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COVARIANT THREE-DIMENSIONAL EQUATION FOR FERMION-ANTIFERMION SYSTEM.

Part I

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I. Introduction

In a series of papers /1–4/ the three-dimensional relativistic formalism was developed for describing the systems composed of two particles with spin 1/2. The description is based on three-dimensional relativistic two-particle equations of Logunov and Tavkhelidze /5/ and Kadyshevsky /6/ in the c.m.s.

A merit of the three-dimensional relativistic spin formalism proposed in /1–4/ is the keeping in the equation kernel all the relativistic information contained in the Feynman matrix elements from which the kernel is constructed. This result is based on a new three-dimensional representation /1–3/ for describing of spin particles interactions in a relativistic case. The transition of the interaction kernel to this three-dimensional representation does not need the application of expansion of Feynman matrix elements in powers of v^2/c^2, like in the case of application of Foldy-Wouthuysen transformation in the Breit equation. In our approach the three-dimensional form is achieved by giving to the four-dimensional matrix elements the sence of a three-dimensional geometric generalization of the quantum-mechanical Breit-Fermi potentials. After this transformation it appears that the Feynman matrix element of one boson exchange, taken in the momentum representation, has the same structure as the Breit-Fermi potential, but the Euclidean elements, such as momentum transfer, been substituted by their analogs from the Lobachevsky momentum space. In this way the three-dimensional formulation is conserved in the relativistic description.

This "succession" of the form of spin two-particle equations takes place also in the configurational representation /4/ if in the relativistic case the Fourier transformation is replaced by the harmonic analysis on the Lorentz group /7/.

The formalism developed in /1–4/ is quite sufficient for calculation of energy levels of relativistic composite systems like vector mesons. However, if we want to use the wave functions, obtained as solutions of three-dimensional equations, for describing the system form factors then we should generalize the spin formalism developed in the c.m.s. to an arbitrary coordinate system [x)]. This problem is just considered in this work.

Our consideration will be based on the diagram technique arising in the covariant Hamiltonian formulation of quantum field theory /6/. Note that in this formalism the covariant three-dimensional equation was earlier obtained for the amplitude of fermion-antifermion elastic scattering /9/. Using this equation and the fermion-antifermion wave function defined in /9/ through the scattering amplitude, the authors of /9/ deduced an equation for the spin wave function for the c.m.s. In this connection, in Sec.2 we present the derivation of a covariant equation for the fermion-antifermion wave function by using an equation for the vertex function, like it has been done in the spinless case in ref./10/.

However, for practical application there still remains an important problem to be solved: is it possible to construct a covariant spin equation which would contain an interaction local in the Lobachevsky momentum space? This question is answered in Sec.3 where as an example one-pion-exchange amplitude would be considered.

2. Equation for a two-fermion wave function and Lobachevsky geometry

Following ref. /10/, we proceed from the equation for R-matrix/6/

$$R(\lambda\tau;\lambda\tau') = -\mathcal{H}(\lambda\tau-\lambda\tau') - \int_0^\infty \mathcal{H}(\lambda\tau-\lambda\tau_1)\frac{d\tau_1}{2\pi(\tau_1-i\epsilon)} R(\lambda\tau_1;\lambda\tau') \qquad (2.1)$$

related to S-matrix by $S = 1 + iR(0;0)$, $R(0;0) = R(\lambda\tau;\lambda\tau')/_{\tau=\tau'=0}$.
Here $\mathcal{H}(\lambda\tau)$ is the Fourier transform of the interaction Hamiltonian; λ, the unit time-like vector [xx)] $\lambda_\mu\lambda^\mu = 1$ with $\lambda_0 > 0$.

Equation (2.1) can be written in the operator form

$$\hat{R} = -\hat{H} - \hat{H}\hat{G}_0\hat{R} \quad ,$$

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x)It should be noted that the covariant equations for the wave function of a two-fermion system were also derived within a single-time formalism in ref./8/.

xx)As for the formulation of equations with λ vector belonging to the light cone, i.e., $\lambda_0^2 - \vec{\lambda}^2 = 0$, see papers /11/.

where the operator matrix elements are defined as follows (in the space of absolute values of the spurion momenta $|\lambda\tau|^2 = \tau^2$)

$$\langle\tau|\hat{R}|\tau'\rangle = R(\lambda\tau;\lambda\tau')\ ;\ \langle\tau|\hat{H}|\tau'\rangle = \mathcal{H}(\lambda\tau-\lambda\tau')$$

$$\langle\tau|\,G_0|\tau'\rangle = \left[2\pi(\tau'-i\varepsilon)\right]^{-1}\delta(\tau-\tau').$$

In order that the equation for the wave function contains the same kernel as that one obtained in /6,9/ for the scattering amplitude, let us perform one iteration of the operator equation and make one substitution

$$R = R' - (1 - HG_0HG_0)^{-1}H.$$

After this we come to an equation /6/

$$R' = K_2 + K_2\,G_0\,R' \qquad (K_2 \equiv HG_0H).$$

Further, in the spin case we define by analogy with /10/ the vertex function $\Gamma^{\sigma_1\sigma_2}_{J,m_J}(p_1,p_2|\mathcal{P},\lambda\tau)$ as a matrix element of the operator $R'(\lambda\tau)=R(\lambda\tau,0)$ (primes will be omitted in what follows):

