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**COVARIANT  
THREE-DIMENSIONAL EQUATION  
FOR FERMION-ANTIFERMION SYSTEM.**

**III. Formulation  
in Relativistic Configurational Representation**

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## 1. Introduction

In two previous papers /1/ a covariant three-dimensional formalism was developed in the momentum space to describe a system of two spin 1/2 particles. It is based on the obtained in /1/ relativistic three-dimensional quasipotential /2/ equation

$$2\Delta_{P,m_2}^0 (M - 2\Delta_{P,m_2}^0) \Psi_{\gamma, m_1}^{\sigma_1 \sigma_2} (\vec{\Delta}_{P,m_2}) = \frac{1}{(2\pi)^3} \sum_{\gamma_1, \gamma_2 = \pm 1/2} \int \frac{d^3 \vec{\Delta}_{K,m_2}}{2\Delta_{K,m_2}^0} \quad (1.1)$$

$$\cdot \sqrt{\gamma_1 \gamma_2} (\vec{\Delta}_{P,m_2} ; \vec{\Delta}_{K,m_2} ; P^2) \cdot \Psi_{\gamma, m_1}^{\gamma_1 \gamma_2} (\vec{\Delta}_{K,m_2})$$

that is a covariant generalization of the equation /3/ obtained earlier for the wave function of a system of two fermions with equal masses  $m_1 = m_2 = m$ . An analogous covariant equation was derived in the momentum space /4/ for a covariantly defined single-time wave function arising in the Logunov-Tavkhelidze quasipotential approach /2/.

As is mentioned in /5/ for the particles of spin 1/2 two-particle equations derived in the single-time Logunov-Tavkhelidze approach /2/ and in the diagram technique of Kadyshevsky /3/ coincide. Therefore methods we present below can be used in both the approaches (see also /6/).

In equation (1.1) vectors  $\vec{\Delta}_{P,m_2}$  and  $\vec{\Delta}_{K,m_2}$  are covariant generalizations of vectors of the particle momenta in the c.m.s. before  $\vec{p}_1 = -\vec{p}_2 = \vec{p}$  and after scattering  $\vec{k}_1 = -\vec{k}_2 = \vec{k}$  introduced earlier in /7,8/ with the help of the relations  $(\Lambda_{\mathcal{P}}(M, \vec{p})) = (\mathcal{P}^0, \vec{p}) ; \Lambda_{\mathcal{P}}^{\mu} = \mathcal{P}^{\mu} / \sqrt{\mathcal{P}^2}$ .

$$\vec{p} \equiv \vec{\Delta}_{P,m_2} = (\Lambda_{\mathcal{P}}^{-1} p_1) = - (\Lambda_{\mathcal{P}}^{-1} p_2) = -\vec{\Delta}_{P_2,m_2},$$

$$\vec{k} \equiv \vec{\Delta}_{K,m_2} = (\Lambda_{\mathcal{P}}^{-1} k_1) = - (\Lambda_{\mathcal{P}}^{-1} k_2) = -\vec{\Delta}_{K_2,m_2}.$$

Time components are defined by

$$\vec{p}_0 = \Delta_{p, m_2 p}^0 = \sqrt{m^2 + (\vec{\Delta}_{p, m_2 p})^2}; \quad \vec{k}_0 = \Delta_{k, m_2 p}^0 = \sqrt{m^2 + (\vec{\Delta}_{k, m_2 p})^2}. \quad (1.3)$$

The aim of this paper is to transform the spin equation (1.1) into a covariant equation in the relativistic configurational representation (introduced earlier for the c.m.s. case in /9/). A similar transformation was made for spinless covariant equations derived within the Hamiltonian formulation of quantum field theory /10/ and in the single-time approach /11/.

In /1/ the wave function of the system with mass  $M$ , total momentum  $\vec{P}$  and total moment  $\mathcal{J}$  ( $m_y$  is the projection of  $\mathcal{J}$  onto axis  $\bar{y}$ ), composed of a fermion and antifermion with momenta  $p_1$  and  $p_2$  ( $\sigma_1$  and  $\sigma_2$  are their polarization indices), was defined as a matrix element of the operator  $R(\alpha\tau)$ , that off energy shell obeys an evolution equation obtained in /12/ and on energy shell it coincides with the scattering amplitude ( $\mathcal{S} = 1 + iR(0)$ ):

$$\frac{\langle p_1 \sigma_1; p_2 \sigma_2 | R(\alpha\tau) | \vec{P}, M; \mathcal{J}, m_y \rangle}{2\Delta_{p, m_2 p}^0 (M - 2\Delta_{p, m_2 p}^0 - i\epsilon)} = (2\pi)^4 \delta^{(4)}(\vec{P} - p_1 - p_2 - \alpha\tau) \quad (1.4)$$

$$\cdot (2P_0 \cdot 2p_{10} \cdot 2p_{20})^{-4/2} \cdot \Psi_{\mathcal{J}, m_y}^{\sigma_1 \sigma_2}(p_1, p_2 | \vec{P}, \alpha\tau)$$

$$2\Delta_{p, m_2 p}^0 = \sqrt{s_p} = \sqrt{(p_1 + p_2)^2}.$$

The quasipotential  $V_{\sigma_1 \sigma_2}^{\sigma_1 \sigma_2}$  is constructed out of matrix elements of the two-particle elastic scattering amplitude, i.e., out of matrix elements  $\langle \vec{p}_1 \sigma_1, \vec{p}_2 \sigma_2 | R(\alpha\tau) | \vec{k}_1 \sigma_1, \vec{k}_2 \sigma_2 \rangle$  of the operator  $R(\alpha\tau)$ .

