14),

$$[\{a_i^{\xi}, a_j^{\eta}\}, a_k^{\epsilon}] = (\epsilon-\xi)\delta_{ik}a_j^{\eta} + (\epsilon-\eta)a_i^{\xi}. \tag{20}$$

These operators are well known in the quantum field theory. They were introduced by Green/11/ as a possible generalization of the statistics of the integer-spin fields and are called paraBose operators. The latter, considered as odd elements, generate a LS /12/

$$\text{lin. env.}\{a_i^{\xi}, \{a_j^{\eta}, a_k^{\epsilon}\} \mid i,j,k = 1,\dots,n;\ \xi,\eta,\epsilon = \pm\}, \tag{21}$$

which is isomorphic to the simple LS $B(0,n)$ /1/. Hence, one can quantize n classical oscillators with Hamiltonian (17), using those representations of $B(0,n)$, for which $\hat{p}_i$ and $\hat{q}_i$ are hermitian operators. If, in addition, one requires the ground state to be non-degenerate, then one has to consider only the Fock representations of the paraBose operators. The latter are labelled by one positive integer, the order of the statistics /13,14/. The representations with degenerate ground states were studied in Ref.15.

We conclude that the Wigner quantization is actually a quantization with paraBose operators. Therefore, it is generalizing the quantum mechanics along the same line as the paraBose statistics extends the quantum field theory (or, may be, the other way around, since the paper of Wigner was published earlier).

III. DYNAMICAL QUANTIZATION OF TWO POINT PARTICLES INTERACTING VIA HARMONIC POTENTIAL

A. Reduction of the Problem

Consider in the frame of the non-relativistic mechanics two point particles with masses m_1 and m_2 and a Hamiltonian

$$H_{tot} = \frac{1}{2}m_1\dot{\vec{r}}_1^{\,2} + \frac{1}{2}m_2\dot{\vec{r}}_2^{\,2} + U(|\vec{r}_1 - \vec{r}_2|). \tag{22}$$

Introduce the centre of mass (CM) coordinates

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}, \quad \vec{r} = \vec{r}_1 - \vec{r}_2 \tag{23}$$

and let μ and m be the total and the reduced masses; $\vec{P} = \mu\dot{\vec{R}}$ and $p = m\dot{r}$ be the total momentum and the internal (the conjugate to $\vec{r}$) momentum, resp.; $r = |\vec{r}_1 - \vec{r}_2|$. Then the energy is a sum of the CM-energy H_{cm} and the internal energy H,

$$H_{tot} = H_{cm} + H, \tag{24}$$

where

$$H_{cm} = \frac{\vec{P}^2}{2\mu}, \quad H = \frac{\vec{p}^2}{2m} + U(r). \tag{25}$$

Similarly, the angular momentum

$$\vec{M}_{tot} = \vec{M}_{cm} + \vec{M} \tag{26}$$

with

$$\vec{M}_{cm} = \vec{R}\times\vec{P}, \quad \vec{M} = \vec{r}\times\vec{p}. \tag{27}$$

According to the definition we have accepted, to quantize the system means to find simultaneous solutions of the Hamiltonian equations, replacing in them the classical variables $\vec{R}$, $\vec{P}$, $\vec{r}$, $\vec{p}$ by operators, i.e.,

$$\dot{\hat{\vec{P}}} = -\frac{\partial\hat{H}_{tot}}{\partial\hat{\vec{R}}}, \quad \dot{\hat{\vec{R}}} = \frac{\partial\hat{H}_{tot}}{\partial\hat{\vec{P}}}, \tag{28}$$

$$\dot{\hat{\vec{p}}} = -\frac{\partial\hat{H}_{tot}}{\partial\hat{\vec{r}}}, \quad \dot{\hat{\vec{r}}} = \frac{\partial\hat{H}_{tot}}{\partial\hat{\vec{p}}} \tag{29}$$