$$\langle\vec{p}_1\sigma_1;\vec{p}_2\sigma_2|R(\lambda\tau)|\vec{\mathcal{P}},M,J,m_J\rangle = \tag{2.2a}$$
$$=(2\pi)^4\,\frac{\delta^{(4)}(\mathcal{P}-p_1-p_2+\lambda\tau)}{\sqrt{2\mathcal{P}_0\cdot 2p_{10}\cdot 2p_{20}}}\cdot\Gamma^{\sigma_1\sigma_2}_{J,m_J}(p_1,p_2|\mathcal{P},\lambda\tau),$$

$$\Gamma^{\sigma_1\sigma_2}_{J,m_J}(p_1,p_2|\mathcal{P},\lambda\tau) = \bar{u}^{\sigma_1}_{\alpha}(p_1)\,\Gamma_{\alpha\beta}(p_1,p_2|\mathcal{P},\lambda\tau;J,m_J)\,v^{\sigma_2}_{\beta}(p_2). \tag{2.2b}$$

Here $|\vec{\mathcal{P}},M;J,m_J\rangle$ is the state vector of a composite system with momentum $\vec{\mathcal{P}}$, mass $M=\sqrt{\mathcal{P}_0^2-\vec{\mathcal{P}}^2}$, intrinsic momentum (spin) J, and its projection m_J onto the z-axis; $\vec{p}_1$ and $\vec{p}_2$ are momenta of the fermion and antifermion. In representation (2.2b) the fermion and antifermion are described by bispinors $u^{\sigma_1}_{\alpha}(p)$ and $v^{\sigma_2}_{\beta}(p)$ ($\alpha,\beta=1,2,3,4$; σ_1,σ_2 are polarization indices equal to $\pm 1/2$) which obey the normalization conditions

$$\bar{u}^{\sigma_1}_{\alpha}(p)\,u^{\sigma_2}_{\alpha}(p) = 2m\,\delta_{\sigma_1\sigma_2}\quad ;\quad \bar{v}^{\sigma_1}_{\alpha}(p)\,v^{\sigma_2}_{\alpha}(p) = 2m\,\delta_{\sigma_1\sigma_2}.$$

The argument of δ-function in (2.2a) contains the 4-vector $\lambda\tau$, which signifies that the vertex function $\Gamma(p_1,p_2|\mathcal{P},\lambda\tau)$ is considered off the energy shell. Indeed, at $\lambda=(1,\vec{0})$ $\tau = \mathcal{P}_0-(p_1+p_2)_0$, so that to the energy shell $\tau=0$ corresponds. The normalization of state vectors will be chosen in the form

$$\langle\vec{p}_1,\sigma_1|\vec{p}_2,\sigma_2\rangle = 2p_{10}\,\delta(\vec{p}_1-\vec{p}_2)\,\delta_{\sigma_1\sigma_2}.$$

Passing in the r.h.s. and l.h.s. of (2.1) to the matrix elements of R and the product KR taken over the same state vectors as (2.2a) we arrive at the relation between Γ and matrix elements of the type $\langle \vec{k}_1\sigma_1, \vec{k}_2\sigma_2, \ldots\ldots \vec{k}_n\sigma_n | R | \vec{\mathcal{P}}, M; J, m_J\rangle$, where $n = 2, 3, \ldots$. For any matrix elements $\langle \vec{k}_1\sigma_1, \ldots\ldots \vec{k}_n\sigma_n | R | \vec{\mathcal{P}}, M; J, m_J\rangle$ with the use of (2.2) one can, in turn, obtain analogous integral relations which form an infinite chain of the coupled integral equations. Applying further a technique developed in [6] for successive elimination of such matrix elements with the aid of other equations of the system [12] we can write in the matrix form (cf. [6]):

$$\langle \vec{p}_1\sigma_1, \vec{p}_2\sigma_2 | R(\lambda\tau) | \vec{\mathcal{P}}, M; J, m_J\rangle = \frac{1}{(2\pi)^7}\sum_{\nu_1\nu_2}\int \langle \vec{p}_1\sigma_1, \vec{p}_2\sigma_2 | \tag{2.3}$$

$$\mathcal{K}(\lambda\tau, \lambda\tau_1) | \vec{k}_1\nu_1, \vec{k}_2\nu_2\rangle \frac{d\vec{k}_1\, d\vec{k}_2\, d\tau_1}{\tau_1 - i\varepsilon} \langle \vec{k}_1\nu_1, \vec{k}_2\nu_2 | R(\lambda\tau_1) | \vec{\mathcal{P}}, M; J, m_J\rangle,$$

where the kernel

$$\mathcal{K} = K_2 \frac{1}{1 - G_0(1 - \Pi_2)K_2} = K_2 + K_2 G_0 (1 - \Pi_2) K_2 + \ldots$$

contains the projection operator Π_2 on two-particle states $|2\rangle = |\vec{k}_1\nu_1, \vec{k}_2\nu_2\rangle$. In the lowest order in coupling constant g the kernel $\mathcal{K}$ coincides with the one-boson exchange matrix element. It should be noted that if the constant g was chosen so that the matrix element K_2 would give the one-boson-exchange potential providing attraction in the fermion-antifermion system, then the subsequent term of the kernel $K_2 G_0 (1 - \Pi_2) K_2$ being proportional to g^2, will, as in [13], result in terms involving repulsion.

