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REPRESENTATIONS OF 1-p-i FUNCTIONALS IN GAUGE FIELD THEORIES

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Generating functionals of one-particle irreducible Green's functions play an important role in Quantum Field Theory. They contain, in a condensed manner, the essential information on the considered theory. In this respect it is quite natural that all questions concerning the renormalization procedure /1/ or the proofs of the light-cone /2,3/ and short distance expansions /2,4/ are based on the consideration of such functionals. For model field theories as ϕ^4 or $\phi^3_{(6)}$ a representation of such functionals is not difficult. The situation changes drastically if gauge fields are taken into account. Of course also here the 1-p-i functionals can be represented in the usual manner

$$\Gamma(a_\mu,\bar{\chi},\chi) = \Sigma \frac{1}{(\ell!)^2}\frac{1}{k!}\int dx_1 \ldots dx_\ell\, dy_1 \ldots dy_\ell\, dz_1 \ldots dz_k\, F_{\mu_1\ldots\mu_k}(x_i,y_j,z_r)\times$$

$$\times\, \bar{\chi}(x_1)\ldots\bar{\chi}(x_\ell)\,\chi(y_1)\ldots\chi(y_\ell)\,a_{\mu_1}(z_1)\ldots a_{\mu_k}(z_k)\,, \qquad (1)$$

where $a_\mu, \bar{\chi}, \chi$ are classical fields corresponding to the gauge field A_μ and the quark fields $\bar{\psi}, \psi$.

Additionally in most cases ghost fields and their corresponding classical fields have to be taken into account in eq. (1). The main disadvantage of such representations is that their coefficient functions cannot be chosen independently, they have to satisfy a complicated set of equations, the so-called Slavnov-Taylor identities /5/. We try to give other representations of these functions which solve by construction the underlying identities and contain therefore independent coefficient functions only. These solutions show the different role of the gauge field which acts in some respect as an ordinary field; on the other hand, the integrated gauge fields contained in the exponentials are needed to guarantee gauge invariance of the considered expression and have no dimension For example, the covariant derivatives in the operators $\bar{\psi}\gamma_{\mu_1}$, $D_{\mu_2}\ldots D_{\mu_n}\psi$ in the light-cone expansion of QCD introduce a lot of gauge fields which are primarily needed to satisfy Slavnov-Taylor identities and gauge invariance /6/. For our considerations it was very important to use axial gauges /7/ which lead to simple Slavnov-Taylor identities. This underlines the importance of such gauges as it is already known from complicated calculations /8/. The question remains whether it would be possible to get similar results for other gauges. The complicated nonlinear structure of the identities in general gauges does not allow such simple proofs.

1. LINEAR SLAVNOV-TAYLOR IDENTITIES

For definiteness we consider the following non-Abelian gauge field theory

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi ,$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{acb} A^c_\mu A^b_\nu ,$$

$$D^{ab}_\mu = \delta^{ab}\partial_\mu - g f^{acb} A^c_\mu ,$$

$$D_{\mu\alpha\beta} = \delta_{\alpha\beta}\partial_\mu + ig t^a_{\alpha\beta} A^a_\mu ,$$

$$A_\mu = A^a_\mu t^a , \quad [t^a, t^b] = i f^{abc} t^c \quad \mathrm{Tr}(t^a t^b) = \frac{1}{2}\delta^{ab},$$

f^{abc} are structure constants of the group SU(3). The generating functional for the complete Green functions reads

$$W = \frac{1}{N}\int DA_\mu D\bar{\psi} D\psi \exp i\{\int dx (n^\mu A^a_\mu)(n^\nu A^a_\nu)\cdot\frac{1}{2\alpha} + \int dx(\mathcal{L} + \mathcal{L}_S)\},$$
$$\mathcal{L}_S = j^a_\mu A^a_\mu + \bar{\xi}\psi + \bar{\psi}\xi .$$