and of the Heisenberg equations

$$\dot{\hat{\vec{P}}} = -\frac{i}{\hbar}[\hat{\vec{P}}, \hat{H}_{tot}], \quad \dot{\hat{\vec{R}}} = -\frac{i}{\hbar}[\hat{\vec{R}}, \hat{H}_{tot}], \tag{30}$$

$$\dot{\hat{\vec{p}}} = -\frac{i}{\hbar}[\hat{\vec{p}}, \hat{H}_{tot}], \qquad \dot{\hat{\vec{r}}} = -\frac{i}{\hbar}[\hat{\vec{r}}, \hat{H}_{tot}] . \tag{31}$$

The operators $\hat{\vec{R}}$, $\hat{\vec{P}}$, $\hat{\vec{r}}$, $\hat{\vec{p}}$ should be determined to be solution of the above eqs. (28-31). By $\hat{H}_{tot}$ we denote the operator, obtained from the classical Hamiltonian H_{tot} after the replacement

$$(\vec{R}, \vec{P}, \vec{r}, \vec{p}) \to (\hat{\vec{R}}, \hat{\vec{P}}, \hat{\vec{r}}, \hat{\vec{p}}) . \tag{32}$$

Independently of the dynamics, the eqs. (28-31) are satisfied with canonical operators. We wish to study some other, dynamically-dependent solutions. Our purpose is not the investigation of the class of all possible operators (32), that satisfy the eqs. (28-31). On the contrary, we shall restrict ourselves only to solutions, which are closely connected to the Lie superalgebra from the class A (for the internal variables). To this end we first assume that the CM-observables can be measured simultaneously with the internal observables. Thus, we accept

Assumption 1. The CM-variables commute with the internal variables, i.e.,

$$[\hat{\vec{R}}, \hat{\vec{r}}] = [\hat{\vec{R}}, \hat{\vec{p}}] = [\hat{\vec{P}}, \hat{\vec{r}}] = [\hat{\vec{P}}, \hat{\vec{p}}] = 0 . \tag{33}$$

Under this assumption the quantization equations resolve into two independent groups. The first one, consisting of eqs. (28) and (30), depends only on the CM-coordinate operator $\hat{R}$ and the momentum operator $\hat{P}$. At this point we make the

Assumption 2. The centre of mass coordinates and momenta are quantized in a canonical way,

$$\begin{aligned} &[\hat{R}_j, \hat{P}_k] = i\hbar\delta_{jk} , \\ &[\hat{R}_j, \hat{R}_k] = [\hat{P}_j, \hat{P}_k] = 0 . \end{aligned} \tag{34}$$

Thus, we are left with the equations

$$\dot{\hat{\vec{p}}} = -\frac{\partial\hat{H}}{\partial\vec{r}}, \qquad \dot{\hat{\vec{r}}} = \frac{\partial\hat{H}}{\partial\vec{p}}, \tag{35}$$

$$\dot{\hat{\vec{p}}} = -\frac{i}{h}[\hat{\vec{p}}, \hat{H}], \qquad \dot{\hat{\vec{r}}} = -\frac{i}{h}[\hat{\vec{r}}, \hat{H}] \tag{36}$$

for the operators $\vec{r}$ and $\vec{p}$, which follow from eqs. (29), (31) and (33).

The eqs. (35) and (36) coincide with the Hamiltonian and the Heisenberg equations of one particle with a Hamiltonian $\hat{H} = \hat{p}^2/2 + U(\hat{r})$. Thus, the problem we are left with is to quantize one point particle moving in a central potential $U(r)$. We shall consider as an example a dynamical quantization only for the case of a harmonic potential,

$$U(r) = \frac{m\omega}{2}\sum_{i=1}^{3} r_i^2 . \tag{37}$$

In the next Sec.III B we quantize dynamically the more general n-dimensional oscillator.