For  $V_{\sigma_1 \sigma_2}^{\sigma_1 \sigma_2}$  corresponding to the one-pion exchange the following representation was found /1,13/:

$$\begin{aligned} & V_{\sigma_1 \sigma_2}^{\sigma_1 \sigma_2}(p_1, p_2; \alpha\tau | k_1, k_2; \alpha\tau) \equiv \langle p_1 \sigma_1; p_2 \sigma_2 | V(\alpha\tau, \alpha\tau) | k_1 \sigma_1; k_2 \sigma_2 \rangle \\ & = \sum_{\sigma'_1, \sigma'_2 = \pm 1/2} D_{\sigma_1 \sigma'_1}^{+1/2} \{V^{-1}(\Lambda_{p_1, p_1})\} D_{\sigma_2 \sigma'_2}^{+1/2} \{V^{-1}(\Lambda_{p_2, p_2})\} \cdot \\ & \cdot \langle \vec{p}_1 \sigma'_1; \vec{p}_2 \sigma'_2 | V(\alpha\tau, \alpha\tau) | \vec{k}_1 \sigma'_1; \vec{k}_2 \sigma'_2 \rangle \cdot D_{\sigma'_1 \sigma_1}^{1/2} \{V^{-1}(\Lambda_{k_1, p_1})\} \cdot D_{\sigma'_2 \sigma_2}^{1/2} \{V^{-1}(\Lambda_{k_2, p_2})\} \cdot D_{\sigma_1 \sigma'_1}^{1/2} \{V^{-1}(\Lambda_{p_1, k_1})\} \cdot D_{\sigma_2 \sigma'_2}^{1/2} \{V^{-1}(\Lambda_{p_2, k_2})\} \end{aligned} \quad (1.5)$$

(summation runs over all repeated indices), where matrices  $D_{\sigma'_i \sigma_i}^{1/2} \{V^{-1}(\Lambda_{p_i, k_i})\}$  describe Wigner rotations of 1/2 spins  $R\{V^{-1}(\Lambda_{p_i, k_i})\}$  ( $\Lambda_{p_i}$  is the matrix of the pure Lorentz transformation - boost:  $\Lambda_{p_i}(m, \vec{\sigma}) = (p_i, \vec{p}_i)$ ).

As has been shown in /1,13/, when as  $V_{\sigma_1 \sigma_2}^{\sigma_1 \sigma_2}$  the one-meson-exchange matrix element is taken  $V_{(2)}$  the matrix element  $\langle \vec{p}_1 \sigma_1; \vec{p}_2 \sigma_2 | V_{(2)}(\alpha\tau, \alpha\tau) | \vec{k}_1 \sigma_1; \vec{k}_2 \sigma_2 \rangle$  obtained after separation by formula (1.5) is a local function in the three-dimensional Lobachevsky momentum space realized on the upper mass hyperboloid

$$p_0^2 - \vec{p}^2 = m^2; \quad k_0^2 - \vec{k}^2 = m^2. \quad (1.6)$$

Our further task is to transform equation (1.1) so that it would contain only the local part of the interaction kernel (§ 2) and then to transform the obtained equation (§ 3) from the momentum to relativistic configurational representation /19/ that is the analog of the coordinate representation used in nonrelativistic quantum mechanics.

## 2. Transformation of the equation to the local form in the Lobachevsky momentum space and covariant summation of spins

As is mentioned in the Introduction, after separating 6 Wigner rotations in (1.5) the remaining part of the matrix element of the interaction kernel is a local function in the Lobachevsky space.

The role of these rotations in (1.5) is the same as of the Wigner rotations entering into the transformation law of the state vector  $|[m, s], \vec{k}, \sigma\rangle$ . The latter is an eigenvalue of two Casimir operators of the Poincare group, ( $W_\mu(\kappa)$  is the relativistic spin vector of Pauli-Lubansky-Bargman-Shirokov, see, e.g. /7,14/):

$$k^2 |[m, s], \vec{k}, \sigma\rangle = m^2 |[m, s], \vec{k}, \sigma\rangle \quad (2.1)$$

$$W^3(\kappa) |[m, s], \vec{k}, \sigma\rangle = -m^2 s(s+1) |[m, s], \vec{k}, \sigma\rangle \quad (2.2)$$

and is Lorentz-transformed by the law

$$U(\Lambda_{p, k}^{-1}) |[m, s], \vec{k}, \sigma\rangle = \sum_{\sigma' = \pm 1/2} D_{\sigma' \sigma}^{1/2} \{V^{-1}(\Lambda_{p, k})\} |[m, s], \vec{k}, \sigma'\rangle. \quad (2.3)$$

The origin of Wigner rotations is that the projection of the spin of a particle  $\sigma$  onto axis  $\bar{x}$  is defined in its rest frame

as an eigenvalue of the third component of the spin operator /14/ x)

$$\hat{S}_i = m^{-1} \hat{W}'_i(k) ; \hat{W}'_\mu(k) = U(\Lambda_k) \hat{W}_\mu(k) U^{-1}(\Lambda_k) \quad (2.4)$$

$$\hat{S}_3 | [m, s], \vec{k} \sigma \rangle = \sigma | [m, s], \vec{k} \sigma \rangle /_{\vec{k}=\vec{0}} ; i = 1, 2, 3. \quad (2.5)$$

The state with momentum  $\vec{k}$  is given by

$$| [m, s], \vec{k} \sigma \rangle = U(\Lambda_k) | [m, s], \vec{0} \sigma \rangle. \quad (2.6)$$

However, since to each momentum, in general, its own rest frame corresponds, the quantization axis for different momenta do not coincide.