Here we have chosen an axial gauge condition with arbitrary gauge parameter α, n_μ is a given constant vector. As usual we consider identities for the functional W. In standard way we transform the variables in the functional integral according to an infinitesimal gauge transform

$$A'^a_\mu = A^a_\mu - D^{ab}_\mu \delta\omega^b, \quad \psi' = \psi + igt^a\psi\delta\omega^a, \quad \bar{\psi}' = \bar{\psi} - ig\bar{\psi}t^a\delta\omega^a .$$

The equation for vanishing of the infinitesimal contributions to W reads

$$[(\partial_\mu j^b_\mu) + j^a_\mu g f^{acb}\frac{\delta}{\delta i j^c_\mu} + \frac{1}{\alpha}(n\cdot\partial)(n^\mu\frac{\delta}{\delta i j^b_\mu})] W(j^a_\nu, \bar{\xi}, \xi) =$$

$$= ig\bar{\xi}t^b\frac{\delta W}{\delta i\bar{\xi}} - ig\left(\frac{\delta W}{\delta i\xi}\right)_R t^b\xi .$$

This relation is now transformed to an identity for the 1-p-i functional Γ. Using standard expressions

$$\frac{\delta\Gamma}{\delta a^a_\mu} = -j^a_\mu, \qquad \frac{\delta\Gamma}{\delta\bar\chi} = -\xi, \qquad \left(\frac{\delta\Gamma}{\delta\chi}\right)_R = -\bar\xi,$$

$$\frac{\delta W}{\delta i j^a_\mu} = a^a_\mu W, \qquad \frac{\delta W}{\delta i\bar\xi} = \chi W, \qquad \left(\frac{\delta W}{\delta i\xi}\right)_R = W\bar\chi,$$

we get

$$-\partial_\mu \frac{\delta\Gamma}{\delta a^b_\mu} - g f^{acb} a^c_\mu \frac{\delta\Gamma}{\delta a^a_\mu} - ig\left(\frac{\delta\Gamma}{\delta\chi}\right)_R t^b\chi + ig\bar\chi t^b \frac{\delta\Gamma}{\delta\bar\chi} + \frac{1}{\alpha}(n\partial)(n^\mu a^b_\mu) = 0. \quad (2)$$

The inhomogenous term $\alpha^{-1}(n\partial)(na^b)$ can be cancelled by introducing the functional $\tilde\Gamma$ according to

$$\Gamma = \tilde\Gamma + \frac{1}{2\alpha}\int dx(n^\mu a^a_\mu)(n^\nu a^a_\nu) \quad (3)$$

satisfying

$$-D^{ba}_\mu(a)\frac{\delta\tilde\Gamma}{\delta a^a_\mu} - ig\left(\frac{\delta\tilde\Gamma}{\delta\chi}\right)_R t^b\chi + ig\bar\chi t^b\frac{\delta\Gamma}{\delta\bar\chi} = 0. \quad (4)$$

This equation states in differential form that the functional $\tilde\Gamma$ is gauge invariant. In this way we have received an important separation of the functional Γ into an explicitly given gauge-dependent part and a gauge-invariant part $\tilde\Gamma$. The remaining task consists in a description of the gauge-invariant part.

2. SOLUTIONS OF SLAVNOV-TAYLOR IDENTITIES

In the following we will derive solutions of equation (4). Thereby we restrict ourselves mainly to solutions which contain at least two field quantities. In principle, as next step we have to add solutions containing at least three field quantities and new independent coefficient functions, and so on. There appear no principal difficulties in doing this, however, with the rising number of field quantities the number of possible SU(3) invariants increases and makes the problem technically more complicated. At first we look for solutions of the pure gauge field identity

$$D^{ab}_\mu \frac{\delta\tilde\Gamma_0}{\delta a^b_\mu} = 0. \quad (5)$$

For this purpose we write the functional $\tilde\Gamma_0$ in the form

$$\tilde\Gamma_0(a) = 1 + \sum_{n=1}^{\infty}\frac{1}{n!}\int dz_1 \dots dz_n f_{n\mu_1 a_1 \dots \mu_n a_n} a^{a_1}_{\mu_1}(z_1)\dots a^{a_n}_{\mu_n}(z_n),$$

$$f_{n\ \cdots\mu_i a_i\cdots\mu_j a_j\cdots}(\ldots z_i,\ldots z_j\ldots) = f_{n\ \cdots\mu_j a_j\cdots\mu_i a_i\cdots}(\ldots z_j\ldots z_i\ldots).$$