B. A-Quantization of n-Dimensional Harmonic Oscillator

Consider an n-dimensional harmonic oscillator with a Hamiltonian

$$H = \sum_{i=1}^{n} \left(\frac{1}{2m} p_i^2 + \frac{m\omega^2}{2} r_i^2\right) . \tag{38}$$

To quantize it we have to replace the canonical phase-space coordinates $(r_1, \dots, r_n, p_1, \dots, p_n)$ by operators $(\hat{r}_1, \dots, \hat{r}_n, \hat{p}_1, \dots, \hat{p}_n)$, which should satisfy the operator Hamiltonian equations $(i = 1, \dots, n)$

$$\dot{\hat{p}}_i = -m\omega^2 \hat{r}_i , \qquad \dot{\hat{r}}_i = \frac{\hat{p}_i}{m} \tag{39}$$

and the Heisenberg equations

$$\dot{\hat{p}}_i = -\frac{i}{\hbar}[\hat{p}_i, \hat{H}], \qquad \dot{\hat{r}}_i = -\frac{i}{\hbar}[\hat{r}_i, \hat{H}] . \tag{40}$$

Eliminating the time derivatives, one concludes that the "classical" and the quantum equations (39) and (40) can be compatible only if

$$\begin{aligned} &[\hat{H}, \hat{p}_k] = i\hbar m\omega^2 \hat{r}_k , \\ &[\hat{H}, \hat{r}_k] = -\frac{i\hbar}{m} \hat{p}_k . \end{aligned} \tag{41}$$

For simplicity we introduce in place of $\hat{r}_i, \hat{p}_i$, $i = 1, \dots, n$ new operators

$$a_k^{\xi} = \left[\frac{(n-1)m\omega}{4\hbar}\right]^{1/2} \hat{r}_k + i\xi\left[\frac{n-1}{4m\omega\hbar}\right]^{1/2} \hat{p}_k , \tag{42}$$

which will be called also creation ($\xi = +$) and annihilation ($\xi = -$) operators (CAO's). In terms of these operators the Hamiltonian (38) and the compatibility conditions (41) read

$$\hat{H} = \frac{\omega\hbar}{n-1}\sum_{i=1}^{n}\{a_i^+ , a_i^-\}, \qquad (43)$$

$$\sum_{i=1}^{n}[\{a_i^+, a_i^-\}, a_k^{\xi}] = -\xi(n-1)\, a_k^{\xi}. \qquad (44)$$

As a solution of the eqs. (44) we choose operators $a_1^{\pm},\dots,a_n^{\pm}$, satisfying the relations

$$\begin{aligned} &[\{a_i^+, a_j^-\}, a_k^+] = \delta_{jk}\, a_i^+ - \delta_{ij}\, a_k^+ \\ &[\{a_i^+, a_j^-\}, a_k^-] = -\delta_{ik}\, a_j^- + \delta_{ij}\, a_k^-, \\ &\{a_i^+, a_j^+\} = \{a_i^-, a_j^-\} = 0. \end{aligned} \qquad (45)$$

These operators were introduced and studied in Refs. 16 and 17. We review shortly some of their properties. The CAO's (45) constitute a basis in the odd part of the special linear Lie superalgebra $A(0,n-1) \equiv sl(1,n)$. The generators

$$e_{ij} = \{a_i^+, a_j^-\}, \quad i,j = 1,\dots,n \qquad (46)$$

span a basis in the even part, which is isomorphic to the general linear Lie algebra $gl(n)$. To underline the link between the LS $A(0,n-1)$ and the operators (45) we call the latter A-operators and the corresponding quantization - an A-quantization.

To write down explicit formulae for the matrix elements of the A-operators is the same problem as to give explicit expressions for the generators of the LS $A(0,n-1)$. At present such formulae do not exist for any (and even for any classified [3]) representation. In Ref. 17 we have studied a class of irreducible representations, which were obtained with the usual for the quantum theory Fock space technique. These Fock representations are labelled by one non-negative integer, the order of the statistics $p=0,1,2,\dots$ As an orthonormed basis in the irreducible representation space $W(n;p)$ one can choose the vectors

$$|p;\Theta_1,\dots,\Theta_n) = (p!)^{-1/2}\Big(\big(p-\sum_{i=1}^{n}\Theta_i\big)!\Big)^{1/2}(a_1^+)^{\Theta_1}\dots(a_n^+)^{\Theta_n}|0>, \qquad (47)$$

where $\Theta_i = 0,1$ and $\sum_{i=1}^{n}\Theta_i \le p$.

