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CLASSICAL AND QUANTUM EQUATIONS OF MOTION FOR TENSOR OF ANGULAR MOMENTUM

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1. Vector-Function Action

In the frame of classical mechanics momentum $\vec{p}$ and angular momentum $\vec{K}$ are considered as an object of similar origin belonging to the group of generalized momenta conjugated in accordance with the generalized coordinates. In paper [1] it is shown, directions of investigation can be changed a little, if quantities $\vec{p}$ and $\vec{K}$ are considered relatively to one and the same coordinates defining angular momentum with the help of momentum $\vec{p}$ and radius-vector $\vec{r}$

$$\vec{K} = [\vec{r} \times \vec{p}]. \tag{1}$$

It is known from HJ[x)] theory that momentum vector can be represented as gradient of any scalar function

$$\vec{p} = \operatorname{grad} S. \tag{2}$$

In this case, it is easy to prove

$$\operatorname{div} \vec{K} = 0. \tag{3}$$

It follows that $\vec{K}$ can be represented as rotation of some vector-function

$$\vec{K} = \operatorname{rot} \vec{U}. \tag{4}$$

The comparison of the expressions (2) and (4) gives an idea that there exist equations of vector-function $\vec{u}$ analogous to HJ equation for function S . The needed equation can be obtained from the analysis of differential forms using the HJ theory, according to which

$$\vec{p} = \operatorname{grad} S, \quad H = -\frac{\partial S}{\partial t}, \tag{5}$$

where H-Hamiltonian of the system.

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x) HJ - Hamiltonian-Jacobi

Knowing quantities $\vec{p}$ and H , also function

$$H = H(\vec{p})$$

one can establish the form of equation for S and definition of its total differential

$$dS = (\vec{p}\,d\vec{r}) - H\,dt. \qquad (6)$$

Let us take into consideration such quantities, which must be known to define vector-function $\vec{u}$. With this purpose, we can present total differential in the following way

$$d\vec{U} = [\mathrm{rot}\,\vec{U}\times d\vec{r}] + \mathrm{grad}(\vec{U}d\vec{r}) + \frac{\partial \vec{U}}{\partial t}dt, \qquad (7)$$

it follows, that for solution of the given task it is necessary and sufficient to know the vectors

$$\vec{K} = \mathrm{rot}\,\vec{U}, \quad \vec{T} = -\frac{\partial \vec{U}}{\partial t} - \mathrm{grad}(\vec{U}\vec{v}) \qquad (8)$$

and also function

$$\vec{T} = \vec{T}(\vec{K}). \qquad (9)$$

According to definitions (1) and (2) vector $\vec{K}$ is vector of angular momentum. In literature vector $\vec{T}$ has not corresponding name; it can be said about this quantity that if $\vec{K}$ is considered as analog of momentum, $\vec{u}$ as analog of S , then $\vec{T}$ gets corresponding interpretation as the analog of Hamiltonian H. Using this analog, one can formulate stationary principle for vector-function action, which can be defined as an integral

$$\vec{U} = \int [\vec{K}\times d\vec{r}] - \vec{T}dt. \qquad (10)$$

According to the stationary principle the equations of motion define such trajectories, for which

$$\delta\vec{U} = 0.$$

Let us calculate variation of integral (10)

$$\delta\vec{U} = \int [\delta\vec{K}\times d\vec{r}] + [\vec{K}\times d(\delta\vec{r})] - \delta\vec{T}dt = 0, \qquad (11)$$

where

$$\delta\vec{T} = [\mathrm{rot}_K\vec{T}\times\delta\vec{K}] + [\mathrm{rot}_r\vec{T}\times\delta\vec{r}], \qquad (12)$$

$$\int [\vec{K}\times d(\delta\vec{r})] = \int [\delta\vec{r}\times d\vec{K}]. \qquad (13)$$

Substituting (12) and (13) in (11) and equating to zero expressions at the same variations, one obtains

$$\frac{d\vec{r}}{dt} = -\mathrm{rot}_K\vec{T}, \quad \frac{d\vec{K}}{dt} = -\mathrm{rot}_r\vec{T}. \qquad (14)$$

The first of equation defines the explicit form of function (9). It follows that

$$\vec{T} = \frac{1}{2}[\vec{K}\times\vec{v}] + \vec{V}(r). \qquad (15)$$

To find out the sense of the second equations let us consider the equation of motion for angular momentum.