Introducing the notation

$$\langle \vec{p}_1\sigma_1, \vec{p}_2\sigma_2 | \mathcal{K}(\lambda\tau, \lambda\tau_1) | \vec{k}_1\nu_1, \vec{k}_2\nu_2\rangle = (2\pi)^4 \delta^{(4)}(\lambda\tau_1 + p_1 + p_2 - \lambda\tau - k_1 - k_2)$$

$$\cdot \left[2p_{10}\cdot 2p_{20}\cdot 2k_{10}\cdot 2k_{20}\right]^{-1} \cdot V^{\sigma_1\sigma_2}_{\nu_1\nu_2}(p_1, p_2; \lambda\tau | k_1, k_2; \lambda\tau_1) =$$

$$= \frac{(2\pi)^4 \delta^{(4)}(\lambda\tau_1 + p_1 + p_2 - k_1 - k_2 - \lambda\tau)}{\sqrt{2p_{10}\cdot 2p_{20}\cdot 2k_{10}\cdot 2k_{20}}} \cdot \bar{u}^{\sigma_1}_{\alpha}(p_1)\, \bar{v}^{\nu_2}_{\xi}(k_2) \cdot V^{\alpha\beta}_{\xi\mu} \cdot u^{\nu_1}_{\mu}(k_1) \cdot v^{\sigma_2}_{\beta}(p_2)$$

and inserting it and (2.1) into (2.3), we arrive at the equation

$$\Gamma^{\sigma_1\sigma_2}_{J,m_J}(p_1,p_2|\mathcal{P},\lambda\tau) = \frac{1}{(2\pi)^3}\sum_{\nu_1\nu_2}\int d\tau_1\, d^4k_1\, d^4k_2\, \theta(k_1^0)\,\delta(k_1^2-m^2)\cdot \tag{2.4}$$
$$\cdot\theta(k_2^0)\,\delta(k_2^2-m^2)\cdot(\tau_1-i\varepsilon)^{-1}\cdot\delta^{(4)}(\mathcal{P}-k_1-k_2-\lambda\tau_1)\cdot$$
$$\cdot V^{\sigma_1\sigma_2}_{\nu_1\nu_2}(p_1,p_2;\lambda\tau|k_1,k_2;\lambda\tau_1)\cdot\Gamma^{\nu_1\nu_2}_{J,m_J}(k_1,k_2|\mathcal{P},\lambda\tau_1)\,.$$

In /9/ it is shown that the kernel in (2.3) $V^{\sigma_1\sigma_2}_{\nu_1\nu_2}(p_1,p_2;\lambda\tau|k_1,k_2;\lambda\tau_1)$ is a matrix element corresponding to the sum of all irreducible (in the sense of one-spurion and two-particle cuttings) diagrams describing the two-fermion interaction with a given Hamiltonian H(p). Graphically, equation (2.4) can be represented as shown in Fig. 1 where dotted lines denote the incoming spurion with momentum $\lambda\tau$ and spurion propagator $(\tau-i\varepsilon)^{-1}$. Inner spinor lines in a diagram technique of Kadyshevsky correspond $S^{+}(k_1)$ and $S^{+}(k_2)$ functions.

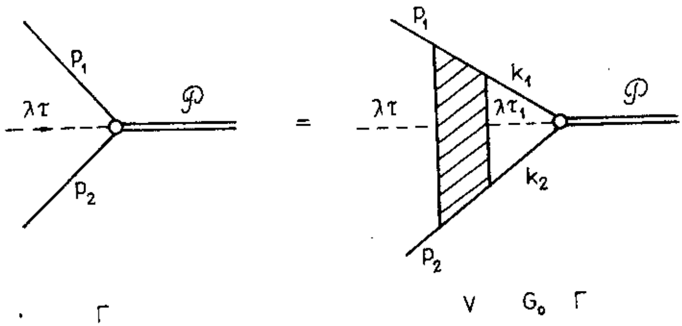

Fig.1

In what follows we will choose the direction of the unit λ - vector along the 4-momentum of a composite system, i.e., equal to the system 4-velocity /6,9/

$$\lambda_\mu = \mathcal{P}_\mu/\sqrt{\mathcal{P}^2} = \mathcal{P}_\mu/M\,.$$

Note that in (2.4) integration over momenta of internal particles runs over the upper sheet of the mass hyperboloid

$$k_0^2 - \vec{k}^2 = m^2 \tag{2.5}$$

and the additional 4-vector $\delta^{(4)}$ in $\lambda\tau$ -function ensures the escape off the energy shell.

In (2.3) we take advantage of the invariance of volume elements $d\Omega_{k_i} = d^4 k_i\, \theta(k_i^0)\, \delta(k_i^2 - m^2)$ in order to pass, in them and δ -function argument, to the vectors $(\overset{\circ}{k}_i)^\mu = (\Lambda_{\mathcal{P}}^{-1} k_i)^\mu$, where $\Lambda_{\mathcal{P}}^{-1}$ is the Lorentz boost to the rest frame of a composite particle: $\Lambda_{\mathcal{P}}^{-1}\mathcal{P} = (M, \vec{0})$, so that, for instance $(\Lambda_{\mathcal{P}} \equiv \Lambda_{\lambda_{\mathcal{P}}})$ /14/

$$\vec{\overset{\circ}{k}}_1 = \overrightarrow{(\Lambda_{\mathcal{P}}^{-1} k_1)} \equiv \vec{\Delta}_{k_1, m\lambda_{\mathcal{P}}} = \vec{k}_1 - m\vec{\lambda}_{\mathcal{P}}\left[k_{10} - \frac{k_1 \lambda_{\mathcal{P}}}{1+\lambda_{\mathcal{P}}^0}\right] \tag{2.6}$$