The CAO's transform the basis vectors according to

$$\begin{aligned} a_k^-|\dots,\Theta_k,\dots) &= \Theta_k(-1)^{\Theta_1+\dots+\Theta_{k-1}}\big(p-\sum_i\Theta_i+1\big)^{1/2}|\dots,\Theta_k-1,\dots), \\ a_k^+|\dots,\Theta_k,\dots) &= (1-\Theta_k)(-1)^{\Theta_1+\dots+\Theta_{k-1}}\big(p-\sum_i\Theta_i\big)^{1/2}|\dots,\Theta_k+1,\dots). \end{aligned} \qquad (48)$$

The rest of the generators (46) can be computed from (46) easily. One can check that within any Fock space the hermitian conjugate of a_i^- equals a_i^+, $(a_i^-)^* = a_i^+$, so that $\hat{r}_i$ and $\hat{p}_i$ are hermitian operators. The Hamiltonian (43) is diagonal in the basis (47). To show this, call the vector $|p;\Theta_1,\dots,\Theta_n) \in W(n,p)$ an m-state and denote it as $|p;m>$ if $\sum_i\Theta_i = m$. Then from (43) and (48) one obtains

$$\hat{H}|p;m> = E_m|p;m>, \qquad (49)$$

where

$$E_m = \frac{\omega\hbar}{n-1}(np - nm + m). \qquad (50)$$

Since m can run only the values $0,1,\dots,\min(n,p)$, the energy of the n-dimensional oscillator with order of the statistics p has $\min(n,p)$ different values. The dimension of the subspace $W_m(n;p)$ of all m-states is

$$\dim W_m(n;p) = \binom{n}{m}, \qquad (51)$$

so that the different (linearly independent) states with energy E_m are $\binom{n}{m}$. In particular the state $|p;0>$ with the highest energy is non-degenerate. A given ground state $|p;\min(n,p)>$ is non-degenerate only if $p \ge n$.

We recall that all considerations are in the Heisenberg picture. The CAO's depend on time, $a_i^{\xi}(t)$, and they have to satisfy also the Hamiltonian equations (39), which in terms of the CAO's read

$$\dot{a}_k^{\xi}(t) = -i\xi\omega\, a_k^{\xi}(t). \qquad (52)$$

Hence,

$$a_k^{\xi}(t) = \exp(-i\xi\omega t)\, a_k^{\xi}(0) \qquad (53)$$

and, if the defining relations (45) for the A-operators hold at a certain time $t=0$, i.e., for $a_i^{\xi} = a_i^{\xi}(0)$, then they hold as equal time relations for any other time t. One easily checks that the Heisenberg equations

$$\dot{a}_k^{\xi}(t) = \frac{i\omega}{n-1}\sum_{j=1}^{n} [\{a_j^+(t), a_j^-(t)\}, a_k^{\xi}(t)] \tag{54}$$

agree at any time with the Hamiltonain equations (52).

From (53) it follows that the generators (46) of $gl(n)$, i.e., of the even part of $A(0,n-1)$ are preserved quantities. Since the m-subspaces $W_m(n;p)$ are invariant and irreducible under $gl(n)$,

$$e_{ij}|p;m\rangle \in W_m(n;p),$$

the corresponding group $GL(n)$ is an invariance group of the Hamiltonian. In the same time the odd generators are shifting the energy so that starting with a given energy state one can obtain a state with any other energy from the spectrum of H. Therefore, the LS $A(0,n-1)$ appear as a spectrum generating algebra of the n-dimensional oscillator.