According to Newton's Second Law

$$\frac{d\vec{K}}{dt} = [\vec{r}\times\vec{F}]. \qquad (16)$$

Let us admit that

$$\vec{F} = -\mathrm{grad}\,\varphi,$$

where φ - potential function. Then

$$\mathrm{div}[\vec{r}\times\vec{F}] = 0,$$

or

$$[\vec{r}\times\vec{F}] = -\mathrm{rot}\,\vec{V}.$$

It follows that equations (14) and (16) are equivalent. The equations (14) are analogous to the Hamiltonian equations. With the help of formulae (8) and (15) one can derive the analog of HJ equation for vector-function. This equation has the form

$$-\frac{\partial\vec{U}}{\partial t} - \mathrm{grad}(\vec{U}\vec{v}) = \frac{1}{2}[\mathrm{rot}\,\vec{U}\times\vec{v}] + \vec{V}(r). \qquad (17)$$

Here we deal with canonical transformations as well as in the frame of the traditional HJ theory

$$\vec{r} = \vec{r}(\vec{r}_0, \vec{K}_0, t), \qquad (18)$$
$$\vec{K} = \vec{K}(\vec{r}_0, \vec{K}_0, t),$$

preserving the form of equations (14). $(\vec{r}, \vec{K})$ and $(\vec{r}_0, \vec{K}_0)$ - joint form of new and old coordinates and component of angular momentum vector, correspondingly. In this case vector-function u plays the role of a producing function.

So far as

$$\vec{U} = \int_{r_0}^{r} [\vec{K}\times d\vec{r}] - \int_{t_0}^{t} \vec{T}dt + \vec{U}_0$$

we get the following equality

$$d\vec{U} = [\vec{K}\times d\vec{r}] - [\vec{K}_0\times d\vec{r}_0] - \vec{T}dt + \vec{T}_0 dt_0 + d\vec{U}_0 \qquad (19)$$

which shows the sense of equations (17). From the equality (19) it follows that

$$\vec{K} = rot_r \vec{U}, \quad \vec{K}_0 = rot_{r_0} \vec{U},$$
$$\vec{T} = -\frac{\partial \vec{U}}{\partial t} - grad(\vec{U}\vec{v}).$$

In a number of cases, the equations (17) and presented formalism can be applied for integration of movement equations (14). This method can turn out to be useful, when in the task external vector fields of $\vec{V}$ type appear interacting only with angular momentum. If external field is introduced in the usual way, e.g.,by means of extendary momentum and energy

$$\begin{aligned} &\vec{p} \to \vec{p} + \frac{e}{c}\vec{A}, \\ &H \to H - e\varphi, \end{aligned} \qquad (20)$$

then the above-worked out formalism is completely equivalent to the HJ theory. Indeed, in this case we can give the following system equations instead of (15):

$$\begin{aligned} &\vec{T} + \frac{1}{2m}[(\vec{p}+\frac{e}{c}\vec{A})\times\vec{K}] = 0, \\ &[(\vec{p}+\frac{e}{c}\vec{A})\times\vec{T}] - (H-e\varphi)\vec{K} = 0, \\ &((\vec{p}+\frac{e}{c}\vec{A})\vec{K}) = 0, \quad ((\vec{p}+\frac{e}{c}\vec{A})\vec{T}) = 0. \end{aligned} \qquad (21)$$

It follows from the system (21) that vectors $\vec{T}$, $\vec{K}$ and are mutually perpendicular. Let us exclude vector $\vec{T}$ from the system, then we shall get the following equation for the vector $\vec{K}$

$$-\frac{1}{2m}[(\vec{p}+\frac{e}{c}\vec{A})\times[(\vec{p}+\frac{e}{c}\vec{A})\times\vec{K}]] - (H-e\varphi)\vec{K} = 0.$$

Opening the brackets according to the well-known formula of vector calculation, we obtain

$$-\frac{1}{2m}(\vec{p}+\frac{e}{c}\vec{A})((\vec{p}+\frac{e}{c}\vec{A})\vec{K}) + \vec{K}\frac{(\vec{p}+\frac{e}{c}\vec{A})^2}{2m} - (H-e\varphi)\vec{K} = 0,$$

but $((\vec{p}+\frac{e}{c}\vec{A})\vec{K}) = 0$; consequently

$$[\frac{1}{2m}(\vec{p}+\frac{e}{c}\vec{A})^2 - (H-e\varphi)]\vec{K} = 0$$

or

$$\frac{1}{2m}(\vec{p}+\frac{e}{c}\vec{A})^2 = (H-e\varphi). \qquad (22)$$

Thus, we got the known formula from nonrealitivistic classical mechanics. In this case the expression given in brackets is equal to determinant of the system (21), and the fact that it is equal to zero supposes nontrivial solutions for the given system.

