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PATH-INTEGRAL EXPRESSION OF DISSIPATIVE DYNAMICS

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The description of dissipative systems via complex potentials is known to be phenomenologically useful; at the same time it can be incorporated (at least in principle and for reasonable potentials) into the standard quantum-mechanical framework/1/. The purpose of this note is to show that the rigorous path-integral theory/2-4/ can be applied to this case; full exposition of the subject will be given in our forthcoming papers. The dissipative system is "singled out" in this approach and its evolution operator is a contractive semigroup; in this sense the method is complementary to the approach based on the influence functional/5-7/ where the state norm is assumed to be preserved. The results presented below are particularly surprising because the heuristic considerations of Feynman/5/ have no direct counterpart here: one describes the corresponding classical systems, e.g., by time-dependent Lagrangians (see/8,9/ and references contained therein) or by the Rayleigh dissipation function/10/ instead of complex Hamiltonians.

Following the idea of Ito/11/, Albeverio and Hoegh-Krohn defined the Feynman integral $I(.)$ as follows: let $\mathcal{H}$ be a real separable Hilbert space of paths with the inner product $(.,.)$ and $f: \mathcal{H} \to \mathbf{C}$ belongs to $\mathcal{F}(\mathcal{H})$, i.e., $f(\gamma) = \int_{\mathcal{H}} e^{i(\gamma,\gamma')} d\mu(\gamma')$, where μ is a finite complex Borel measure on $\mathcal{H}$, then

$$I(f) = \int_{\mathcal{H}} \exp\left(-\frac{i}{2}\|\gamma\|^2\right) d\mu(\gamma). \tag{1}$$

Choosing now $\mathcal{H} = AC_0[J^t; \mathbf{R}^d]$, the space of all absolutely continuous $\mathbf{R}^d$-valued paths in $J^t = [0,t]$ with square-integrable derivatives and $\gamma(t) = 0$, equipped with the norm $\|\gamma\|^2 = \int_0^t \dot{\gamma}^2(\tau)\,d\tau$, we can use the definition (1) to express solution of the Schrödinger equation with absorptive potentials from the class $\mathcal{F}(\mathbf{R}^d)$, i.e., such that are Fourier transforms of finite Borel measures on $\mathbf{R}^d$:

<u>Theorem 1:</u> The continuous contractive semigroup $V = \exp(-iHt)$ corresponding to the pseudo-Hamiltonian $H = -\Delta + V$, $D(H) = D(-\Delta)$, with $V \in \mathcal{F}(\mathbf{R}^d)$ and

$$\operatorname{Im} V(x) \leq 0, \quad x \in \mathbf{R}^d \tag{2}$$

acts on an arbitrary $\phi \in \mathcal{F}(\boldsymbol{R}^d)$ according to the formula

$$(V_t \phi)(x) = I(g_{x,t}), \tag{3a}$$

where

$$g_{x,t}(\gamma) = \exp\{-i\int_0^t V(\gamma(\tau)+x)\,d\tau\}\,\phi(\gamma(0)+x). \tag{3b}$$

Sketch of the proof: The proof of (3) for real $V \in \mathcal{F}(\boldsymbol{R}^d)$ given in Refs.[2,3] consists essentially of two parts. First the path integral on the rhs of (3a) is evaluated by expansion of the exponent in (3b) and application of (1) to each term of the series; it can be done for any complex $V \in \mathcal{F}(\boldsymbol{R}^d)$ as well. Further the resulting series is identified with the Dyson expansion of $V_t \phi$ in the interaction picture. These considerations remain valid whenever the Dyson expansion is consistent, in particular for a bounded potential which obeys (2) (cf. Ref.[12], sec. X.12). ■

In order to deal with some other potentials including unbounded ones, one has to extend the definition (1). This can be done by the polygonal-path method of Truman[4], where the projection operators $P(\sigma)$:

$$(P(\sigma)\gamma)(\tau) = \gamma(\tau_i) + (\gamma(\tau_{i+1}) - \gamma(\tau_i))\,\delta_i^{-1}(\tau - \tau_i), \quad \tau \in [\tau_i, \tau_{i+1}] \tag{4}$$

corresponding to partitions $\sigma = \{0 = \tau_0 < \tau_1 < \ldots < \tau_n = t\}$ of J^t with $\delta_i = \tau_{i+1} - \tau_i$ are used to define

$$I_\infty(f) = \lim_{m\to\infty} I(f \circ P(\sigma^{(m)})). \tag{5}$$

for those f for which the limit exists; here $\{\sigma^{(m)}\}$ is the sequence of "equidistant" partitions: $\delta_i^{(m)} = t/m$. The definition (5) extends (1): it holds $I(f) = I_\infty(f)$ for any $f \in \mathcal{F}(\mathcal{H})$; let us remark that the same is true if an arbitrary "crumbling" sequence (i.e., $\lim_{m\to\infty}\delta_i^{(m)} = 0$) of partitions $\{\sigma^{(m)}\}$ is used instead the "equidistant" one. Now we are ready to formulate the assertion concerning the damped oscillator:

Theorem 2: The statement of Theorem 1 remains valid if $I(.)$ is replaced by $I_\infty(.)$ and

$$V(x) = x.Bx, \quad B = A - iW, \tag{6}$$

where A, W are real positive symmetric $d \times d$ matrices, W strictly positive. In this case, moreover, the functional integral can be evaluated:

$$I_\infty(g_{x,t}) = \int_{\mathbf{R}^d} G(x,y,t)\phi(y)\,dy, \tag{7a}$$

$$G(x,y,t) = (2\pi i)^{-d/2}(\det(\Omega^{-1}\sin\Omega t))^{-1/2}\exp\{\frac{i}{2}[x\cdot\Omega(\mathrm{tg}\,\Omega t)^{-1}x+ +y\cdot\Omega(\mathrm{tg}\,\Omega t)^{-1}y]-iy\cdot\Omega(\sin\Omega t)^{-1}x\}, \tag{7b}$$

where $\Omega=(2B)^{1/2}$, giving thus the Green function explicitly.

Sketch of the proof: Due to strict positivity of W the "cylindrical" integrals of (5) can be evaluated as follows[13]:

$$I(g_{x,t}\circ P(\sigma))=(2\pi i)^{-nd/2}\int_{\mathbf{R}^{nd}}\exp\{\frac{i}{2}\|\gamma_\sigma\|^2 - -i\int_0^t(\gamma(\tau)+x).B(\gamma_\sigma(\tau)+x)d\tau\}\phi(\gamma(0)+x)\,dm(\gamma_\sigma), \tag{8}$$

where $\gamma_\sigma=P(\sigma)\gamma$ and m is the Lebesgue measure on $\mathbf{R}^{nd}$. Substituting $\phi(x)=\int_{\mathbf{R}^d} e^{ix\cdot y}\,d\nu(y)$, γ_σ from (4) and $\xi_i=\gamma(\tau_i)+x$, and assuming $\delta_0=\delta_1=\ldots=\delta$, we get

$$I(g_{x,t}\circ P(\sigma))=(2\pi i)^{-nd/2}\exp(\frac{i}{2}x.C(\delta)x)\int_{\mathbf{R}^d}d\nu(y)\int_{\mathbf{R}^{nd}}\exp(i\xi\cdot\eta+ +\frac{i}{2}\xi\cdot M_n(\delta)\xi)\,d\xi, \tag{9}$$

where $M_n=M_n(\delta)$ is the $nd\times nd$ matrix

$$M_n=\begin{pmatrix} C & -D & 0 & 0 & \ldots & 0\\ -D & 2C & -D & 0 & \ldots & 0\\ 0 & -D & 2C & -D & \ldots & 0\\ \ldots & & & & & \ldots\\ 0 & \ldots & & & -D & 2C\end{pmatrix}, \tag{10}$$

with $C=C(\delta)=\delta^{-1}-\frac{2}{3}B\delta$, $D=D(\delta)=\delta^{-1}+\frac{1}{3}B\delta$, further $\xi=(\xi_0,\ldots,\xi_{n-1})$, $\eta=(y,0,\ldots,0,-Dx)$. The last integral in (9) contains complex matrix M_n so the standard tricks are not applicable, however, by straightforward calculation we get for it again $(2\pi i)^{nd/2}(\det M_n)^{-1/2}\exp(-\frac{i}{2}\eta\cdot M_n^{-1}\eta)$. Substituting this to (9) and using the fact that $C(\delta)$ and $D(\delta)$ commute mutually, we can write

$$I(g_{x,t} \circ P(\sigma)) = (\det[\tilde{d}(M_n)])^{-1/2} \exp\{\tfrac{i}{2} x \cdot H(\tilde{d}(M_n))^{-1} x\} \times$$

$$\int_{R^d} d\nu(y) \exp\{-\tfrac{i}{2} y \cdot \tilde{d}(K_{n-1})(d(M_n))^{-1} y + i y \cdot \tilde{D}^n (\tilde{d}(M_n))^{-1} x\}, \tag{11}$$

where $\tilde{D} = \delta D$, $\tilde{d}(M_n) = \delta^n d(M_n)$ and $d(M_n)$ is the $d \times d$ matrix representing the "block determinant" of M_n, further K_{n-1} is the lower right $(n-1)d \times (n-1)d$ submatrix of M_n and finally