$$(\overset{\circ}{k}_1)_0 = (\Lambda_{\mathcal{P}}^{-1} k_1)_0 \equiv \Delta^0_{k_1, m\lambda_{\mathcal{P}}} = \sqrt{m^2 + \vec{\overset{\circ}{k}}_1{}^2} = k_1^\mu \lambda_{\mathcal{P}\mu} \tag{2.7}$$

$$\delta^{(4)}\left[\mathcal{P} - \lambda_{\mathcal{P}}\tau_1 - k_1 - k_2\right] = \delta^{(4)}\left[\left(1 - \frac{\tau_1}{M}\right)\Lambda_{\mathcal{P}}^{-1}\mathcal{P} - \Lambda_{\mathcal{P}}^{-1}(k_1 + k_2)\right] =$$

$$= \delta\left[M - \tau_1 - (\overset{\circ}{k}_1)_0 - (\overset{\circ}{k}_2)_0\right] \cdot \delta^{(3)}\left(\vec{\overset{\circ}{k}}_1 + \vec{\overset{\circ}{k}}_2\right) \tag{2.8}$$

The spatial part of the vector $\overset{\circ}{k}_{1\mu}$ has the meaning of a covariant generalization of the momentum vector of a particle in the c.m.s. introduced earlier in /14/. Indeed, as is seen from (2.8) $\vec{\overset{\circ}{k}}_1 = -\vec{\overset{\circ}{k}}_2$ that because the vectors $(\overset{\circ}{k}_i)_\mu$ are on the mass shell the equation (2.5) ensures the equality of invariant time components $(\overset{\circ}{k}_1)_0 = (\overset{\circ}{k}_2)_0$. In what follows we shall supply with the index zero from above all the quantities representing the covariant generalization to an arbitrary system of quantities defined in the c.m.s.

After integration in (2.4) over $d^4 k_2$ and $d\tau_1$ we get the equation

$$\Gamma^{\sigma_1\sigma_2}_{J, m_J}(p_1, p_2 | \mathcal{P}, \lambda\tau) = \frac{1}{(2\pi)^3}\sum_{\nu_1,\nu_2}\int \frac{d^3\overset{\circ}{k}_1}{2(\overset{\circ}{k}_1)_0} \cdot \frac{1}{2(\overset{\circ}{k}_1)_0\left[M - 2(\overset{\circ}{k}_1)_0 - i\varepsilon\right]} \tag{2.9}$$

$$\cdot V^{\sigma_1\sigma_2}_{\nu_1\nu_2}(p_1, p_2; \lambda\tau | k_1, k_2; \lambda\tau) \cdot \Gamma^{\nu_1\nu_2}_{J, m_J}(k_1, k_2 | \mathcal{P}, \lambda\tau),$$

where $\tau_1 = M - 2(\overset{\circ}{k}_1)_0$; $\tau = M - 2(\overset{\circ}{p}_1)_0$ and the quantities k_1 and k_2 in the r.h.s. are connected with $(\overset{\circ}{k}_{1,2})^\mu$ by the transformation $(k_{1,2})^\mu = (\Lambda_{\mathcal{P}}\overset{\circ}{k}_{1,2})^\mu$ and $\overset{\circ}{k}_2{}^\mu = \left[(\overset{\circ}{k}_1)_0, -\vec{\overset{\circ}{k}}_1\right]$.

In the next section we shall need some quantities which will be considered as the covariant generalization to an arbitrary reference frame used earlier in refs. /1-4/ for the c.m.s. objects from the Lobachevsky space /15/, realized on the upper sheet of the hyperboloid (2.5). For instance, the vector considered in /7/ as the difference of two vectors $\vec{p}$ and $\vec{k}$ in the Lobachevsky space[x)] (or the transfer-momentum vector in the Lobachevsky space)

$$\vec{k}(-)\vec{p} \equiv \vec{\Delta}_{k,p} = \overrightarrow{(\Lambda_p^{-1}k)} = \vec{k} - \frac{\vec{p}}{m}\left(k_0 - \frac{kp}{m+p_0}\right) \quad (2.10)$$

and its time component may be introduced also for covariantly defined vectors of the particle momenta in the c.m.s. (2.6) and (2.7) by the relations /17/

$$\vec{\Delta}_{\overset{\circ}{k}_1,\overset{\circ}{p}_1} \equiv \vec{\overset{\circ}{k}}_1(-)\vec{\overset{\circ}{p}}_1 = \vec{\overset{\circ}{k}}_1 - \frac{\vec{\overset{\circ}{p}}_1}{m}\left[(\overset{\circ}{k}_1)_0 - \frac{\vec{\overset{\circ}{k}}_1\cdot\vec{\overset{\circ}{p}}_1}{m+(\overset{\circ}{p}_1)_0}\right] \quad (2.11)$$

$$(\overset{\circ}{\Delta}_{\overset{\circ}{k}_1,\overset{\circ}{p}_1})_0 = \sqrt{m^2 + \vec{\overset{\circ}{\Delta}}{}^2_{\overset{\circ}{k}_1,\overset{\circ}{p}_1}} = \frac{(\overset{\circ}{k}_1)^\mu(\overset{\circ}{p}_1)_\mu}{m} = \frac{(\overset{\circ}{k}_1)_0(\overset{\circ}{p}_1)_0 - \vec{\overset{\circ}{k}}_1\cdot\vec{\overset{\circ}{p}}_1}{m} \quad (2.12)$$