Using the functional equation (5) and the explicit representation given above we derive a set of coupled differential equations for the coefficient functions of this functional

$$\frac{\partial}{\partial z_\nu} f_{n+1\ \nu a\mu_1 a_1\cdots\mu_n a_n}(z,z_1,\ldots,z_n) = g\delta(z-z_1) f^{aa_1b} f_{n\ \mu_1 b\mu_2 a_2\cdots\mu_n a_n}(z_1,z_2,\ldots,z_n)$$

$$+ g\delta(z-z_2) f^{aa_2b} f_{n\ \mu_1 a_1\mu_2 b\cdots\mu_n a_n}(z_1,z_2,\ldots,z_n)+\ldots$$

$$+ g\delta(z-z_n) f^{aa_nb} f_{n\ \mu_1 a_1\cdots\mu_{n-1}a_{n-1}\mu_n b}(z_1,z_2,\ldots,z_n). \tag{6}$$

$$\frac{\partial}{\partial z_\nu} f_{1\ \nu a}(z) = 0. \tag{7}$$

For theories without spontaneous symmetry breaking we have $f_{1\ \mu a} = 0$, so that the system (6,7) starts with $f_{1\ \mu a} = 0$ $\frac{\partial}{\partial z_\nu} f_{2\ \nu a\mu_1 a_1}(z,z_1)=0$. If only a special solution of this system is found, then we denote this solution by $\{\Gamma_0^{(2)}=\{f_{n\ (\mu a)}^{(2)}\}$. Of course, it consists of the infinite set of coefficients $f_{n\ (\mu a)}^{(2)}$ which are determined by the first nonvanishing coefficient function satisfying $\frac{\partial}{\partial z_\nu} f_{2\ \nu a\mu_1 a_1}(z,z_1)=0$. This gives rise to a solution which contains at least two field quantities. As next step we have to consider a solution of our system (6,7) starting with $f_{1\ \mu_1 a_1} = 0$, $f_{2\ \mu_1 a_1\mu_2 a_2} = 0$, $\frac{\partial}{\partial z_\nu} f_{3\ \nu a\mu_1 a_1\mu_2 a_2}(z,z_1,z_2)=0$. We denote this solution by $\tilde\Gamma_0^{(3)}$ because it contains at least three field quantities. This procedure can be continued. The complete solution reads

$$\tilde\Gamma_0 = 1 + \sum_{k=2}^{\infty} \tilde\Gamma_0^{(k)} . \tag{8}$$

Of course at each step of the solution all possible SU(3) invariants have to be taken into account.

Let us consider the first step of this iteration procedure explicitly. By geometrical intuition two gauge invariant expressions(with at least two field quantities) are possible

$$\int dz_1 dz_2 G_{\mu_1\nu_1\mu_2\nu_2}(z_1,z_2) F^d_{\mu_1\nu_1}(z_1) \tilde P^{da}(z_1 z_2) F^a_{\mu_2\nu_2}(z_2).$$

$$\tilde P^{da}(z_1,z_2) = \{P\exp - ig\int_{z_2}^{z_1} \tilde a_\mu\, dx^\mu\}^{da}, \quad \tilde a_\mu = ia^a_\mu t^{bac},$$

$$2\int dz_1 dz_2 G_{\mu_1\nu_1\mu_2\nu_2}(z_1,z_2)\,\mathrm{Tr}\{P(z_2,z_1)F_{\mu_1\nu_1}(z_1)P(z_1,z_2)F_{\mu_2\nu_2}(z_2)\},$$