By terminology of authors of /7,15/, where this question is studied, spin indices "sit" on their momenta so that the Wigner rotation  $D_{\sigma\sigma'}^{1/2} \{ V^{-1}(\Lambda_p, k) \}$  realizes in (2.3) "the remove" of the spin index  $\sigma$  from momentum  $k$  onto the transformed momentum  $\Lambda_{\vec{p}}^{-1} k$ .

Analogous Wigner rotations enter into the obtained in /7,16/ transformation (which we shall use below) from the wave function of two free particles  $\Psi_{\sigma_1 \sigma_2}^{s_1 s_2}(p_1, p_2)$  (with momenta  $p_1$  and  $p_2$ , spins  $S_1$  and  $S_2$  and their projections) to the wave function of the whole system characterized by the mass  $M = \sqrt{(p_1 + p_2)^2}$ , total momentum  $P = p_1 + p_2$ , total spin  $J$  and its projection  $m_J$ . The spin function  $\Psi_{\sigma_1 \sigma_2}^{s_1 s_2}(p_1, p_2)$  depends on 8 variables  $p_1, p_2, \sigma_1$  and  $\sigma_2$ . Therefore the set of 6 variables  $M, P, J, m_J$  should be completed by two variables; as the latter it is convenient to take the sum of spins and orbital moment in the o.m.s. For this, according to /7/, it is first necessary to transform both spins to the c.m.s. with the use of transformations  $D_{\sigma_1 \sigma_1'}^{s_1} \{ V^{-1}(\Lambda_p, p_1) \}$  and  $D_{\sigma_2 \sigma_2'}^{s_2} \{ V^{-1}(\Lambda_p, p_2) \}$  then to pass over from momenta  $\vec{p}_1$  and  $\vec{p}_2$  to the total momentum  $\vec{P}$  and vector  $\vec{\beta} = \vec{P} / P$  (1.3) which is a covariant generalization of the particle-momentum vector in the o.m.s. (xx). The action

x) The components of the spin vector  $S_i$  are related to the components of the relativistic spin vector  $W_\mu(k)$  by  $W'_\mu(k) = (\Lambda_k^{-1})^\mu_\nu W_\nu(k)$ , where  $W_0(k) = \vec{\sigma} \cdot \vec{k}$ ,  $W(k) = m\vec{\sigma} + (k_0 + m)^{-1} \vec{k} (\vec{k} \cdot \vec{\sigma})$ .

xx) In what follows we shall denote by small zeros at top the values that have the meaning of covariant generalizations of their analog in o.m.s. in a sense of (1.2) transformation.

of the total angular momentum of the system

$$\hat{L}^{\kappa} = \frac{1}{2} \epsilon^{\kappa l m} \hat{M}_{l m} ; M_{l m} = M_{l m}^{(1)} \otimes I^{(2)} + I^{(1)} \otimes M_{l m}^{(2)} \quad (2.7)$$

(  $M_{\mu\nu}$  are generators of the Lorentz group,  $\mu, \nu = 0, 1, 2, 3$ ;  $\kappa, l, m = 1, 2, 3$ )

$$(\hat{L}) \Psi_{\sigma_1 \sigma_2}^{s_1 s_2}(p_1, p_2) = \left\{ -i \left[ \vec{p}_1 \times \frac{d}{d\vec{p}_1} \right] - i \left[ \vec{p}_2 \times \frac{d}{d\vec{p}_2} \right] \right\} \Psi_{\sigma_1 \sigma_2}^{s_1 s_2}(p_1, p_2) \quad (2.8)$$

can be represented as a sum of actions of the orbital momentum of the whole system as a unique object and orbital momenta of the relative motion /16/ of two particles

$$(\hat{L}) \Psi_{\sigma_1 \sigma_2}^{s_1 s_2}(P, \vec{\beta}) = \left\{ -i \left[ \vec{P} \times \frac{d}{d\vec{P}} \right] - i \left[ \vec{P} \times \frac{d}{d\vec{\beta}} \right] \right\} \Psi_{\sigma_1 \sigma_2}^{s_1 s_2}(P, \vec{\beta}). \quad (2.9)$$

Since under the Lorentz transformations the relative orbital moment is transformed only by the Wigner rotations, its modulus  $l_{rel}$  and eigenvalues

$$\hat{l}_{rel}^2 = - \left[ \vec{P} \times \frac{d}{d\vec{P}} \right]^2 = i n r ; \hat{l}_{rel}^2 Y_{l m} \left( \frac{\vec{P}}{P} \right) = l(l+1) Y_{l m} \left( \frac{\vec{P}}{P} \right) \quad (2.10)$$

are invariants. After the transformation to the o.m.s. of spins  $S_1$  and  $S_2$  they can be summed with the orbital momentum of the relative motion to give the total spin of the system

$$J = (S_1 + S_2) + l_{rel} = S + l_{rel}. \quad (2.11)$$

As a result, the transformation from the wave function of two particles to the wave function of the system as a whole has the form /7,16/ ( in our case  $S_1 = S_2 = 1/2$ ;  $m_1 = m_2 = m$  )

$$\Psi_{\sigma_1 \sigma_2}^{1/2 1/2}(p_1, p_2) = \sum_{\sigma_1', \sigma_2' = \pm 1/2} D_{\sigma_1 \sigma_1'}^{1/2} \{ V^{-1}(\Lambda_p, p_1) \} D_{\sigma_2 \sigma_2'}^{1/2} \{ V^{-1}(\Lambda_p, p_2) \} \Psi_{\sigma_1' \sigma_2'}^{1/2 1/2}(P, \vec{\beta}) \quad (2.12)$$