C. Quantization of the Two Particle System

We shall apply the results of the previous section to quantize the internal motion of the two particle system with potential (37). In this case $n=3$ and in terms of the A-operators

$$a_k^{\xi} = (2\hbar)^{-1/2}(m\omega)^{1/2}\hat{r}_k + i\xi(2m\omega\hbar)^{-1/2}\hat{p}_k \tag{55}$$

the internal Hamiltonian reads

$$\hat{H} = \frac{1}{2m}\hat{\vec{p}}^2 + \frac{m\omega}{2}\hat{\vec{r}}^2 = \frac{\omega\hbar}{2}\sum_{i=1}^{3}\{a_i^+, a_i^-\}. \tag{56}$$

For the operator of the squared distance between the particles $\hat{\vec{r}}^2 = \hat{r}_1^2 + \hat{r}_2^2 + \hat{r}_3^2$ and the squared internal momentum $\hat{\vec{p}}^2 = \hat{p}_1^2 + \hat{p}_2^2 + \hat{p}_3^2$ one obtains

$$\hat{\vec{r}}^2 = \frac{\hbar}{2m\omega}\sum_{i=1}^{3}\{a_i^+, a_i^-\}, \tag{57}$$

$$\hat{\vec{p}}^2 = \frac{m\omega\hbar}{2}\sum_{i=1}^{3}\{a_i^+, a_i^-\}. \tag{58}$$

The operator $\sum_{i=1}^{3}\{a_i^+, a_i^-\}$ is element from the centre of $gl(3)$ and therefore $\hat{H}$, $\hat{\vec{r}}^2$ and $\hat{\vec{p}}^2$ commute with each other and with any other element from the universal enveloping algebra of $gl(3)$.

The linear envelope of the components of the orbital momentum (27)

$$\hat{M}_k = \frac{i\hbar}{2}\sum_{\ell,m}\epsilon_{k\ell m}\{a_\ell^+, a_m^-\} \tag{59}$$

gives (as a real algebra) the algebra $so(3)$. Remark that, contrary to the canonical case, the momentum is measured in units $h/2$. The square of the momentum

$$\hat{\vec{M}}^2 = \frac{\hbar^2}{2}\sum_{i,j,k}(\epsilon_{ijk})^2\{a_j^+, a_k^-\}\{a_k^+, a_j^-\} \tag{60}$$

is the Casimir operator of $so(3)$ and hence all operators

$$\hat{H}, \hat{\vec{r}}^2, \hat{\vec{p}}^2, \hat{\vec{M}}^2, \hat{M}_3 \tag{61}$$

commute with each other and can be measured simultaneously. If $|p;k\rangle$ is a k-state in a representation with order of the statistic p, then

$$\hat{H}|p;k\rangle = \frac{\omega\hbar}{2}(3p-2k)|p;k\rangle$$

$$\hat{\vec{r}}^2|p;k\rangle = \frac{\hbar}{2m\omega}(3p-2k)|p;k\rangle \tag{62}$$

$$\hat{\vec{p}}^2|p;k\rangle = \frac{m\omega\hbar}{2}(3p-2k)|p;k\rangle$$

$$\hat{\vec{M}}^2|p;k\rangle = 0 \quad \text{for } k=0,3$$
$$= \frac{\hbar^2}{2} \quad \text{for } k=1,2.$$

There is only one state, the state $|p;0,0,0\rangle$ corresponding to the maximum distance between the particles and the maximum of the internal energy

$$r_{max} = \left(\frac{3\hbar p}{2m\omega}\right)^{1/2}, \quad E_{max} = \frac{3}{2}\omega\hbar p. \tag{63}$$

This state carries momentum zero. If $p \geq 3$, then $|p;1,1,1\rangle$ is a ground state; it is non-degenerate, with zero momentum and corresponds to the minimal distance and energy

$$r_{min} = \left(\frac{3\hbar(p-2)}{2m\omega}\right)^{1/2}, \quad E_{min} = \frac{3}{2}\omega\hbar(p-2). \tag{64}$$

If, however, $p=1$ or 2, then the ground state is degenerate; there are different states with the same energy and momentum 1 (in units $\hbar/2$). For all possible irreducible representations of the LS $A(0,2)$, i.e., for a fixed order of the statistics p the distance between the particles is restricted from above, the particles are confined.