$$P(z_1,z_2) = P\exp - ig\int_{z_2}^{z_1} a_\mu dx^\mu.$$

For the minimal number of field quantities both expressions are identical to $\int dz_1 dz_2 G_{\mu_1\nu_1\mu_2\nu_2}(z_1,z_2)\hat F_{\mu_1\nu_1}(z_1)\hat F_{\mu_2\nu_2}(z_2)$, $\hat F^a_{\mu\nu} = a^a_{\nu,\mu} - a^a_{\mu,\nu}$, so that only one expression has to be taken into account. We choose

$$I_0^{(2)} = \int dz_1 dz_2\, G_{\mu_1\nu_1\mu_2\nu_2} F^d_{\mu_1\nu_1}(z_1)\,\tilde P^{da}(z_1,z_2) F^a_{\mu_2\nu_2}(z_2). \tag{9}$$

In comparison with $a_\mu = a^a_\mu t^a$ the matrix $t^a_{\alpha\beta}$ is substituted by the matrix $\theta^a_{bc} = if^{bac}$ that obeys the same commutation relations and belongs to the adjoint representation. For completeness we show that the original expression

$$I_0^{(2)} = \frac{1}{2!}\int dz_1 dz_2\, f^{(2)}_{\mu_1 a_1 \mu_2 a_2} a^{a_1}_{\mu_1} a^{a_2}_{\mu_2} \quad \text{with} \quad \frac{\partial}{\partial z_1^{\mu_1}}\cdot f^{(2)}_{\mu_1 a_1 \mu_2 a_2}(z_1,z_2) = 0,$$

$\frac{\partial}{\partial z_2^{\mu_2}}\cdot f^{(2)}_{\mu_1 a_1 \mu_2 a_2}(z_1,z_2) = 0$ corresponds to our ansatz (9) for the minimal number of field quantities. Indeed using the representation

$$f^{(2)}_{\mu_1 a_1 \mu_2 a_2} = 2\delta^{a_1 a_2}\epsilon_{\mu_1\nu_1\sigma_1\rho}\,\epsilon_{\mu_2\nu_2\sigma_2\rho}\,\frac{\partial}{\partial z_1^{\sigma_1}}\,\frac{\partial}{\partial z_2^{\sigma_2}} G_{\nu_1\nu_2}(z_1,z_2)$$

we get

$$\int dz_1 dz_2 f^{(2)}_{\mu_1 a_1 \mu_2 a_2} a^a_{\mu_1} a^a_{\mu_2} = \frac{1}{2}\int dz_1 dz_2\, \epsilon_{\mu_1\nu_1\sigma_1\rho}\,\epsilon_{\mu_2\nu_2\sigma_2\rho}\, G_{\sigma_1\sigma_2}\hat F^a_{\mu_1\nu_1}(z_1)\hat F^a_{\mu_2\nu_2}(z_2)$$

if partial integration is allowed. This last point will not be discussed here. It could be related to possible contributions from different topological sectors.

At last we have to check that the ansatz (9) satisfies eq. (5). With the help of

$$D^{ba}_\mu(a)\frac{\delta \tilde P^{cd}(z_1,z_2)}{\delta a^a_\mu(x)} = -f^{cbd'}\delta(x-z_1)\tilde P^{d'd}(z_1,z_2) + \delta(x-z_2)\tilde P^{cc'}(z_1,z_2) f^{c'bd}$$

and

$$D^{ab}_\mu(a)\frac{\delta F^d_{\mu\nu}(z)}{\delta a^b_\mu(x)} = -gf^{acd}\Big\{(a^c_\nu(x) - a^c_\nu(z))\frac{\partial}{\partial z_\mu} - (a^c_\mu(x) - a^c_\mu(z))\frac{\partial}{\partial z_\nu} - gf^{ceg}a^e_\mu a^g_\nu\Big\}\delta(x-z)$$

this can be easily seen.