2. Quantum Equation of Motion for Angular Momentum Vector

Though the above-presented formalism as well as the HJ method can be of some use at solving mechanics problems, however, their main value from the modern point of view is that they play essential role in developing new theory. In particular the classical mechanics conception of the HJ type was the starting point in developing of quantum mechanics. Looking forward, let us note, that the generalization of the HJ equations in the spirit of L.de Broglie leads us to the Schrödinger equation, then the application of this procedure to the equations (21) brings to the Pauli equation for the spin 1. Thus we can say that the presence of the spin is included in classical equations (21), but electromagnetic interaction does not reveal it. As it is known, the spin effects appear only in the quantum mechanics.

In the equations (21) let us make transit into quantum mechanics according to the well-known receipt, replacing

p_i for operator $-i\hbar\frac{\partial}{\partial x^i}$;

H for operator $i\hbar\frac{\partial}{\partial t}$.

The system (21) transforms in the following system of differential equations

$$
\begin{aligned}
&\vec{T} + \frac{1}{2m}\left[\left(-i\hbar\frac{\partial}{\partial \vec{r}} + \frac{e}{c}\vec{A}\right)\times\vec{K}\right] = 0,\\
&\left[\left(-i\hbar\frac{\partial}{\partial \vec{r}} + \frac{e}{c}\vec{A}\right)\times\vec{T}\right] - \left(i\hbar\frac{\partial}{\partial t} - e\varphi\right)\vec{K} = 0,\\
&\left(\left(-i\hbar\frac{\partial}{\partial \vec{r}} + \frac{e}{c}\vec{A}\right)\vec{K}\right) = 0, \quad \left(\left(-i\hbar\frac{\partial}{\partial \vec{r}} + \frac{e}{c}\vec{A}\right)\vec{T}\right) = 0.
\end{aligned} \tag{23}
$$

As it should be, in quasi-classical approximation equations (23) transform into the system (21). The solving in the quasi-classical approximation can be presented in the form

$$
\begin{aligned}
\vec{K} &= e^{iS/\hbar}\left[\vec{K}_0 + \hbar\vec{K}_1 + \ldots\right],\\
\vec{T} &= e^{iS/\hbar}\left[\vec{T}_0 + \hbar\vec{T}_1 + \ldots\right].
\end{aligned} \tag{24}
$$

Substituting (24) in (23) and equating to zero the coefficients at different degrees $\hbar$, we obtain

$$
\begin{aligned}
&\vec{T}_0 + \frac{1}{2m}\left[\left(\frac{\partial S}{\partial \vec{r}} + \frac{e}{c}\vec{A}\right)\times\vec{K}_0\right] = 0,\\
&\left[\left(\frac{\partial S}{\partial \vec{r}} + \frac{e}{c}\vec{A}\right)\times\vec{T}_0\right] - \left(-\frac{\partial S}{\partial t} - e\varphi\right)\vec{K}_0 = 0,\\
&\left(\left(\frac{\partial S}{\partial \vec{r}} + \frac{e}{c}\vec{A}\right)\vec{K}_0\right) = 0, \quad \left(\left(\frac{\partial S}{\partial \vec{r}} + \frac{e}{c}\vec{A}\right)\vec{T}_0\right) = 0,
\end{aligned}
$$

e.i., equations (21).

In the absence of the external field the system of differential equations of the first order (23) is identical to the Schrödinger equation. The including of the external field leads to the appearance of the additional components in the Schrödinger equation. The appearance of the additional components is connected with noncommutativity of the component-operator

$$\pi_i = -i\hbar\frac{\partial}{\partial x_i} + \frac{e}{c}A_i. \tag{25}$$

Let us consider permanent electromagnetic field. The operator (25) satisfies the following commutation relations

$$
\begin{aligned}
[\pi_x, \pi_y] &= -i\hbar\frac{e}{c}\mathcal{H}_z,\\
[\pi_y, \pi_z] &= -i\hbar\frac{e}{c}\mathcal{H}_x,\\
[\pi_z, \pi_x] &= -i\hbar\frac{e}{c}\mathcal{H}_y,
\end{aligned} \tag{26}
$$

where $\mathcal{H}_x, \mathcal{H}_y, \mathcal{H}_z$ are components of magnetic field. Let us exclude vector T from the equation (23). Then with the help of (26) one can obtain the following equations for the K