The circumstance that the internal distance can be measured simultaneously with the energy does not contradict the uncertainty relation. The operator of every coordinate can be diagonalized. Since, however, the Cartesian coordinates do not commute with each other, they cannot be diagonalized all together. Thus, the two particles are moving as the ends of a stick with a certain length, which orientation in the space cannot be localized.

We have considered a very restricted class of representations of $\vec{r}$ and $\vec{p}$, corresponding to the Fock representations of the generalized creation and annihilation operators (55). Since the Hamiltonian and the relative distance are elements from the center of the even subalgebra $gl(3)$, in the arbitrary representation of $A(0,2)$ the eigenvalue of $\hat{H}$ (resp. $\hat{\vec{r}}^2$) will be the same within every $gl(3)$ multiplet. Therefore, the spectrum of H and $\vec{r}^2$ will be finite also in the general case (if $\hat{\vec{r}}$ and $\hat{\vec{p}}$ are hermitian operators) and in fact will have no more than four different values. The momentum can be arbitrary with integer and half-integer values within one representation of $A(0,2)$.

The realization of the creation and annihilation operators (and hence of $\vec{p}$ and $\vec{r}$) within the same Lie superalgebra is not unique. In Ref. 18 we have considered another realization in the framework of the quantum field theory, leading to infinite-dimensional Fock representations.

I would like to thank Professor H.D.Doebner for the valuable discussions and suggestions he made and also for the warm hospitality at the Institute of Theoretical Physics in Clausthal-Zellerfeld.

REFERENCES

1. See, for instance, Gantchev A., Palev T. J.Math.Phys., 1980, 21, p.797, where the Lie-superalgebraical properties are studied in the more general context of paraBose operators. The Bose operators are expressed by $\hat{p}_i$ and $\hat{q}_i$ through the relations (18).
2. Throughout the paper we use the notation from Kac V.G. Adv.Math., 1977, 26, p.8 for the simple Lie-superalgebras.
3. Kac V.G. Lecture Notes in Math., 1978, 676, p.597.
4. See, for instance, Segal I.E. Field Quantization. Proc. Conf. Perspectives in Field Theory. Stockholm, Sept., 1980.
5. Wigner E.P. Phys.Rev., 1950, 77, p.711.
6. Yang L.M., Phys.Rev., 1951, 84, p.788.
7. Ohnuki Y., Kamefuchi S. J.Math.Phys., 1978, 19, p.67.

8. Sharma J.K., Mehta C.L., Sudarshan E.C.G. J.Math.Phys., 1978, 19, p.2089.
9. Mukunda N. et al. J.Math.Phys., 1980, 21, p.2386.
10. Ehrenfest P. Zeits. f.Phys., 1927, 45, p.455.
11. Green H.S. Phys.Rev., 1953, 90, p.270.
12. Omote M., Ohnuki Y., Kamefuchi S. Progr.Theor.Phys., 1976, 56, p.1948.
13. Jordan T.F., Mukunda N., Pepper S.V. J.Math.Phys., 1963, 4, p.1089.
14. Greenberg O.W., Messiah A.M.L. Phys.Rev., 1965, 138B, p.1155.
15. Palev T.D. Ann.Inst.Henri Poincare, 1975, 23, p.49.
16. Palev T.D. J.Math.Phys., 1980, 21, p.1293.
17. Palev T.D. JINR, P2-11943, Dubna, 1978.
18. Palev T.D. J.Math.Phys., 1980, 21, p.2560.

Received by Publishing Department
on June 18 1981.