where

$$\Psi_{\sigma_1' \sigma_2'}^{1/2 1/2}(P, \vec{\beta}) = \sum_{S, m_S} \sum_{l, m_l} \sum_{J, m_J} \langle l, m_l m_S | J, m_J \rangle \chi_{l m_l}^{1/2 1/2}(\vec{\beta}) \Phi_{m_S}^{J l S}(P, |\vec{P}|) \quad (2.13)$$

$$\langle l, m_l m_S | J, m_J \rangle \cdot \chi_{l m_l}^{1/2 1/2}(\theta, \varphi) \cdot \Phi_{m_S}^{J l S}(P, |\vec{P}|)$$

and  $\theta$  and  $\varphi$  - are spherical angles of the vector  $\vec{e}_p = \vec{p}/|\vec{p}| = (\sin\theta \sin\varphi, \sin\theta \cos\varphi, \cos\theta)$ .

Coming after these remarks back to eq. (1.1) and expression (1.5) for the quasipotential kernel, we conclude that in (1.5) Wigner rotations  $D_{\sigma_1 \sigma_2}^{1/2} \{V^{-1}(\Lambda_{\mathcal{P}}, p_i)\}$ ,  $i=1,2$  "remove" the spin indices  $\sigma_1, \sigma_2$  from momenta  $\vec{p}_1$  and  $\vec{p}_2$  onto momenta  $\vec{p}_1^* = \vec{p}$  and  $\vec{p}_2^* = -\vec{p}$  and  $D_{\nu_1 \nu_2}^{1/2} \{V^{-1}(\Lambda_{\mathcal{P}}, k_i)\}$  and  $D_{\nu_1 \nu_2}^{1/2} \{V^{-1}(\Lambda_{\mathcal{P}}, k_i)\}$  "remove"  $\nu_1$  and  $\nu_2$  onto momenta  $\vec{k}_1 = \vec{k}$  and  $\vec{k}_2 = -\vec{k}$ . Therefore, indices  $\nu_1 \nu_2$  "sit" on momentum  $\vec{k}$ ; and indices  $\sigma_1 \sigma_2$  and  $\sigma_1 \sigma_2$ , on momentum  $\vec{p}$ . Now, since as a result of the boson exchange the momentum  $\vec{p}$  transforms to the momentum  $\vec{k} \neq \vec{p}$  then Wigner rotations  $D_{\nu_1 \nu_2}^{1/2} \{V^{-1}(\Lambda_{\vec{k}}, p_i)\}$  and  $D_{\nu_1 \nu_2}^{1/2} \{V^{-1}(\Lambda_{\vec{k}}, p_i)\}$  realise a further "remove" of spin indices on the momentum  $\vec{p}$ . As a result, we arrive at the matrix elements  $V_{\sigma_1 \sigma_2, \nu_1 \nu_2}^{\sigma_1 \sigma_2, \nu_1 \nu_2} = \langle p_1 \sigma_1 p_2 \sigma_2 | V(\alpha\tau, \lambda\tau) | k_1 \nu_1 k_2 \nu_2 \rangle$  whose spin indices are all sitting on the same momentum  $\vec{p}$ . Just this "separated" of the kinematic Wigner rotations amplitude is local in the Lobachevsky momentum space, i.e., depends on the difference of two vectors  $\vec{k}$  and  $\vec{p}$  in the Lobachevsky space  $\vec{\Delta}_{\vec{k}, \vec{p}} = \vec{k} - \vec{p}$  (see /1,13/).

So, we have seen that 4  $D^{1/2}$  matrices with Wigner rotations in (1.5) to the right from

$$V_{\nu_1 \nu_2, \sigma_1 \sigma_2}^{\sigma_1 \sigma_2, \nu_1 \nu_2}(\vec{p}, \vec{k} \leftarrow \vec{p}) = \langle p_1 \sigma_1 p_2 \sigma_2 | V(\alpha\tau, \lambda\tau) | k_1 \nu_1 k_2 \nu_2 \rangle$$

provide automatically the transfer of entering into (1.5) under the summation sign dummy spin indices onto the momentum  $\vec{p}$ . Consequently, if now in the l.h.s. of eq. (1.1), using the transformation inverse to (2.12)

$$\Psi_{\nu_1 \nu_2, \sigma_1 \sigma_2}^{\sigma_1 \sigma_2, \nu_1 \nu_2}(\vec{p}, \vec{p}) = \sum_{\sigma_1, \sigma_2} D_{\sigma_1 \sigma_2}^{1/2} \{V^{-1}(\Lambda_{\mathcal{P}}, p_i)\} \quad (2.14)$$

$$D_{\sigma_1 \sigma_2}^{1/2} \{V^{-1}(\Lambda_{\mathcal{P}}, p_i)\} \Psi_{\nu_1 \nu_2, \sigma_1 \sigma_2}^{\sigma_1 \sigma_2, \nu_1 \nu_2}(p_1, p_2)$$

we transfer indices  $\sigma_1$  and  $\sigma_2$  onto the momentum  $\vec{p} (\equiv \vec{\Delta}_{\mathcal{P}, m_2 \mathcal{P}})$  and make the same transformation with the r.h.s. integral part of (1.1), i.e., "remove" indices  $\sigma_1$  and  $\sigma_2$  of  $\langle p_1 \sigma_1, p_2 \sigma_2 | V | k_1 \nu_1 p; k_2 \nu_2 p \rangle$  onto the momentum  $\vec{p}$ , then eq. (1.1) can be represented in the form