In analogy with the construction in ref./3/ of the half-transfer momentum out of the transfer momentum vector $\vec{k}(-)\vec{p}$, we define a half-transfer momentum corresponding to $\vec{\overset{\circ}{k}}(-)\vec{\overset{\circ}{p}}$ as follows /18/

$$\vec{\overset{\circ}{\varkappa}}_1 = \{\vec{\overset{\circ}{k}}_1(-)\vec{\overset{\circ}{p}}_1\}\cdot\sqrt{\frac{m}{2\{m+(\overset{\circ}{\Delta}_{\overset{\circ}{k}_1,\overset{\circ}{p}_1})_0\}}}, \quad (2.13)$$

$$(\overset{\circ}{\varkappa}_1)_0 = \sqrt{m^2 + \vec{\overset{\circ}{\varkappa}}{}^2_1}.$$

Since, as has been mentioned, in accordance with (2.8) in any coordinate system $\vec{\overset{\circ}{k}}_1 = -\vec{\overset{\circ}{k}}_2 = \vec{\overset{\circ}{k}}$; $\vec{\overset{\circ}{p}}_1 = -\vec{\overset{\circ}{p}}_2 = \vec{\overset{\circ}{p}}$, then

$$\vec{\Delta}_{\overset{\circ}{k}_1,\overset{\circ}{p}_1} \equiv \vec{\overset{\circ}{k}}_1(-)\vec{\overset{\circ}{p}}_1 = -\{\vec{\overset{\circ}{k}}_2(-)\vec{\overset{\circ}{p}}_2\} \equiv -\vec{\Delta}_{\overset{\circ}{k}_2,\overset{\circ}{p}_2} = \vec{\Delta}_{\overset{\circ}{k},\overset{\circ}{p}}, \quad (2.14)$$

$$\vec{\overset{\circ}{\varkappa}}_1 = -\vec{\overset{\circ}{\varkappa}}_2 = \vec{\overset{\circ}{\varkappa}}. \quad (2.15)$$

Four- vector $(\overset{\circ}{p}_{1,2})^\mu = (\Lambda_{\mathcal{P}}^{-1}p_{1,2})^\mu$ and $(\overset{\circ}{k}_{1,2})^\mu$ belong to the same mass hyperboloid (2.8) as vectors $p_{1,2}$ and $k_{1,2}$ do. Therefore they can be parametrized by spherical coordinates as follows:

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x) In contrast to the Euclidean difference of two vectors $\vec{k}-\vec{p}$ the difference in the Lobachevsky space will be denoted by symbol (-)/7/.

$$(\overset{\circ}{p})_0 = m\, ch\chi_p\ ;\ \vec{\overset{\circ}{p}} = m\,\vec{e}_p\, sh\chi_p\ ;\ \vec{e}_p = \frac{\vec{\overset{\circ}{p}}}{|\vec{\overset{\circ}{p}}|}\ , \tag{2.16}$$

$$(\overset{\circ}{k})_0 = m\, ch\chi_k\ ;\ \vec{\overset{\circ}{k}} = m\,\vec{e}_k\, sh\chi_k\ ,\ \vec{e}_k = \frac{\vec{\overset{\circ}{k}}}{|\vec{\overset{\circ}{k}}|}\ . \tag{2.17}$$

In these coordinates the equality (2.12) represents a formula for the cosine of a compound angle in the Lobachevsky trigonometry /15/

$$ch\chi_{\overset{\circ}{\Delta}} = \sqrt{1 + \frac{(\vec{\overset{\circ}{k}}(-)\vec{\overset{\circ}{p}})^2}{m^2}} = ch\chi_k\, ch\chi_p - sh\chi_k \cdot sh\chi_p (\vec{e}_k\, \vec{e}_p)$$

and the relation between the momentum transferred $(\overset{\circ}{\Delta})_0 = m\, ch\chi_\Delta$ $\vec{\overset{\circ}{\Delta}} = m\,\vec{e}_\Delta\, sh\chi_\Delta$ and the half-transfer (2.13) takes the form

$$(\overset{\circ}{\varkappa})_0 = m\, ch\left(\frac{\chi_\Delta}{2}\right) = m\sqrt{\frac{m + (\Delta_{\overset{\circ}{k},\overset{\circ}{p}})_0}{2m}}\ , \tag{2.18}$$

$$\vec{\overset{\circ}{\varkappa}} = m\,\vec{e}_\Delta\, sh\left(\frac{\chi_\Delta}{2}\right) = \vec{\Delta}_{\overset{\circ}{k},\overset{\circ}{p}}\sqrt{\frac{m}{2[m + (\Delta_{\overset{\circ}{k},\overset{\circ}{p}})_0]}}\ .$$

The four-dimensional transferred momentum squared $t = (k-p)^2$ is expressed through the momentum squared in the Lobachevsky space as follows:

$$\begin{aligned} t &= (k-p)^2 = 2m^2 - 2kp = 2m^2 - 2(\Lambda_{\mathcal{P}}^{-1}k)^\mu(\Lambda_{\mathcal{P}}^{-1}p)_\mu = \\ &= 2m^2 - 2\overset{\circ}{k}{}^\mu \overset{\circ}{p}_\mu = 2m^2 - 2m\sqrt{m^2 + (\vec{\overset{\circ}{k}}(-)\vec{\overset{\circ}{p}})^2} = \\ &= 2m^2 - 2m\sqrt{m^2 + (\vec{\Delta}_{\overset{\circ}{k},\overset{\circ}{p}})^2}\ . \end{aligned} \tag{2.19}$$

In view of (2.19), (2.13), (2.18), the invariant Feynman propagator assumes the form

$$\frac{1}{\mu^2 - (p-k)^2} = \frac{1}{\mu^2 - 2m^2 + 2m\sqrt{m^2 + (\vec{\Delta}_{\overset{\circ}{k},\overset{\circ}{p}})^2}} = \frac{1}{\mu^2 + 4\overset{\circ}{\varkappa}{}^2}\ . \tag{2.20}$$

In the nonrelativistic limit $\vec{k}(-)\vec{p} \to \vec{k} - \vec{p}\ ;\ \vec{\overset{\circ}{\varkappa}} \to (\vec{k} - \vec{p})/2$.