As next step, we have to look for expressions which contain 3-point functions as the lowest coefficient functions. There are two expressions

$$\int dz_1 dz_2 dz_3\, G^{(1)}_{(\mu,\nu)}(z_i) \mathrm{Tr}\{P(z_3,z_1) F_{\mu_1\nu_1}(z_1) P(z_1,z_2) F_{\mu_2\nu_2}(z_2) P(z_2,z_3) F_{\mu_3\nu_3}(z_3)\},$$

$$\int dz_1 dz_2 dz_3\, G^{(2)}_{(\mu,\nu)}(z_i) F^{a_1}_{\mu_1\nu_1}(z_1) \tilde{P}^{a_1 d}(z_1,z_2) f^{d a_2 d_3} F^{a_2}_{\mu_2\nu_2}(z_2) \tilde{P}^{d_2 a_3}(z_2,z_3) F^{a_3}_{\mu_3\nu_3}(z_3). \tag{10}$$

corresponding to different SU(3) invariants, but we will not outline this in detail. The integration path in the exponential is completely arbitrary. Variations of this path give contributions to higher coefficient functions which can be taken into account in the next step of the iteration procedure.

Turning back to the original identity (4) spinor fields have to be taken into account. Also in this case the general solution can be discussed in the previous manner. For shortness we write down the simplest gauge-invariant expression containing two spinor fields only

$$\int dx\, dy\, H_{st}(x,y) \bar{\chi}(x) P(x,y) \chi_t(y), \quad H_{st}.$$

With the help of the relation

$$D^{ab}_{\mu} \frac{\delta P(x,y)}{\delta a^b_\mu(z)} = ig\delta(z-x)\, t^a P(x,y) - ig\delta(z-y)\, P(x,y)\, t^a$$

it is easily seen that this is a solution of eq. (4). Again we have to construct gauge-invariant expressions containing more than two spinor fields. We will not do this in detail, as we think the construction is clear: one has to form SU(3) invariants by taking into account that the exponentials connect SU(3) transformations $\hat{\Omega}(x)$ at different points $\hat{P}(x,y) = \hat{\Omega}(x)\hat{\Omega}^{-1}(y)$ ($\hat{\Omega}$ belongs to the same representation as $\hat{a}_\mu$).

As final result we collect the first terms of the complete representation of the functional Γ

$$\Gamma = 1 + \frac{1}{\alpha} \int dx (n^\mu a^a_\mu)(n^\nu a^a_\nu) + \Gamma^{(2)} + \Gamma^{(3)} + \dots,$$

$$\Gamma^{(2)} = \int dz_1 dz_2\, G_{\mu_1\nu_1\mu_2\nu_2}(z_1,z_2) F^d_{\mu_1\nu_1}(z_1) \tilde{P}^{da}(z_1,z_2) F^a_{\mu_2\nu_2}(z_2) +$$

$$+ \int dx\, dy\, H_{rs}(x,y) \chi_r(x) P(x,y) \chi_s(y),$$

$$\Gamma^{(3)} = \int dz\, dx\, dy\, H(x,y,z)_{rs\mu\nu}\, \bar{\chi}_r(x) P(x,z) F_{\mu\nu}(z) P(z,y) \chi_s(y) +$$

$$+ \int dz_1 dz_2 dz_3\, G^{(1)}_{(\mu,\nu)}(z_i) \mathrm{Tr}(P(z_3,z_1) F_{\mu_1\nu_1}(z_1) P(z_1,z_2) F_{\mu_2\nu_2}(z_2) P(z_2,z_3) F_{\mu_3\nu_3}(z_3) +$$

$$+\int dz_1 dz_2 dz_3\, G^{(2)}_{(\mu,\nu)}(z_i) F^{a}_{\mu_1\nu_1}(z_1)\tilde{P}^{ad}(z_1,z_2) f^{dbc} F^{b}_{\mu_2\nu_2}(z_2)\tilde{P}^{cg}(z_2,z_3) F^{g}_{\mu_3\nu_3}(z_3)\,.$$

$$P(z_1,z_2) = P\exp(-ig\int_{z_2}^{z_1} a_\mu dx^\mu\,, \qquad \tilde{P}^{ab}(z_1,z_2) = [P\exp(-ig\int_{z_2}^{z_1}\tilde{a}_\mu dx^\mu)]^{ab}$$

H, G arbitrary coefficient functions.