$$2 p_0 (M - 2 p_0) \Psi_{\nu_1 \nu_2, \sigma_1 \sigma_2}^{\sigma_1 \sigma_2, \nu_1 \nu_2}(\vec{p}) = \frac{1}{(2\pi)^3} \sum_{\nu_1 \nu_2, \sigma_1 \sigma_2 = \pm 1/2} \quad (2.15)$$

$$\int \frac{d^3 \vec{k}}{2(k^0)} V_{\nu_1 \nu_2, \sigma_1 \sigma_2}^{\sigma_1 \sigma_2, \nu_1 \nu_2}(\vec{p}; \vec{k} \leftarrow \vec{p}) \Psi_{\nu_1 \nu_2, \sigma_1 \sigma_2}^{\sigma_1 \sigma_2, \nu_1 \nu_2}(\vec{k})$$

where all spin indices "sit" on the same momentum  $\vec{p}$ . As a result, under the Lorentz transformations all spin indices  $\sigma_i \nu_i$  and  $\nu_i \sigma_i$  in eq. (2.15) will be transformed by the same matrix of the small Lorentz group.

After that we have transferred all the spin indices in (2.15) to the same momentum, we can, according to /7,16/ sum up spins and define the wave function with the total spin

$$\Psi_{\nu_1 \nu_2, \sigma_1 \sigma_2}^{\sigma_1 \sigma_2, \nu_1 \nu_2}(\vec{p}) = \sum_{\sigma_1 \sigma_2, \nu_1 \nu_2 = \pm 1/2} \langle \frac{1}{2}, \frac{1}{2} | \sigma_1 \sigma_2 | S \sigma \rangle \Psi_{\nu_1 \nu_2, \sigma_1 \sigma_2}^{\sigma_1 \sigma_2, \nu_1 \nu_2}(\vec{p}) \quad (2.16)$$

Consequently, for wave functions with the transferred spins spin combinations take the form of quantum-mechanical relations

$$S=0, \sigma_p^0 = 0 \quad \Psi_{\nu_1 \nu_2, \sigma_1 \sigma_2}^{\sigma_1 \sigma_2, \nu_1 \nu_2}(\vec{p}) = (\sqrt{2})^{-1} \{ \Psi_{\nu_1 \nu_2, \sigma_1 \sigma_2}^{1/2, -1/2}(\vec{p}) - \Psi_{\nu_1 \nu_2, \sigma_1 \sigma_2}^{-1/2, 1/2}(\vec{p}) \} \quad (2.17)$$

$$S=1, \sigma_p^0 = 0 \quad \Psi_{\nu_1 \nu_2, \sigma_1 \sigma_2}^{\sigma_1 \sigma_2, \nu_1 \nu_2}(\vec{p}) = (\sqrt{2})^{-1} \{ \Psi_{\nu_1 \nu_2, \sigma_1 \sigma_2}^{1/2, 1/2}(\vec{p}) + \Psi_{\nu_1 \nu_2, \sigma_1 \sigma_2}^{-1/2, -1/2}(\vec{p}) \} \quad (2.18)$$

$$S=1, \sigma_p^0 = -1 \quad \Psi_{\nu_1 \nu_2, \sigma_1 \sigma_2}^{\sigma_1 \sigma_2, \nu_1 \nu_2}(\vec{p}) = \Psi_{\nu_1 \nu_2, \sigma_1 \sigma_2}^{-1/2, -1/2}(\vec{p}) \quad (2.19)$$

$$S=1, \sigma_p^0 = 1 \quad \Psi_{\nu_1 \nu_2, \sigma_1 \sigma_2}^{\sigma_1 \sigma_2, \nu_1 \nu_2}(\vec{p}) = \Psi_{\nu_1 \nu_2, \sigma_1 \sigma_2}^{1/2, 1/2}(\vec{p}) \quad (2.20)$$

Note that sometimes relations (2.16)-(2.20) are used for describing composite systems without transferring the spin indices on the same relative momentum. This approximation is valid for systems with the nonrelativistic relative motion of constituents (e.g., in positronium) when the matrices of Wigner rotations describing, under Lorentz transformations, different rotations of spin indices which are sitting on different momenta differ slightly from unit matrices. However, in describing the light

vector and pseudoscalar mesons ( $\pi, \rho, \omega, \varphi$ ), where it is certainly known that the motion of constituent valence quarks is relativistic, a correct separation of spin projections can be performed only after the operation (2.14) of transfer of spin indices on the same momentum. Otherwise, the neglect of laws of the relativistic spin kinematics may result in "entangling" of different spin components of the wave function of a two-quark system while passing over to another reference frame (just this operation is used for describing the meson form factor defined as a matrix element of the current sandwiched between wave functions with momenta  $\mathcal{P}$  and  $\mathcal{P} + \mathcal{Q}$ ).