The wave function of the system of a fermion and antifermion with momenta $\vec{k}_1$ and $\vec{k}_2$, spin polarizations σ_1 and σ_2, which form a bound state with momentum $\mathcal{P}$, mass M, total spin $\mathcal{J}$ and its projection onto the z-axis m_J, will be given by the formula

$$\Psi^{\sigma_1\sigma_2}_{J,m_J}(k_1,k_2/\mathcal{P},\chi_{\mathcal{P}}\tau) = \frac{\Gamma^{\sigma_1\sigma_2}_{J,m_J}(k_1,k_2/\mathcal{P},\chi_{\mathcal{P}}\tau)}{2\Delta^\circ_{k,m\chi_{\mathcal{P}}}[M - 2\Delta^\circ_{k,m\chi_{\mathcal{P}}} - i\varepsilon]}\ . \tag{2.21}$$

The substitution of (2.21) into (2.9) changes the equation for the Γ function to the form

$$2\Delta^{0}_{k,m\lambda_{P}}\left(M-2\Delta^{0}_{k,m\lambda_{P}}\right)\Psi^{\sigma_1\sigma_2}_{J,m_J}\left(k_1,k_2/P,\lambda_P\tau\right)= \tag{2.22}$$

$$=\frac{1}{(2\pi)^3}\sum_{\nu_1,\nu_2=\pm\frac{1}{2}}\int\frac{d^3\vec{\Delta}_{k',m\lambda_P}}{2\Delta^{0}_{k',m\lambda_P}}\cdot V^{\sigma_1\sigma_2}_{\nu_1\nu_2}\left(k_1,k_2;\lambda_P\tau/k_1',k_2';\lambda_P\tau'\right)\cdot\Psi^{\nu_1\nu_2}_{J,m_J}\left(k_1',k_2'/P\lambda_P\tau'\right)$$

of the equation obtained earlier /9/ for c.m.s. As it was shown in /16/ eq.(2.22) has the same form as the spin equation that was deduced in the framework of Logunov-Tavkhelidze single-time approach. Equation (2.22) can be considered also as of the generalization to the spin case of the Kadyshevsky covariant spinless equation /6(III),17/.

3. Three-dimensional form of the kernel in the one-meson-exchange approximation

Let us consider the kernel of equation (2.9) $V^{\sigma_1\sigma_2}_{\nu_1\nu_2}(p_1p_2;\lambda\tau|k_1,k_2;\lambda\tau_1)$ in the second order in coupling constant corresponding to diagrams drawn in Fig.2 (annihilation terms are omitted). For the Hamiltonian $\mathcal{H}_{int}=g:\bar{\Psi}(x)\gamma_5\Psi(x)\varphi(x):$ this kernel has the form /9/ $t=(k_1-p_1)^2$:

$$V^{\sigma_1\sigma_2}_{(2)\,\nu_1\nu_2}\left(p_1,p_2;\lambda\tau/k_1,k_2;\lambda\tau_1\right)=-\frac{g^2}{\sqrt{\mu^2-t+\frac{1}{4}(\tau-\tau_1)^2}}\,;$$

$$\cdot\left[\frac{1}{2}(\tau+\tau_1)+\sqrt{\mu^2-t+\frac{1}{4}(\tau-\tau_1)^2}\right]^{-1}\cdot\bar{u}^{\sigma_1}(p_1)\gamma_5u^{\nu_1}(k_1)\cdot\bar{v}^{\nu_2}(k_2)\gamma_5v^{\sigma_2}(p_2). \tag{3.1}$$

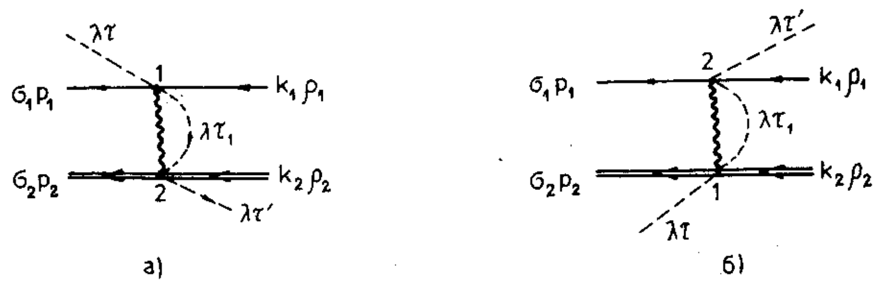

Fig.2

In virtue of the conservation of 4-velocity of the system/6(III)/

$$\lambda_{\mathcal{P}} \equiv \frac{(p_1+p_2)}{\sqrt{(p_1+p_2)^2}} = \frac{(k_1+k_2)}{\sqrt{(k_1+k_2)^2}} = \lambda_k$$

and the conservation of 4-momentum (including the spurion lines) $p_1+p_2-\lambda\tau = k_1+k_2-\lambda\tau_1$ which gives the relation $\tau-\sqrt{S_p}=\tau_1-\sqrt{S_k}$ transforms on the energy shell $\tau=\tau_1=0$ into the Feynman matrix element

$$T^{\sigma_1\sigma_2}_{(2)\,\lambda_1\lambda_2}(p_1,p_2|k_1,k_2) = -g^2\frac{\bar{u}^{\sigma_1}(p_1)\gamma_5 u^{\lambda_1}(k_1)\cdot\bar{v}^{\lambda_2}(k_2)\gamma_5 v^{\sigma_2}(p_2)}{\mu^2-(p_1-k_1)^2}. \quad (3.2)$$