$$H = C\tilde{d}(M_n) - \delta^{-1} \tilde{D}^2 \tilde{d}(M_{n-1}), \tag{12}$$

where M_{n-1} is the upper left $(n-1)d \times (n-1)d$ submatrix of M_n. One obtains easily

$$\tilde{d}(M_n) = C\tilde{d}(K_{n-1}) - \delta^{-1} \tilde{D}^2 \tilde{d}(K_{n-2}) \tag{13}$$

and the following recursive relation

$$\tilde{d}(K_{n-1}) = 2\delta C \tilde{d}(K_{n-2}) - \tilde{D}^2 \tilde{d}(K_{n-3}). \tag{14}$$

Further we put

$$\tilde{d}(K_{n-1}) = n\delta - \tfrac{2}{3} n \binom{n}{2} \delta^3 B + \sum_{j=2}^{\infty} a_j^{(n-1)}(\delta) (\tfrac{1}{3} B)^j \tag{15}$$

with $a_j^{(n-1)} = 0$ for $j > n-1$, then the relation (14) gives

$$a_j^{(n-1)} = 2a_j^{(n-2)} - a_j^{(n-3)} - 4\delta^2 a_{j-1}^{(n-2)} - 2\delta^2 a_{j-1}^{(n-3)} - \delta^4 a_{j-2}^{(n-3)}. \tag{16}$$

We take the following ansatz

$$a_j^{(n-1)}(\delta) = 3 \frac{(j+1)!}{(2j+1)!} \binom{n}{j+1} \delta^{2j+1} 2^j \sum_{k=0}^{j} \alpha_k^j n^k, \tag{17}$$

then (16) gives for the highest k

$$\alpha_j^j = (-1)^j 3^{j-1}, \quad \alpha_{j-1}^j = (-1)^{j-1} 3^{j-2} \binom{j}{2},$$

$$\alpha_{j-2}^j = (-1)^j 3^{j-2} \binom{j+2}{4}, \ldots \tag{18}$$

Now one has to check that the series (15) with $\delta = t/n$ converges uniformly w.r.t. n, then the relations (17,18) imply

$$\lim_{n \to \infty} \tilde{d}(K_{n-1}(t/n)) = \Omega^{-1} \sin \Omega t, \tag{19}$$

where $\Omega = (2B)^{1/2}$ (choice of branch of the square root is clearly irrelevant). Using further (12,13) we get

$$\lim_{n\to\infty} \tilde{d}(M_n(t/n)) = \cos\Omega t, \tag{20}$$

$$\lim_{n\to\infty} H(t/n) = -\Omega \sin\Omega t , \tag{21}$$

then we substitute from (19-21) to (12) and use the definition (5) obtaining thus

$$I_\infty(g_{x,t}) = (\det(\cos\Omega t))^{-1/2} \exp(-\frac{i}{2} x\cdot\Omega \operatorname{tg}\Omega t x) \int_{R^d} d\nu(y) \tag{22}$$

$$\cdot \quad \exp\{-\frac{i}{2} y\cdot\Omega^{-1} \operatorname{tg}\Omega t y + i y\cdot(\cos\Omega t)^{-1} x\}.$$

Using the matrix functional calculus rules[14] one can check that the function ψ: $\psi(x,t) = I_\infty(g_{x,t})$ solves the Schrödinger-type equation with the potential (6), mass $m = 1/2$ and initial condition $\psi(x,0) = \phi(x)$. Finally, by Fourier transformation of the integrand in (22) and interchange of integrations we obtain (7). ■

In conclusion let us make some remarks. The pseudo-Hamiltonian approach[1] used here makes possible to avoid usual peculiarities of time-independent Lagrangians. Comparing to Refs.[9,15] we do not assume any driving force (stochastic or not); on the other hand we study oscillators of arbitrary dimension (the generalizations to $d > 1$ is non-trivial in the damped case, because the matrices A, W are not necessarily simultaneously diagonalizable). Finally, notice that (7) gives correct propagator (including the phase factor) in the non-damped limit: if $d = 1$ and $\Omega = \omega_1 - i\omega_2$, then

$$\lim_{\omega_2\to 0+} \left(\frac{\Omega}{\sin\Omega t}\right)^{1/2} = \left(\frac{\omega_1}{|\sin\omega_1 t|}\right)^{1/2} \exp\{-\frac{\pi i}{2} \operatorname{Ent} \frac{\omega_1 t}{\pi}\}, \tag{23}$$

where the exponential factor is just the Maslov correction. This shows one more way how to prove the "extended" Feynman formula[16] for the undamped oscillator.

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