By using the relations for Clebsch-Gordon coefficients

$$\sum_{S_1' S_2'} \langle \frac{1}{2} \frac{1}{2} S_1 S_2 | S' \sigma' \rangle \langle \frac{1}{2} \frac{1}{2} S_1' S_2' | S' \sigma' \rangle = \delta_{S_1 S_1'} \cdot \delta_{S_2 S_2'}$$

$$\sum_{S_1' S_2'} \langle \frac{1}{2} \frac{1}{2} S_1 S_2 | S \sigma \rangle \langle \frac{1}{2} \frac{1}{2} S_1' S_2' | S' \sigma' \rangle = \delta_{S S'} \cdot \delta_{\sigma \sigma'}$$

we rewrite equation (2.15) as

$$2p_0 (M - 2p_0) \Psi_{S, m_S}^{S \sigma p_0}(\vec{p}) = \sum_{S'=0,1} \sum_{\sigma'=-S'}^{S'} \frac{1}{(2\pi)^3} \int \frac{d\vec{k}}{2(k)_0} V_{S \sigma p_0}^{S' \sigma' p_0}(\vec{p}, \vec{k} \leftarrow \vec{p}) \Psi_{S', m_{S'}}^{S' \sigma' p_0}(\vec{k}) \quad (2.21)$$

where we have defined

$$V_{S \sigma p_0}^{S' \sigma' p_0}(\vec{p}, \vec{k} \leftarrow \vec{p}) = \sum_{S_1 p_1, S_2 p_2 = \pm 1/2} \sum_{S_1' p_1', S_2' p_2' = \pm 1/2} \langle \frac{1}{2} \frac{1}{2}; S_1 p_1, S_2 p_2 | S \sigma \rangle V_{S_1 p_1, S_2 p_2}^{S_1' p_1', S_2' p_2'}(\vec{p}, \vec{k} \leftarrow \vec{p}) \langle \frac{1}{2} \frac{1}{2}; S_1' p_1', S_2' p_2' | S' \sigma' \rangle$$

It is known that for some processes the symmetry properties of a two-particle system lead to the observation of the total spin. In particular, in the scattering of identical particles singlet-triplet transitions are forbidden by the Pauli principle. In other system, like, e.g., the neutron-proton system, this takes place due to the isotopic - invariance of strong interactions or CP-invariance of electromagnetic interactions, like in the positronium. For such systems we may take the kernel of eq. (2.21) to be diagonal in the total spin of a system

$$V_{S \sigma p_0}^{S' \sigma' p_0}(\vec{p}, \vec{k} \leftarrow \vec{p}) = \delta_{S S'} \cdot V_{S \sigma p_0}^{S \sigma p_0}(\vec{p}, \vec{k} \leftarrow \vec{p}) \quad (2.23)$$

### 3. Transformation of the equation into the relativistic configurational representation

In ref. /9/ it is proposed to transform to the configurational representation with the use of functions composing a complete set in the Lobachevsky space, i.e. on the mass hyperboloid (1.6). Such functions have been found in ref. /18/ on the basis of work /17/ and have the form (notation of ref. /9/):

$$\xi(\vec{\Delta}_{p, m \Delta p}, \vec{r}) = \left[ \frac{(\Delta_{p, m \Delta p})^\mu n_\mu}{m} \right]^{-1-i r m}, \quad n_\mu = (\pm, \vec{n}), \quad (3.1)$$

$$\vec{n}^2 = 1, \quad \vec{r} = r \vec{n}.$$

Their relations of completeness and orthogonality are as follows

$$\frac{1}{(2\pi)^3} \int \xi^*(\vec{p}, \vec{r}) \xi(\vec{k}, \vec{r}) d\vec{r} = m^{-1} p_0 \delta^{(3)}(\vec{p} - \vec{k}) \quad (3.2a)$$

$$\frac{1}{(2\pi)^3} \int \xi^*(\vec{p}, \vec{r}) \xi(\vec{p}', \vec{r}') \frac{d^3 \vec{p}'}{m^{-1} p'_0} = \delta^{(3)}(\vec{r} - \vec{r}'). \quad (3.2b)$$

The functions (3.1) realize the principal series of unitary irreducible representations of the Lorentz group - group of motions of the Lobachevsky space and correspond to the following eigenvalues, of the Casimir operator of the Lorentz group  $\hat{C}_4 = \frac{1}{4} M_{\mu\nu} M^{\mu\nu}$  ( $M_{\mu\nu} = p_\mu \frac{\partial}{\partial p^\nu} - p_\nu \frac{\partial}{\partial p^\mu}$  are generators of  $SO(3,1)$ ):

$$\hat{C}_4 \xi(\vec{p}, \vec{r}) = \left( \frac{1}{m^2} + r^2 \right) \xi(\vec{p}, \vec{r}), \quad 0 \leq r < \infty. \quad (3.3)$$

In the nonrelativistic limit  $\xi(\vec{p}, \vec{r}) \rightarrow e^{i \vec{p} \cdot \vec{r}}$ . The group parameter  $r$  was proposed in /9/ to consider as a relativistic generalization of the relative-coordinate modulus.

For the relativistic plane waves (3.1) in /9/ there was defined the free Hamiltonian operator

$$\hat{H}_0 \xi(\vec{p}, \vec{r}) = (p_0) \xi(\vec{p}, \vec{r}), \quad (3.4)$$

which has the form of the differential-difference operator

$$\hat{H}_0 = m \operatorname{ch} \left( \frac{i}{m} \frac{\partial}{\partial r} \right) + \frac{i}{r} \operatorname{sh} \left( \frac{i}{m} \frac{\partial}{\partial r} \right) - \frac{\Delta_{\theta, \varphi}}{2 m r^2} \exp \left( \frac{i}{m} \frac{\partial}{\partial r} \right) \quad (3.5)$$

( $\Delta_{\theta, \varphi}$  is the Laplace operator on a sphere,  $\theta$  and  $\varphi$  - are spherical angles of the unit vector  $\vec{n}$ ).