Spin structures of matrix elements (3.1)and(3.2)are identical because additional spurion propagators $(\tau_1-i\epsilon)^{-1}$ appear in the Kadyshevsky technique as Fourier transforms of entering into T-product scalar $\theta[x^0-y^0]$ functions written in the invariant variables. Therefore, we first consider the spin structures.

The bispinor $u(p_1)$ x) entering into (3.1),(3.2) can be expressed through the bispinor in the fermion rest frame

$$u^{\sigma}(p_1) = S(p_1)u^{\sigma}(0) = S(\mathcal{P})\cdot S^{-1}(\mathcal{P})\cdot S(p_1)u^{\sigma}(0) \quad (3.3)$$

Matrices $S(\mathcal{P})$ and $S(p_1)$ correspond to the pure Lorentz transformations

$$S(p) = \sqrt{(p_0+m)/2m}\,\left(1+\vec{\alpha}\vec{p}/p_0+m\right);\quad \vec{\alpha}=\gamma_0\vec{\gamma},$$

which do not compose a group. Their product is not, in general, a pure Lorentz transformation onto a resultant vector, it rather contains also the Wigner rotation $V(\Lambda_{\mathcal{P}},k)$ defined by

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x)

Bispinors $u^{\sigma}(k)$ and $v^{\sigma}(k)$ enter into the spinor-field expansion over the creation and annihilation operators as follows:

$$\Psi(x) = \frac{1}{(2\pi)^{3/2}}\sum_{\sigma=-1/2}^{1/2}\int\frac{d^3\vec{k}}{2k_0}\left\{e^{ikx}b^+_{\sigma}(k)v^{\sigma}(k)+e^{-ikx}a_{\sigma}(k)u^{\sigma}(k)\right\},$$

$$\bar{\Psi}(x) = \frac{1}{(2\pi)^{3/2}}\sum_{\sigma=-1/2}^{1/2}\int\frac{d^3\vec{k}}{2k_0}\left\{e^{ikx}a_{\sigma}(k)\bar{u}^{\sigma}(k)+e^{-ikx}b_{\sigma}(k)\bar{v}^{\sigma}(k)\right\},$$

$$S^{-1}(p)\cdot S(k)=S(\Lambda_p^{-1}k)\cdot I\otimes D^{1/2}\{V^{-1}(\Lambda_p,k)\},$$

where I is a unit 4x4 matrix, and the matrix for the Wigner rotation of spin 1/2 has the form /14(I)/

$$D^{1/2}\{V^{-1}(\Lambda_p,k)\}=\sqrt{\frac{(p_0+m)(k_0+m)}{2m(\Delta^0_{p,k}+m)}}\left\{1-\frac{\vec{p}\vec{k}+i\vec{\sigma}[\vec{p}\times\vec{k}]}{(p_0+m)(k_0+m)}\right\}.$$

As a result, the formula (3.2) can be rewritten in the form

$$u^{\sigma}(p)=S(\mathcal{P})\cdot S(\Delta_{p,m\lambda_{\mathcal{P}}})\cdot I\otimes D^{1/2}\{V^{-1}(\Lambda_{\mathcal{P}},k)\}\,u^{\sigma}(0) \qquad (3.4)$$

and, correspondingly, the currents

$$\bar{u}^{\sigma_1}(p_1)\gamma_5 u^{\lambda_1}(k_1)=\bar{u}^{\sigma_1}(0)\,D^{1/2}\{V^{-1}(\Lambda_{\mathcal{P}},p_1)\}\cdot S^{-1}(\Delta_{p_1,m\lambda_{\mathcal{P}}})\cdot$$
$$\cdot S^{-1}(\mathcal{P})\gamma_5 S(\mathcal{P})\cdot S(\Delta_{k_1,m\lambda_{\mathcal{P}}})\cdot D^{1/2}\{V^{-1}(\Lambda_{\mathcal{P}},k_1)\}\,u^{\lambda_1}(0). \qquad (3.5)$$

A similar result follows also for the term $\bar{v}^{\lambda_2}(k_2)\gamma_5 v^{\sigma_2}(p_2)$ if by the transformation $v=C\bar{u}^{c}_{\;T}$; $\bar{v}=u^{c}\cdot C^{-1}_{\;T}\ (C=\gamma_0\gamma_2)$ one passes to the charge-conjugate bispinor u^c of the same form as u. In virtue of the dependence of matrix $S(\mathcal{P})$ on $\vec{\alpha}=\gamma_0\vec{\gamma}$ it may be interplaced with γ_5. As a result