$$i\hbar\frac{\partial}{\partial t}\vec{K} = \hat{H}_0\vec{K} + \mu\left[\vec{\mathcal{H}}\times\vec{K}\right], \tag{27}$$

where

$$\mu = \frac{\hbar e}{2mc},$$

$$\hat{H}_0 = \frac{1}{2m}\left[\pi_x^2 + \pi_y^2 + \pi_z^2\right] - e\varphi.$$

Let us present vector-function K in the form of a column

$$\hat{K} = \begin{pmatrix} K_x \\ K_y \\ K_z \end{pmatrix},$$

then we can rewrite equation (27) in the following way

$$i\hbar\frac{\partial}{\partial t}\hat{K} = \hat{H}_0\hat{K} + \mu(\mathcal{H}\hat{\tau})\hat{K}, \tag{28}$$

where $\hat{\tau}$ is the operator with components

$$\hat{\tau}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \hat{\tau}_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \hat{\tau}_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{29}$$

As is known, the operator $\hat{\tau}$ is generator of SU (2) group. Dimensionality of this group is three, Lie algebra in the joint presentation is given by antisymmetrical matrix (3x3) of form (29). The matrices (29) satisfy commutation relations

$$[\tau_i, \tau_j] = \varepsilon_{ijk}\tau^k,$$

ε_{ijk} -total antisymmetrical tensor, $\varepsilon_{123} = 1$.

The equation (28) is the Pauli equation for the spin 1. It can be obtained directly replacing the Pauli equation operator $\hat{\tau}$ for operator $\hat{\sigma}$ (Pauli matrices). But above-given way of obtaining equation (28) is of methodological interest, as it shows the connection between the quantum equation of the Pauli type with classical equations (21). The equations (21) describe movement of the angular momentum in the external field. The following fact is of interest, the transition to quantum equations from (21) led to the appearance of spin 1 in the system, e.i., quantization of the angular momentum. The wave function in the case of the free movement has the form

$$\vec{K} = \vec{K}_0 e^{\frac{i}{\hbar}(p\tau - Et)}.$$

Thus amplitude of the wave function corresponds to the vector of the angular momentum in the probability interpretation.

The movement in the permanent magnetic field can be taken as an example [2]. If magnetic field is rather weak then we can neglect the components in the operator $\hat{H}_0$, containing square of the vector potential. In this case the following approximate expression for Hamiltonian can be obtained

$$\hat{H} = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) - e\varphi + \frac{e}{2mc}\left[(\hat{M} + \hbar\hat{\tau})\vec{\mathcal{H}}\right], \quad (30)$$

$\hat{M}$ - operator of orbital momentum.

In the task of spherical symmetry depending on the magnetic field additional part of energy operator commutate with main part. So the addition to the energy level in a magnetic field is in the summing to it an eigenvalue of additional component. If axis z is directed along a magnetic field, then the addition will be

$$\Delta E = \frac{e\hbar}{2mc}(M \pm \Delta M)\mathcal{H}_z, \quad (31)$$

where M is eigenvalue of operator $\hat{M}_z$, and $\Delta M = \pm 1, 0$ is eigenvalue of operator $\hat{\tau}$.

3. Classical and Quantum Equation of Motion for Angular Momentum in STR*)

In the frame of STR the following expression for antisymmetrical tensor of the angular momentum with the help of 4-dimentional vectors of momentum p_i and coordinates x_i

$$M_{ij} = x_i p_j - x_j p_i,$$
$$(x_i) = (x, y, z, ct), \quad (p_i) = (p_x, p_y, p_z, \frac{E}{c}), \quad (32)$$

whith generalized expression (1) in the case of relativity [3].

Reveal tensor (32) in the matrix form

$$(M_{ij}) = \begin{pmatrix} 0 & K_1 & -K_2 & T_1 \\ -K_1 & 0 & K_3 & T_2 \\ K_2 & -K_3 & 0 & T_3 \\ -T_1 & -T_2 & -T_3 & 0 \end{pmatrix}. \quad (33)$$

Thus vectors K and T in STR unite in one antisymmetrical tensor, where

$$\vec{K} = [\vec{\tau} \times \vec{p}], \quad \vec{T} = \vec{\tau}\frac{E}{c} - (tc)\vec{p}. \quad (34)$$

In the HJ theory, where

$$p_i = \frac{\partial S}{\partial x^i},$$

the expression (32) can be presented in the form

$$M_{ik} = \frac{\partial \mathcal{U}_i}{\partial x^k} - \frac{\partial \mathcal{U}_k}{\partial x^i}, \quad (35)$$

generalizing formulae (8).