Under Lorentz transformations of the vector  $\Delta_{p, m \Delta p}$  only its spatial part (1.2) is transformed

$$\Delta_{p', m_{\mathcal{D}'}}^{\circ} = \Delta_{p, m_{\mathcal{D}}}^{\circ} = p_{\mu} \lambda_{\mathcal{D}}^{\mu} \quad ; \quad p' = L p \quad ; \quad \mathcal{D}' = L \mathcal{D} \quad (3.6)$$

$$(\Delta_{p', m_{\mathcal{D}'}}^{\circ})_{ij} = R_{ij} \{ V^{-1}(L^{-1}, \mathcal{D}') \} (\Delta_{p, m_{\mathcal{D}}}^{\circ})_{ij} \quad ; \quad ij' = 1, 2, 3, \quad (3.7)$$

which undergoes the Wigner rotation only. Due to the Lorentz invariance of the scalar product  $(\Delta_{p, m_{\mathcal{D}}})_{\mu} \lambda_{\mathcal{D}}^{\mu}$  *invariant* into (3.1), this means that under the Lorentz transformations  $L$  the unit vector  $\vec{n}$ , which plays the role of the direction vector of a relativistic analog of the relative coordinate  $\vec{r} = r \vec{n}$ , transforms by the law:

$$n'_i = R_{ij} \{ V(L^{-1}, \mathcal{D}') \} n_j \quad ; \quad ij' = 1, 2, 3 \quad (3.8)$$

The modulus of the "relativistic relative coordinate"  $r = |\vec{r}|$  conjugated in (3.1) to the covariantly defined vector of a particle momentum in the c.m.s.  $\vec{A}_{p, m_{\mathcal{D}}}$  (1.2), (1.3), is a Lorentz invariant as it (of. 3.3) numbers eigenvalues of the Casimir invariant operator of the Lorentz group  $\hat{C}_4$ .

As is shown in /17/, owing to the invariance of the modulus of the relativistic relative coordinate  $r$  the operator introduced in /19/

$$\hat{A} = [\vec{r} \times \vec{A}_{p, m_{\mathcal{D}}}] = \hat{A}_{nonrel.} \cdot \exp\left(\frac{i}{m} \frac{\partial}{\partial r}\right) \quad (3.9)$$

(where the operator  $\hat{A}_{nonrel.}$  in spherical coordinates has the same components as the moment operator in nonrelativistic quantum mechanics and  $\vec{A}_{p, m_{\mathcal{D}}}$  is the momentum operator /20/;  $\vec{A}_{p, m_{\mathcal{D}}}$   $\cdot \xi(\vec{A}_{p, m_{\mathcal{D}}}, \vec{r}) = \vec{A}_{p, m_{\mathcal{D}}} \cdot \xi(\vec{A}_{p, m_{\mathcal{D}}}, \vec{r})$  transforms by the same law as the covariantly defined operator of the relative (internal) orbital moment (2.10). For this reason the operator

$$\hat{A}^2 = -\Delta_{\theta, \varphi} \cdot \exp\left(\frac{i}{m} \frac{\partial}{\partial r}\right) \quad (3.10)$$

in  $\hat{H}_0$  (3.5) is a relativistic invariant what, with the invariance of  $\tau$ , makes invariant the whole operator of  $\hat{H}_0$ .

After making in eq. (2.15) the transformation with functions (3.1)

$$\Psi_{\sigma_1 \sigma_2}(\vec{r}) = \frac{1}{(2\pi)^3} \int \frac{d^3 \vec{p}}{2(\vec{p})_0} \cdot \xi(\vec{p}, \vec{r}) \cdot \Psi_{\sigma_1 \sigma_2}(\vec{p}) \quad (3.11)$$

it takes the form <sup>xx)</sup> (with allowing for (3.4)):

$$2\hat{H}_0 (M - 2\hat{H}_0) \Psi_{j, m_j}^{\sigma_1 \sigma_2}(\vec{r}) = \frac{1}{(2\pi)^3} \sum_{j_1, j_2 = \pm 1/2} \int d^3 \vec{r}_1 \cdot V_{j_1 j_2}^{\sigma_1 \sigma_2}(\vec{r}, \vec{r}_1) \cdot \Psi_{j_1 m_{j_1}}^{\sigma_1 \sigma_2}(\vec{r}_1) \quad (3.12)$$

where

$$V_{j_1 j_2}^{\sigma_1 \sigma_2}(\vec{r}, \vec{r}_1) = \frac{1}{(2\pi)^6} \int \frac{d^3 \vec{p}}{2(\vec{p})_0} \cdot \frac{d^3 \vec{k}}{2(\vec{k})_0} \cdot \xi(\vec{p}, \vec{r}) V_{j_1 j_2}^{\sigma_1 \sigma_2}(\vec{p}, \vec{k}(-\vec{p})) \xi(\vec{k}, \vec{r}_1) \quad (3.13)$$