$$\bar{u}^{\sigma_1}(p_1)\gamma_5 u^{\lambda_1}(k_1)=\bar{u}^{\sigma_1}(0)\cdot D^{1/2}\{V^{-1}(\Lambda_{\mathcal{P}},p_1)\}\cdot\gamma_5\cdot S(\Delta^0_{p,k})\qquad (3.6)$$
$$\cdot D^{1/2}\{V^{-1}(\Lambda_{\Delta_{k_1,m\lambda_{\mathcal{P}}}};\Delta_{p_1,m\lambda_{\mathcal{P}}})\}\cdot D^{1/2}\{V^{-1}(\Lambda_{\mathcal{P}},k_1)\}\,u^{\lambda_1}(0),$$

where the auxiliary function $D^{1/2}\{V^{-1}(\Lambda_{\Delta_{k_1,m\lambda_{\mathcal{P}}}},\Delta_{p_1,m\lambda_{\mathcal{P}}})\}$ becomes the function $D^{1/2}\{V^{-1}(\Lambda_k,p)\}$ of the c.m.s. parametrization /1-3/. Further, following the results of ref. /18/, the expression for the amplitude (3.2) (and for (3.4)) will be written in the form

$$\langle \vec{p}_1\sigma_1;\vec{p}_2\sigma_2|T_{(2)PS}|\vec{k}_1\nu_1;\vec{k}_2\nu_2\rangle = (T_{(2)})^{\sigma_1\sigma_2}_{\nu_1\nu_2}(\vec{p}_1\vec{p}_2;2\tau|\vec{k}_1,\vec{k}_2;2\tau_1)=$$

$$=\sum \mathcal{D}^{1/2}_{\sigma_1\sigma_{1p}^{\circ}}\{V^{-1}(\Lambda_{\mathcal{P}},p_1)\}\ \mathcal{D}^{1/2}_{\sigma_2\sigma_{2p}^{\circ}}\{V^{-1}(\Lambda_{\mathcal{P}},p_2)\}\cdot$$

$$\cdot\langle \vec{p}_1\sigma_{1p}^{\circ};\vec{p}_2\sigma_{2p}^{\circ}|T_{(2)PS}|\vec{k}_1\nu_{1p}^{\circ};\vec{k}_2\nu_{2p}^{\circ}\rangle\cdot \tag{3.7}$$

$$\cdot\ \mathcal{D}^{1/2}_{\nu_{1p}^{\circ},\nu_{1k}^{\circ}}\{V^{-1}(\Lambda_{k_1^{\circ}},p_1^{\circ})\}\cdot \mathcal{D}^{1/2}_{\nu_{1k}^{\circ},\nu_1}\{V^{-1}(\Lambda_{\mathcal{P}},k_1)\}\cdot$$

$$\cdot\ \mathcal{D}^{1/2}_{\nu_{2p}^{\circ},\nu_{2k}^{\circ}}\{V^{-1}(\Lambda_{k_2^{\circ}},p_2^{\circ})\}\ \mathcal{D}^{1/2}_{\nu_{2k}^{\circ},\nu_2}\{V^{-1}(\Lambda_{\mathcal{P}},k_2)\},$$

(over the repeated indices x) summation is implied), where the matrix element remaining after separating the Wigner rotations xx)

$$\langle p_1\sigma_{1p}^{\circ};p_2\sigma_{2p}^{\circ}|T_{(2)PS}|k_1\nu_{1p}^{\circ};k_2\nu_{2p}^{\circ}\rangle=$$

$$=-4g^2(\mu^2+4\overset{\circ}{\vec{\varkappa}}{}^2)^{-1}(\vec{\sigma}_1\overset{\circ}{\vec{\varkappa}})_{\sigma_{1p}^{\circ},\nu_{1p}^{\circ}}\cdot(\vec{\sigma}_2\overset{\circ}{\vec{\varkappa}})_{\sigma_{2p}^{\circ},\nu_{2p}^{\circ}} \tag{3.8}$$

$$(\vec{\sigma}_1\overset{\circ}{\vec{\varkappa}})_{\nu_1\nu_2}\equiv\xi^{*\nu_1}(\vec{\sigma}_1\overset{\circ}{\vec{\varkappa}})\xi_{\nu_2}\quad;\quad t=(k-p)^2=-4\overset{\circ}{\vec{\varkappa}}{}^2,$$

has the "absolute form" (see /3/) and depends only on the vector of half-transfer momentum $\overset{\circ}{\vec{\varkappa}}$. This fact will be important for constructing from (3.8) of a quasipotential, local in the relativistic configurational representation, in an arbitrary reference frame. Note also that according to /19/ the Wigner rotation has the

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x) In (3.7) somewhat complicated notation is introduced for spin indices, however, it is convenient for interpretation. Namely, the momentum in a row with the spin polarization signifies, e.g., that this polarization index is "sitting" on the momentum ($\overset{\circ}{p}$ for σ_{1p}°, ect.

xx) $u^{\sigma}(o)=\sqrt{m}\,(\xi_\sigma,\xi_\sigma)$, where ξ_σ are the two-component Pauli spinors with the normalization condition $\xi^{+\sigma}\xi_{\sigma'}=\delta_{\sigma\sigma'}$.

geometrical meaning of the angle of rotation of the spin vector that occurs after its parallel translation along a closed triangle in the Lobachevsky space.

Conclusion

Let us summarize our results. In Sec. 2 the covariant three-dimensional equation is obtained for the fermion-antifermion wave function with the kernel defined by using the diagram technique appearing in a covariant formulation of quantum field theory. In Sec. 3 from this kernel in the one-meson-exchange approximation we have extracted the part local in the Lobachevsky momentum space.

Our further task is to transform eq. (2.11) to a form, that would contain only the local part of the kernel given by (3.8), and then to write the equation in the relativistic configuration representation.

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