3. WARD IDENTITIES IN QUANTUM ELECTRODYNAMICS

Here all expressions are simpler as in non-Abelian gauge theorie. Without group-theoretical and topological complications it is possible to write down all expressions completely. The foregoing results remain valid if we substitute $f^{abc}\to 0\; t^a\to 1$. The Ward identity for the 1-p-i functional takes the form

$$-\partial_\mu \frac{\delta\Gamma}{\delta a_\mu} - ig(\frac{\delta\Gamma}{\delta\chi})_R\chi + ig\bar{\chi}\frac{\delta\Gamma}{\delta\bar{\chi}} + \frac{1}{\alpha}(n\partial)(na) = 0\,.$$

The complete solution is written in the form

$$\Gamma = \Gamma + \frac{1}{2!}\int dz_1 dz_2\, E_{2\,\mu_1\mu_2}\, a_{\mu_1} a_{\mu_2} + \frac{1}{4!}\int dz_1 \ldots dz_4\, E_{4\,\mu_1\ldots\mu_4}\, a_{\mu_1}\ldots a_{\mu_4} + \ldots$$

$$+\int dxdy\, G_{rs}(x,y)\bar{\chi}_r(x)\, P(x,y)\chi_s(y) + \int dxdy\,dz\, G_{rs\mu}\,\bar{\chi}_r(x) P(x,y)\,\chi_s(y)\, a_\mu + \ldots$$

$$+\int dx_1 dy_1 dx_2 dy_2\, H_{r_1 s_1 r_2 s_2}\bar{\chi}_{r_1}(x_1)P(x_1,y_1)\chi_{s_1}(y_1)\bar{\chi}_{r_2}(x_2)P(x_2,y_2)\chi_{s_2}(y_2) + \ldots$$

$$+\ldots + \frac{1}{2\alpha}\int dx(n^\mu a_\mu)(n^\nu a_\nu)\,.$$

$$P(x,y) = \exp -ig\int_y^x a_\mu dx^\mu$$ E, G, H arbitrary coefficient functions

$$\frac{\partial}{\partial z_i^{\mu_i}} E_{\mu_1\ldots\mu_i\ldots\mu_n} = 0\,, \qquad \frac{\partial}{\partial z_i^{\mu_i}} G_{(r,s)\,\mu_1\ldots\mu_i\ldots\mu_n} = 0$$

$$\frac{\partial}{\partial z_i^{\mu_i}} H_{(r,s)\,\mu_1\ldots\mu_i\ldots\mu_n} = 0\,,\ldots \qquad H_{r_1 s_1 r_2 s_2}(x_1,y_1,x_2,y_2) = H_{r_2 s_1 r_1 s_2}(x_2,y_1,x_1,y_2)\,,\ldots$$

Looking at this expression we remark that the gauge field plays two different roles. The fields inside the exponential are mainly needed to guarantee gauge invariance whereas the physical important fields outside the exponential belong to transversal coefficient functions. In principle they can be represented by the field strength tensor. The same expression for Γ (with changed last term) can be obtained also for other gauges in QED.

REFERENCES

1. Zavialov O.I. Renormalized Feynman Diagrams. "Nauka", M., 1979.
2. Anikin S.A., Zavialov O.I. Ann.Phys., 1978, 166, p.135-166.
3. Bordag M., Robaschik D. Nucl.Phys., 1980, B169, p.445-460.
4. Zimmermann W. Brandeis Lectures 1970, S.Deser et al., Eds., MIT Press, Cambridge, 1971.
5. Slavnov A.A., Fadeev L.D. Introduction to Quantum Theory of Gauge Fields. "Nauka", Moscow, 1978.
6. Kartchev N.I. Renormalization in Massless Theories and Operator Product Expansion in Quantum Field Theory, talk at the Int.Conf.on Generalized Functions and Their Application in Math.Physics, Moscow, Nov. 1980.
7. Kummer W. Renormalization of Non-Abelian Gauge Fields. Schladming Lecture 1976, P.Urban Ed., Springer Verlag 1976.
8. Ellis R.K. et al. Phys.Lett., 1978, 78B, p.281-284.

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