The part of quasipotential  $V$  that depends only on the momentum transfer vector squared in the Lobachevsky space  $V_{(j_1) j_1 j_2}^{\sigma_1 \sigma_2} [(\vec{k}(-\vec{p}))^2]$  <sup>xx)</sup> transformed into the relativistic configurational representation

$$V_{(j_1) j_1 j_2}^{\sigma_1 \sigma_2} [(\vec{k}(-\vec{p}))^2] = \int \xi(\vec{k}(-\vec{p}), \vec{r}) V_{(j_1) j_1 j_2}^{\sigma_1 \sigma_2}(|\vec{r}|) d^3 \vec{r} \quad (3.14)$$

depends only on the modulus of vector  $r = |\vec{r}|$ . Therefore, for it we may apply in (3.12) the integral form of the addition theorem of "relativistic plane waves" /9/ ( $d\omega_{\vec{k}} = \sin\theta d\theta d\mathcal{S}$ )

$$\int d\omega_{\vec{k}} \xi(\vec{k}(-\vec{p}), \vec{r}) = \int d\omega_{\vec{k}} \xi(\vec{k}, \vec{r}) \xi^*(\vec{p}, \vec{r}) \quad (3.15)$$

and allowing for (3.2) we may take it out of the integral sign in (3.12) in the local form

$$2\hat{H}_0 (M - 2\hat{H}_0) \Psi_{j, m_j}^{\sigma_1 \sigma_2}(\vec{r}) = \sum_{j_1 j_2} V_{(j_1) j_1 j_2}^{\sigma_1 \sigma_2}(r) \Psi_{j_1 m_{j_1}}^{\sigma_1 \sigma_2}(\vec{r}) +$$

$$+ \sum_{j_1 j_2} \frac{1}{(2\pi)^3} \int d^3 \vec{r}_1 V_{j_1 j_2}^{\sigma_1 \sigma_2}(\vec{r}, \vec{r}_1) \Psi_{j_1 m_{j_1}}^{\sigma_1 \sigma_2}(\vec{r}_1) \quad (3.16)$$

<sup>x)</sup> In what follows we omit in (2.15) the symbol  $\vec{p}$  for polarization indices, which indicates that all spin indices are quantized on the same axis given by vector  $\vec{p}$ . Indeed the presence in  $\sigma_{L\vec{p}}$  polarization indices of symbol  $\vec{p}$  is pure conventional since the quantities  $\sigma_{i\vec{p}}$  and  $j_{i\vec{p}}$  assume the numerical values  $\sigma_{i\vec{p}}, j_{i\vec{p}} = \pm 1/2$ .

<sup>xx)</sup> We denote the remaining part by  $V_{(j_2) j_1 j_2}^{\sigma_1 \sigma_2}$ .

The remaining part of the potential  $V_{(2)}$  after the transformation (3.14) becomes dependent not only on the modulus  $|\vec{r}_2|$  but also on the direction  $\vec{n}_2$  defined by the spin structures of  $V_{(2)}$ .

Let us apply to (3.13) the equality

$$\xi(\vec{k}, \vec{r}) = \xi(\vec{k} \leftarrow \vec{p}, \vec{r} \cdot \vec{n}_{\Lambda_p}) \cdot \xi(\vec{p}, \vec{r}) \quad (3.17)$$

which contains the unit vector

$$\vec{n}_{\Lambda_p} = (\vec{p}_0 - \vec{p} \cdot \vec{n})^{-1} [m\vec{n} - \vec{p} + \vec{p}(\vec{p} \cdot \vec{n}) / (p_0 + m)] \quad (3.18)$$

and taking advantage of the invariance of the volume element  $d^3\vec{k}/2(k_0)$  =  $d^3(\vec{k} \leftarrow \vec{p}) / 2(k \leftarrow \vec{p})_0$ ;  $(\vec{k} \leftarrow \vec{p})_\mu = (\Lambda_p \vec{k})_\mu$  perform integration over  $d^3(\vec{k} \leftarrow \vec{p}) / 2(k \leftarrow \vec{p})_0$ .

As a result, for  $V_{(2)}^{G_1 G_2}(\vec{r}_1, \vec{r}_2)$  we obtain from (3.13) the expression

$$V_{(2)}^{G_1 G_2}(\vec{r}_1, \vec{r}_2) = \frac{1}{(2\pi)^3} \int \frac{d^3\vec{p}}{2(p_0)} \xi(\vec{p}, \vec{r}_1) V_{(2)}^{G_1 G_2}(\vec{p}, \vec{r}_2 \cdot \vec{n}_{\Lambda_p}) \xi(\vec{p}, \vec{r}_2) \quad (3.19)$$

The part  $V_{(2)}$  depends upon  $\vec{p}$  through its spin structure and vector  $\vec{n}_{\Lambda_p}$ . In the first case is readily eliminated from integration over  $\vec{p}$  in (3.19) by changing  $\vec{p}$  to the momentum operator  $\hat{p}$  (for the explicit form see /6/):

$$\hat{p} \xi(\vec{p}, \vec{r}) = \vec{p} \xi(\vec{p}, \vec{r}) \quad (3.20)$$

what allows us to localize the  $\vec{p}$ -dependent orbital terms of the potential  $V$ .

As is shown in a previous paper [1, 11] the interaction kernel  $V$  constructed out of matrix elements of the one-boson-exchange amplitude in an arbitrary reference frame does not differ in form from the expression found earlier in the c.m.s. /20/. This makes it possible to apply here a method proposed in /21/ by which one can completely localize in  $r$ -space the kernel part  $V_{(2)}^{G_1 G_2}$  which contains the structures with spin-orbital coupling and partially localize the tensor forces.

#### 4. Conclusion

We have obtained main equations for the wave function of a system of two spin 1/2 particles transformed into a convenient form for the practical use. These are the equation in the momentum representation (2.15) with the kernel  $V_{(2)}^{G_1 G_2}(\vec{p}, \vec{k} \leftarrow \vec{p})$  local in the Lobachevsky space and equation (3.16) in the relativistic

configurational representation. In a subsequent publication we shall consider in detail the problem of diagonalization of the equation in 2-space (3.16) and also the transition to invariant partial equations containing invariant eigenvalues of the relative orbital moment and relativistic spin.

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