Let us obtain equations of 4-vector-function $\mathcal{U}_i$ in the frame of HJ theory. With this aim we can multiplay (33) from the left on p^i and sum it up. Taking into consideration the general relation of the classical relativistic mechanics

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*) STR - special theory of relativity

$$p^{\ell}p_{\ell} = -m_o^2c^2, \qquad (36)$$

we get

$$m_o^2c^2X_K + p_K S = p^{\ell}M_{\ell K},$$
$$S = p^{\ell}X_{\ell}. \qquad (37)$$

Eq. (35), being substituted in the system (37), turns into the following equation of HJ type for 4-vector-function U_1 generalizing equation (17) for the case of STR

$$m_o^2c^2X_K + p_K\frac{\partial U_{\ell}}{\partial X_{\ell}} = p^{\ell}\left(\frac{\partial U_{\ell}}{\partial X^K} - \frac{\partial U_K}{\partial X^{\ell}}\right). \qquad (38)$$

The relations (35) give the idea of the existence of the Maxwell type wave equations for function U_1. In fact, if the first pair of Maxwell's equations can be obtained from (35), then we have the equations (37) as the analog of the second pair (in eikonal approximation). It is interesting, in this case the following expression plays the role of "current"

$$J_K = m_o^2c^2X_K + p_K S,$$

it can be easily checked

$$p^{\ell}J_{\ell} = 0.$$

Quantum-mechanical wave equations can be obtained, naturally, as well as in the nonrelativistic case, replacing operator

$$p_i \to \pi_i \equiv -i\hbar\frac{\partial}{\partial x^i} + \frac{e}{c}A_i \qquad (39)$$

instead of vector p_i.

Thus the equations (35) and (37) can be rewritten in the form

$$M_{iK} = \pi_K U_i - \pi_i U_K,$$
$$m_o^2c^2U_K + \pi_K S = \pi^{\ell}M_{\ell K}, \qquad (40)$$
$$S = \pi^{\ell}U_{\ell}.$$

In the absence of interaction the system of equations (40) is equivalent to the equation of Klein-Gordon. Supposing S =0, equations (40) transform into the well-known equations of Proca [4], however, unlike the latter, external field in (40) is included correctly. It is interesting, that one of generalizations of the Proca equations (Strückelberg formalism) has the form of (40). The generalizations of the Proca equations undertaken to remove contradiction in the procedure of interaction including. The detailed discussion of this problem can be found in paper [5].

The equations systems of the first order (40) can be written in the form of equations of the second order for (u_1,s). They are

$$(\pi^{\ell}\pi_{\ell} + m_o^2c^2)U_K + [\pi_K, \pi^{\ell}]U_{\ell} = 0,$$
$$(\pi^{\ell}\pi_{\ell} + m_o^2c^2)S + \frac{i\hbar e}{2c}F_{K\ell}U^K = 0, \qquad (41)$$
$$F_{K\ell} = \frac{\partial A_{\ell}}{\partial X^K} - \frac{\partial A_K}{\partial X^{\ell}}.$$

Let us consider the case of permanent magnetic field $\vec{\mathcal{H}}$. Here we obtain the following equation for the spatial part of the vector u = (u, u_o):

$$(\hat{H}^2 - \hat{p}^2)\vec{U} - \frac{i\hbar e}{c}[\vec{\mathcal{H}} \times \vec{U}] = m_o^2c^2\vec{U},$$

$$\hat{H} = i\hbar\frac{\partial}{\partial t} - e\varphi, \quad \hat{p} = -i\hbar\frac{\partial}{\partial X} + \frac{e}{c}\vec{A}.$$

As is shown above, the interaction of spin 1 with magnetic field corresponds to the expression in square brackets.

Ordinarily it is assumed that the system of wave functions (40) corresponds to elementary particles (e.g.,vector mesons) with spin equal to one. However, correspondence is known not to be so adequate as, for example, correspondence of the

Dirac equations to electron. The above-given way of deriving the equations (40) allows one to interpret them as wave relativity equations for the tensor of angular momentum. Usually it was assumed, that only the HJ equations had the corresponding classical analogy in the form of expression (36) or relativistic equation of HJ. The system Dirac, Proca, etc., were formed by factorization of Klein-Gordon operator. In the present paper we have derived equations of Proca type for spin 1 according to the principle of correspondence of the classical mechanics. We can propose the similar prosedure can be used for the equation of Dirac type at the corresponding modification of method. However, the realization of it will be discussed in the following paper.

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